De la columna más alta de Euler a los rascacielos ideales: el punto de vista matemático

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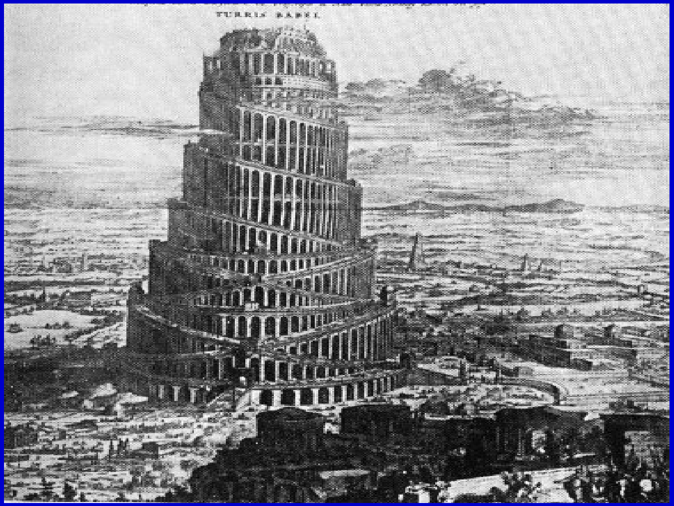
Universidad de Alcalá,

1 de diciembre de 2004

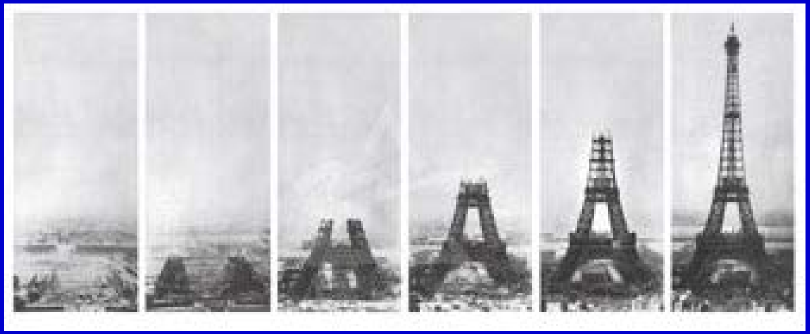
1.Introducción

Sueños permanentes:

Edificios cada vez más altos …



…. y monumentos y torres
cada vez más altas.



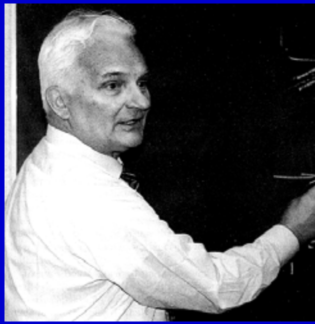
Contenido de la charla

2. La columna más alta de Euler

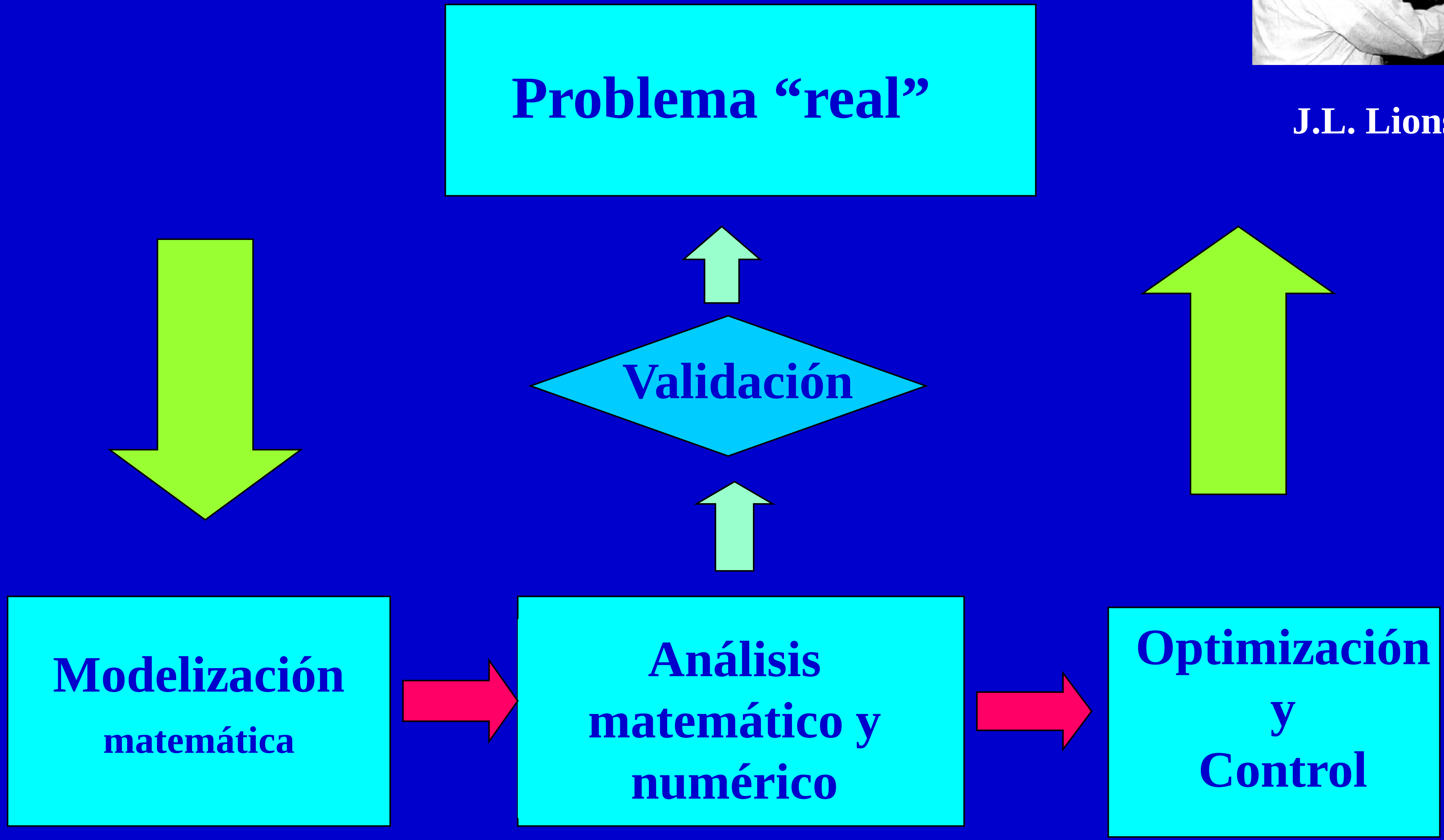
Leonhard Euler (1707-1783)

3. Columnas reforzadas y heterogéneas: rascacielos

Exposición en términos de la "universal trilogy" de la Matemática Aplicada

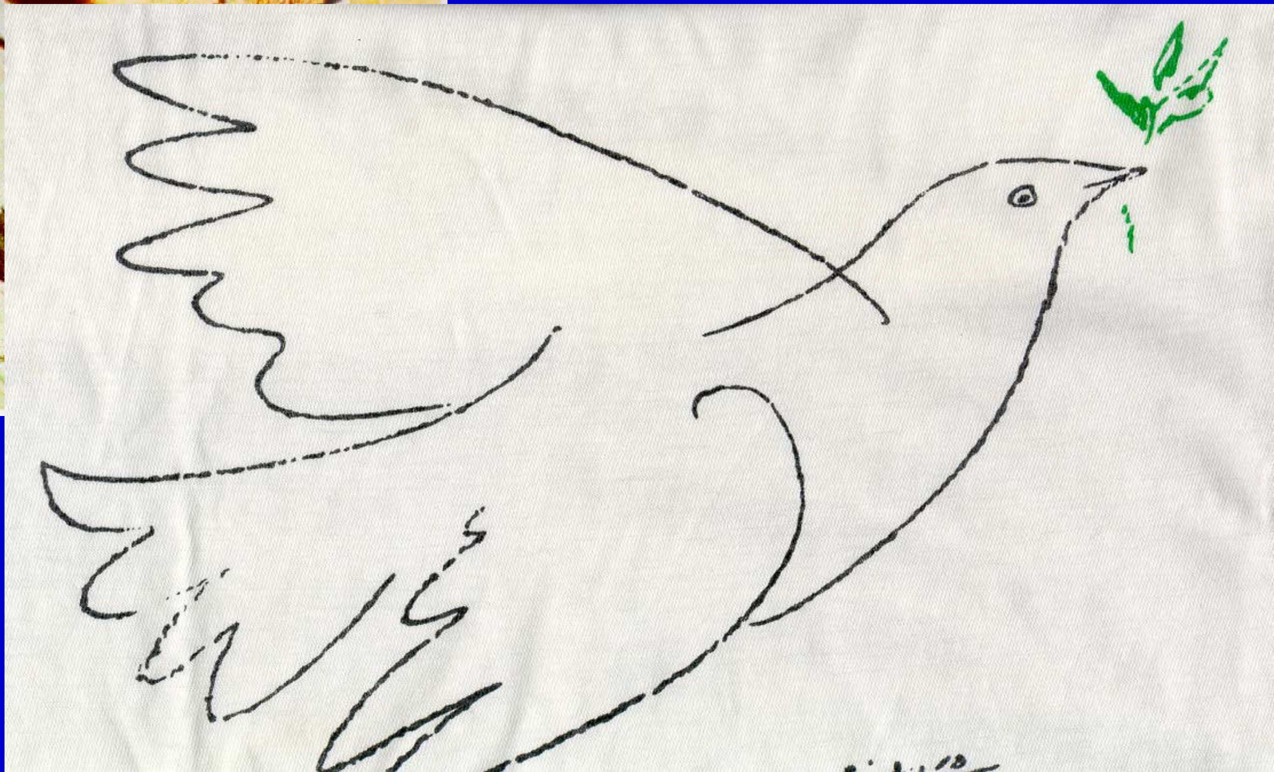


J.L. Lions

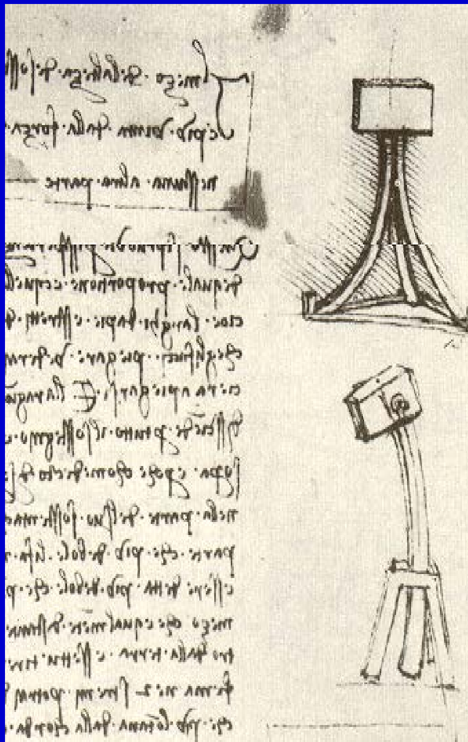


2. La columna más alta de Euler

2.1. Modelización: puntos similares en Arte



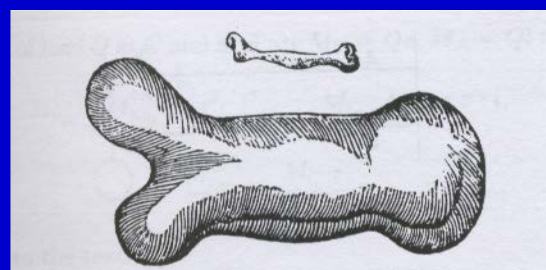
Estudios pioneros

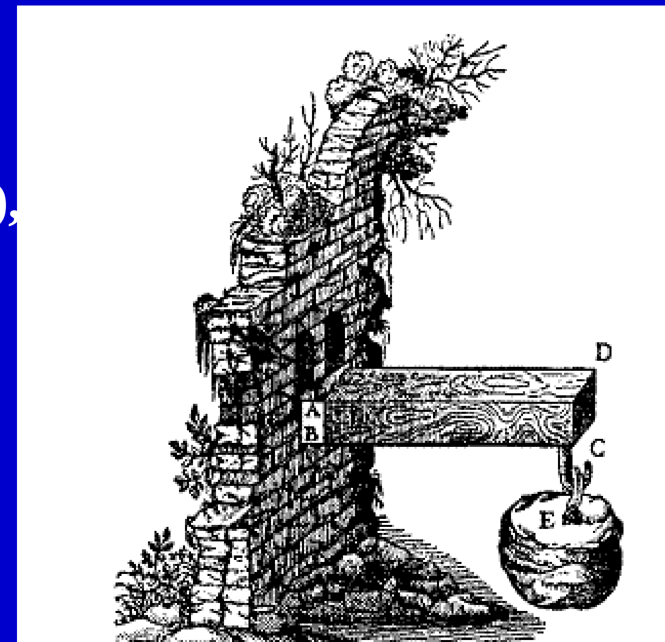


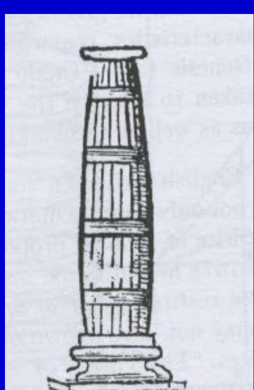
Leonardo da Vinci (1452-1519)

Galileo Galilei (1562-1642),

1638,



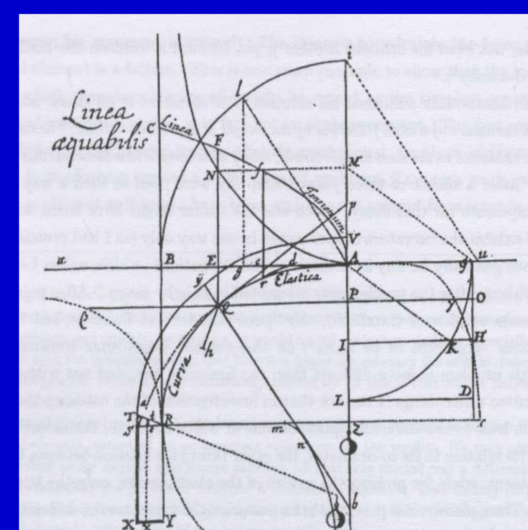




Ignace-Gaston Pardies

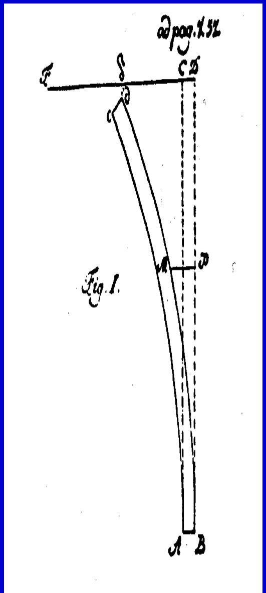
1673

Jacques Bernoulli (1654-1705),
1690

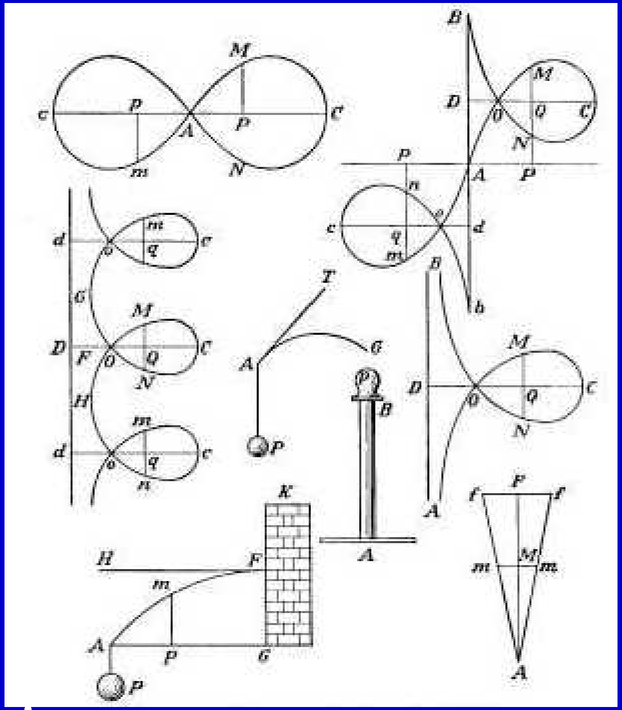


Jean Bernoulli (1667-1748):

On the curvature of extensible strings
(1691, ..., 1742)

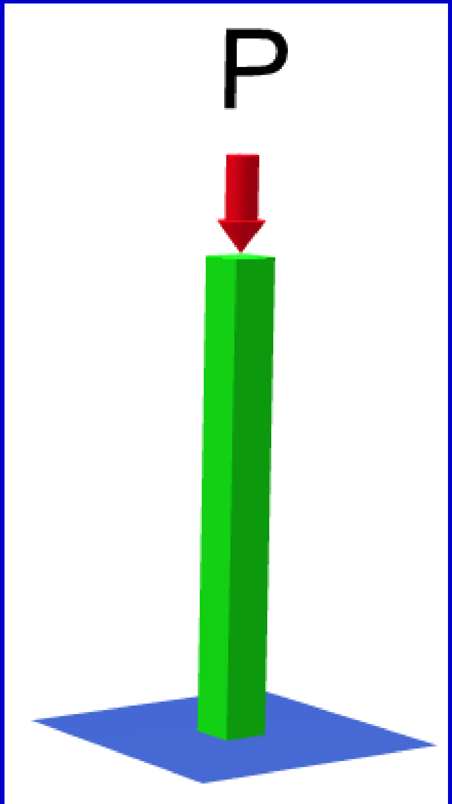


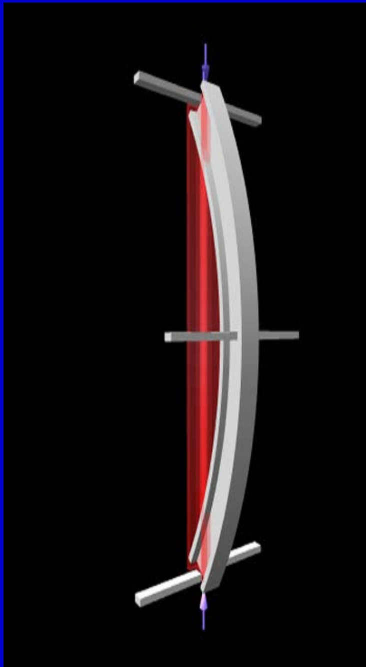
Euler, 1727



Modulo de extension

(modulo de Young)

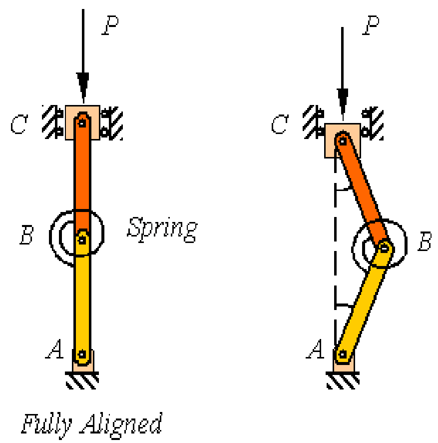




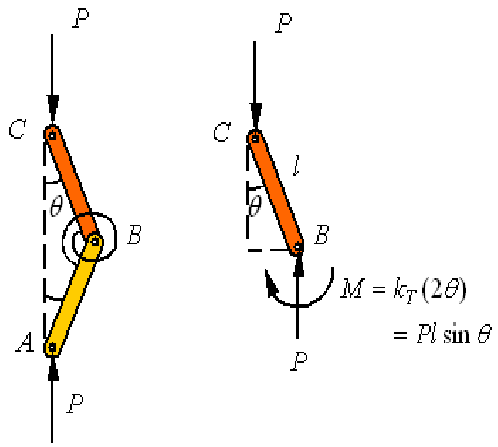
Buckling of Columns

Basic idea:

Consider a column that is constructed from two pin-connected links with a torsional spring connected between the two links as shown in the figure. As long as the two bars AB and BC are perfectly aligned, the system is in equilibrium and one theoretically can increase the load until the beams fail in compression.



In reality, the two members can never be perfectly aligned so the system supports the load by the aid of the torsional spring and takes a shape such as shown in the right figure above.



Since the member *ABC* is a two-force member, the loads applied at *A* and *C* must be equal and along the line connecting *A* to *C* as shown in the above left figure. The free-body-diagram of AB shown on the right side of the figure above indicates that for equilibrium to hold, the miss-alignment angle θ must increase until the moment in the torsional spring increases to balance the couple developed by the two vertical forces. This requires that

$$k_T(2\theta) = Pl \sin\theta$$

where k_T is the stiffness of the torsional spring and the reader notes that the torsional spring is twisted twice the miss-alignment angle θ. Assuming small miss-alignment angles so that one can replace $\sin\theta$ by θ, one gets

$$(2k_T - Pl)\theta = 0$$

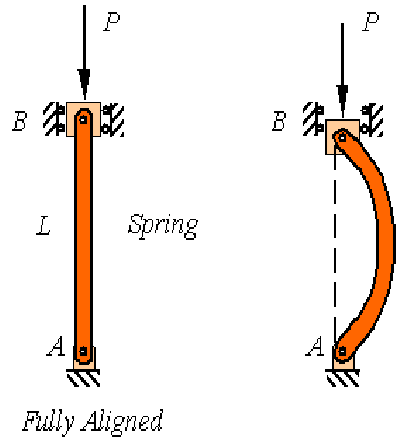
Obviously, $\theta = 0$ is a solution to this equation. This solution represents the trivial solution that reflects the perfectly aligned system. But, this system has a non-trivial solution where the term in the parenthesis becomes zero to require

$$P_{cr} = \frac{2k_T}{l}$$

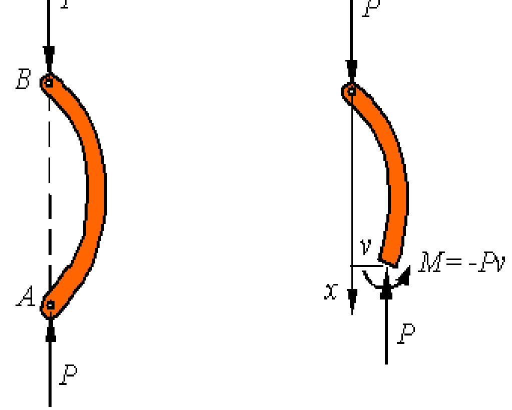
The load calculated in this way is called the critical load, designated by the subscript "cr". For loads smaller than the critical load, the system will have accelerations that are consistent with bringing the system back into alignment. For loads above this critical load the system has accelerations consistent with increasing the miss-alignment angle, resulting in the collapse of the system. Therefore, the system is considered to be capable of carrying loads up to the critical load.

Buckling in a simply supported column:

Consider the pin-connected column *AB* of length *L* as shown in the following figure. Similar to the example above, if the column is fully aligned, the applied compressive load *P* can be increased until one reaches the compressive strength of the material. Yet, in reality the column will fail due to buckling as shown in the figure on the right long before this load is reached.



The analysis of the buckling of a continuous column is similar to the example given above to motivate the problem. Since the column is a two-force member, the reaction loads at the two pins are equal and directed along the line connecting the two pins as shown in the figure to the left below. The free-body-diagram of a segment of the column is also drown below and it is clear from this diagram that for the member to be in equilibrium the bending moment must balance the couple created by the misalignment of the two loads.



Designating the out of plane displacement of the column by v, the bending moment must be M=-Pv. One can combine this with the beam deflection equation

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

to get the equation for the column as

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = 0$$

2.2. Sobre el análisis matemático

This is a second order homogeneous ordinary differential equation with constant coefficients that has a solution of the form

$$v = C_1 \sin(\lambda x) + C_2 \cos(\lambda x)$$

where C_1 and C_2 are constants to be fit to the boundary conditions and λ must be restricted to satisfy the differential equation. The boundary conditions for this pin-supported column are that the displacement is zero at both supports. Therefore,

$$v = 0 \quad at \quad x = 0 \quad \Rightarrow \quad 0 = C_2$$
$$v = 0 \quad at \quad x = L \quad \Rightarrow \quad 0 = C_1 \sin(\lambda L)$$

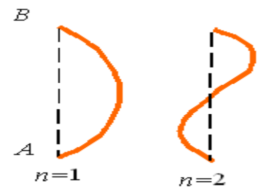
Obviously, if both C_1 and C_2 are zero one obtains the trivial solution v=0 for the fully aligned beam. For the beam to have a nontrivial solution (buckled solution), one must select $\sin(\lambda L) = 0$ that results in requirement that $\lambda L = 0, \pi, 2\pi, \ldots$ that yield

$$\lambda = \frac{n\pi}{L}$$

for any integer n. This results in the solution

$$v = C_1 \sin(n\pi \frac{x}{L})$$

As can be seen from the figure, different values of n represent different modes of buckling.



In addition to the boundary conditions, the solution must satisfy the differential equation. Substitution of this solution into the differential equation gives

$$-\lambda^2 C_1 \sin(\lambda x) + \frac{P}{EI} C_1 \sin(\lambda x) = 0$$

Reorganization yields

$$(\frac{P}{EI} - \lambda^2) C_1 \sin(\lambda x) = 0$$

Clearly, if C_1 is zero, one arrives at the trivial solution v=0 that satisfies the differential equation, and which is associated with the fully aligned beam, but there is a non-trivial solution when the term in the round parentheses goes to zero. Therefore, to get a nontrivial solution to the buckling problem, the axial load must satisfy the relation $P = EI\lambda^2$, which results in the expression for the critical load given by

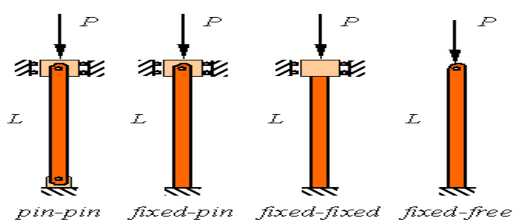
$$P_{cr} = \frac{n^2 \pi^2 EI}{L^2}$$

Obviously, the smallest critical load is associated with n=1. Therefore, the column will buckle at the load associated with the first buckling mode if the column is not restricted from taking the shape associated with this mode.

Obviously, the smallest critical load is associated with n=1. Therefore, the column will buckle at the load associated with the first buckling mode if the column is not restricted from taking the shape associated with this mode.

Different supports:

The buckling of columns with a variety of different support conditions are shown in the following figure and can be analyzed using similar procedures to the simply supported column studied above.

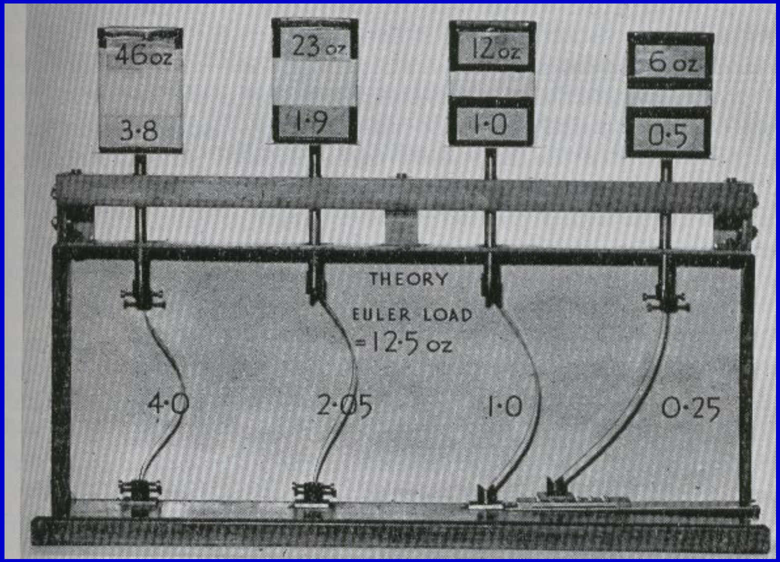


The results for the other columns are similar to the pin-pin supported column analyzed above with only the replacement of the actual length of the column with an effective length. If L is the actual length of the column and L_e is the effective length of the column, then the critical buckling load for the column is given by

$$P_{cr} = \frac{n^2 \pi^2 EI}{L_e^2}$$

where the effective length L_e is given by

$$L_e = \begin{cases} L & pin-pin \\ 0.7L & fixed-pin \\ 0.5L & fixed-fixed \\ 2L & fixed-free \end{cases}$$

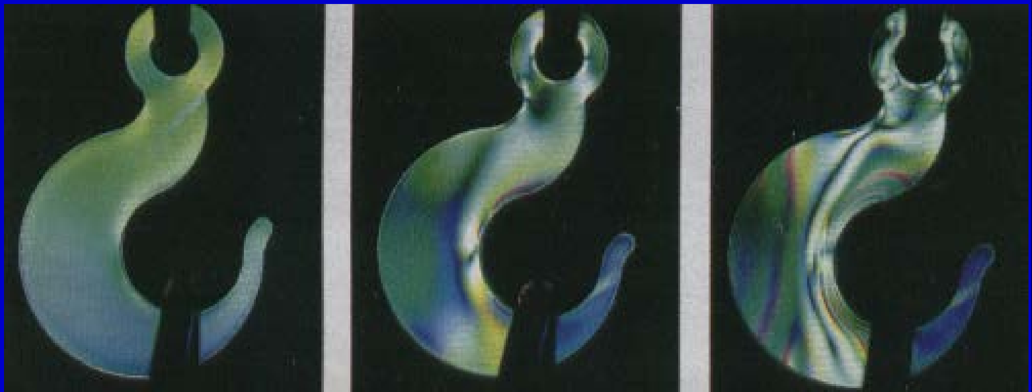


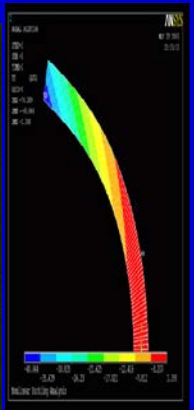
Diferentes perfiles

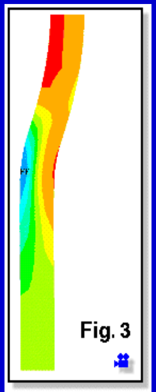
Validación: regreso a la modelización. Sección no despreciable (y no homogénea)

Teoría de la Elasticidad tridimensional

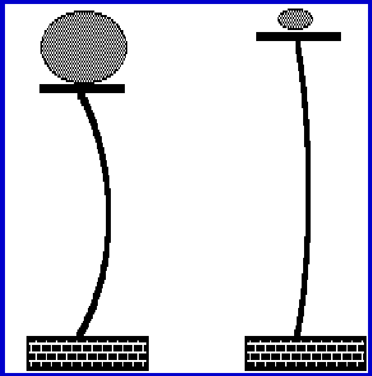
Navier, St. Venant,…

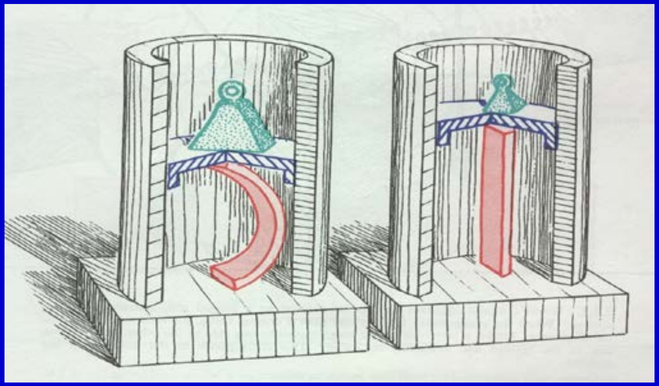


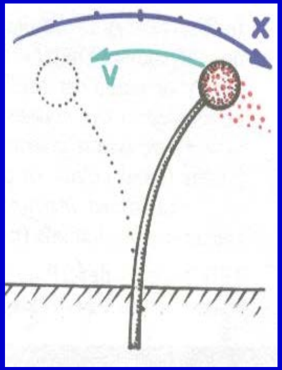


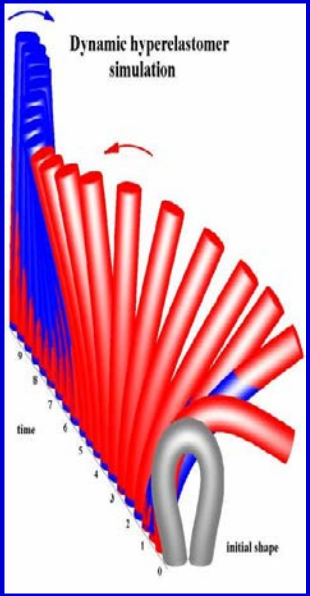


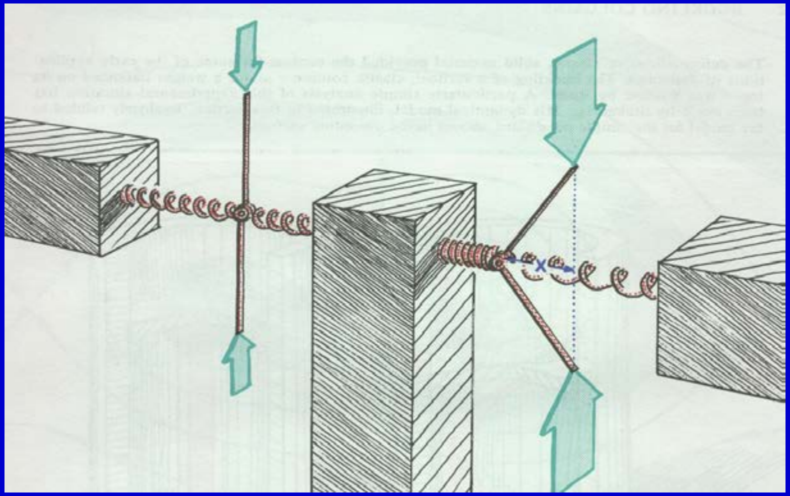
Observaciones sobre la dinámica (J.J. Stoker 1950)

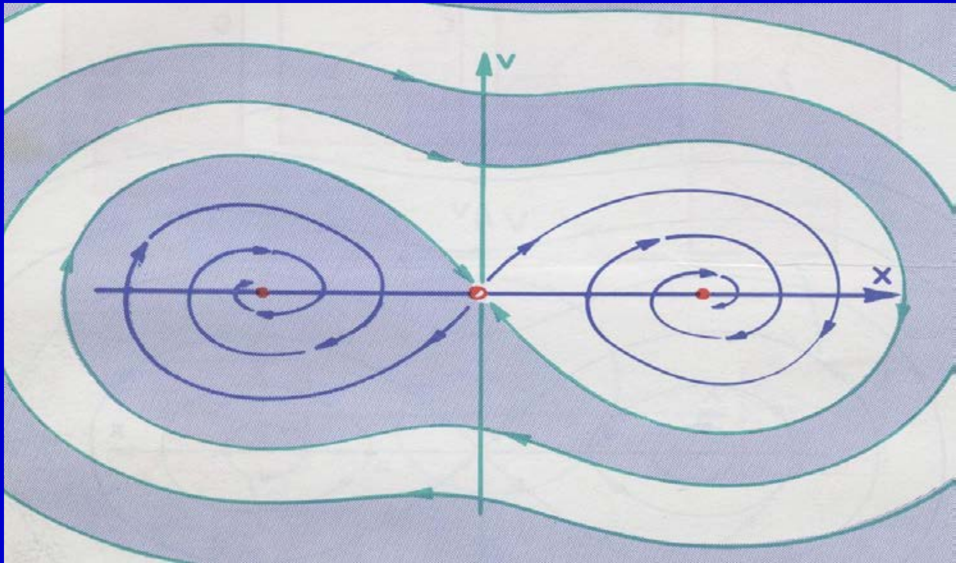




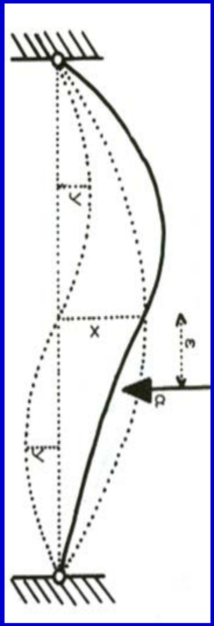


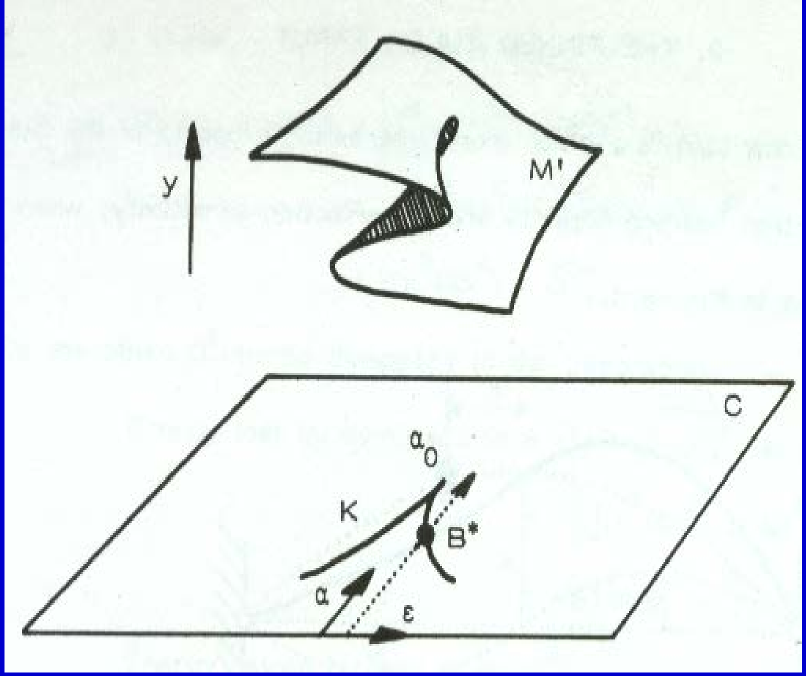






Catastrophe Theory, E. C. Zeeman:
Selected papers 1972-1977

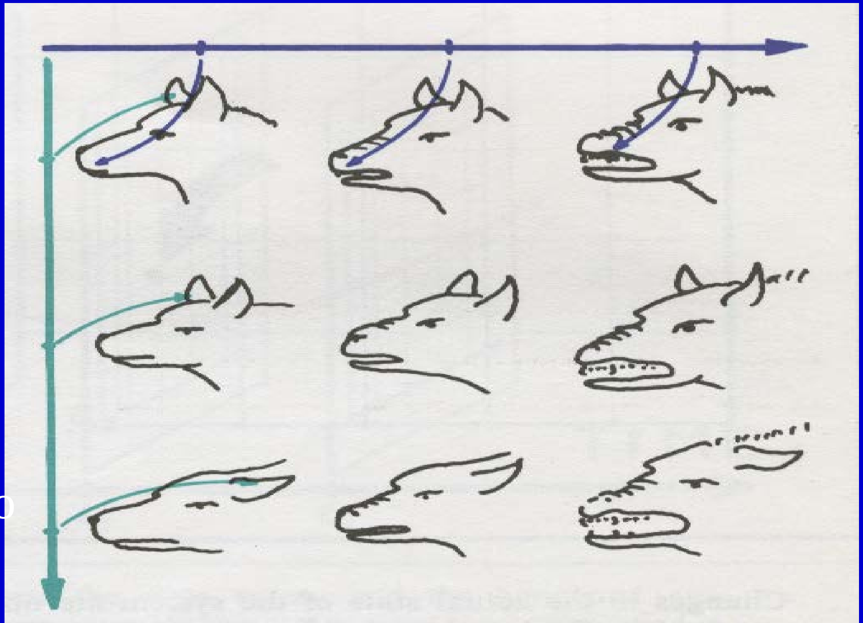


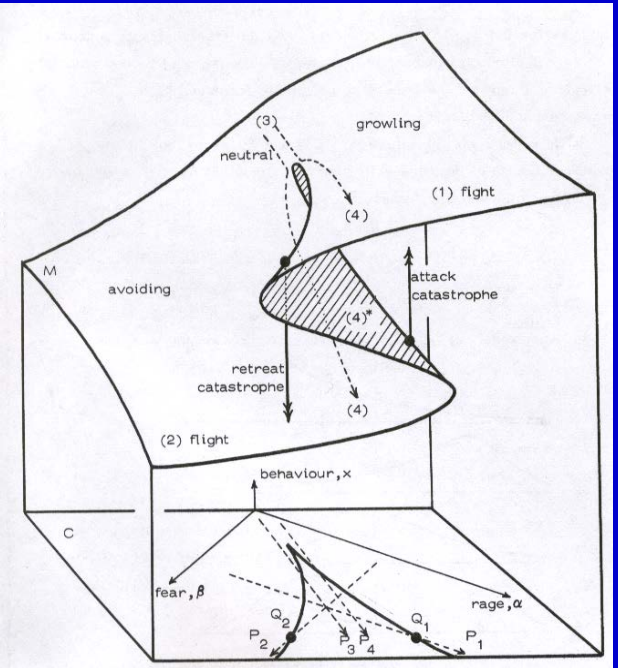


Psicología: mecanismo de agression

rabia

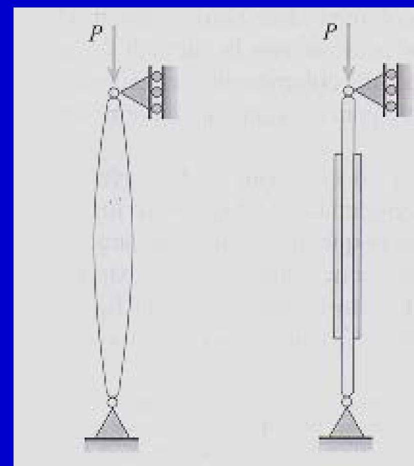
miedo

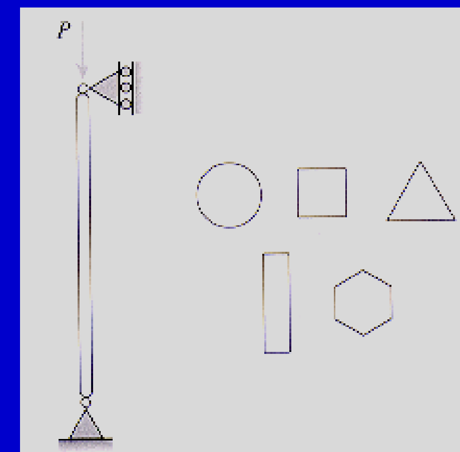




2.3. Optimización y Control

La mejor columna "elastica": M. Vitruvio (I b. C.) *De Architecura* 25b. C., L.B. Alberti (1404-1472) 1450, Euler 1744, J. Lagrange (1736-1813) 1773,...





La más alta columna

L. Euler, *Leonhardi Euleri Opera Omnia*, Scientiarum Naturalium Helveticae edenda curvaverunt F. Rudio, A. Krazer, P. Stackel. Lipsiae et Berolini, Typis et in aedibus B. G. Teubneri, 1911—.

B. Keller and F. I. Niordson, "The Tallest Column", Journal of Mathematics and Mechanics, 1986

S. J. Cox and M. L. Overton, "Qn the optimal design of columns against buckling," SIAM 1. on Math. Anal. 1992

Fig. 2 Cross-sectional area of strongest clamped-free column as a function of distance along column. Left end is clamped and right end is free.

$$H_c = (9.(1.8663)^2 \frac{EI}{4\rho A})^{1/3}$$

E = modulo de Young $\rho = densidad$

$I(z) = \alpha A^2(z)$ $y(z)$ $EI(z)y''(z)$

$$E\alpha A^2(z)y''(z) = \int_z^H \rho A(\tilde{z})[y(\tilde{z}) - y(z)]\,d\tilde{z}, \quad 0 < z < H,$$ $y(0) = y'(0) = 0.$

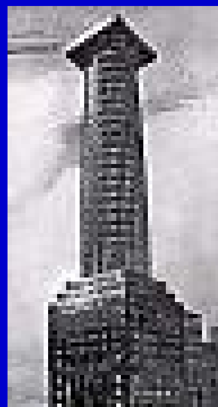
$x = z/H, \quad a(x) = HA(xH)/V, \quad \eta(x) = y(xH)/H,$ $\lambda = \rho H^4/\alpha EV$

$$a^2(x)\eta'' = \lambda \int_x^1 a(\tilde{x})[\eta(\tilde{x}) - \eta(x)]\,d\tilde{x}, \quad \eta(0) = \eta'(0) = 0,$$ $\int_0^1 a\,dx = 1$

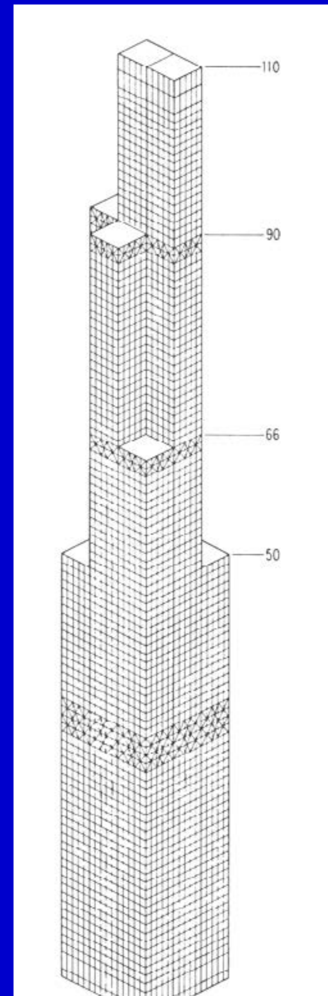
$u(x) = \eta'(x)$

$$-(a^2(x)u'(x))' = \lambda\left(\int_x^1 a(t)\,dt\right)u(x), \quad 0 < x < 1, \quad u(0) = a^2(1)u'(1) = 0$$

3. Columnas reforzadas y heterogéneas: rascacielos

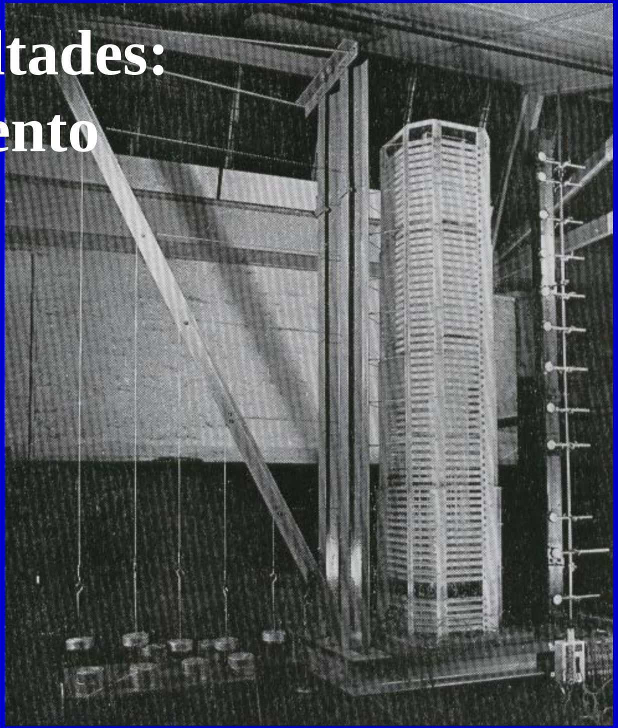


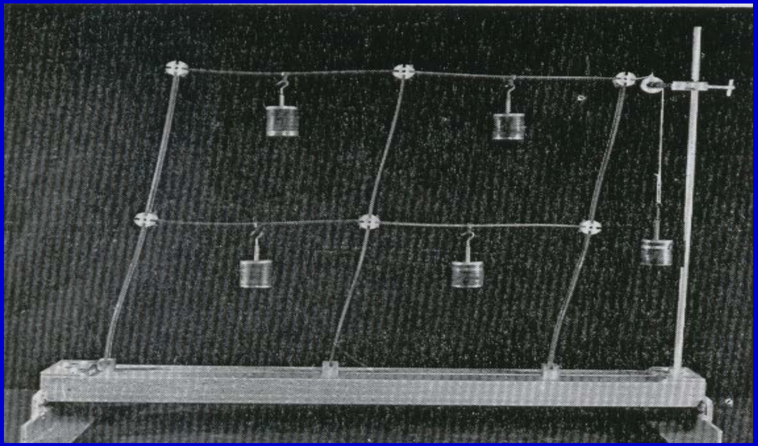
Lost, Chicago, 1920

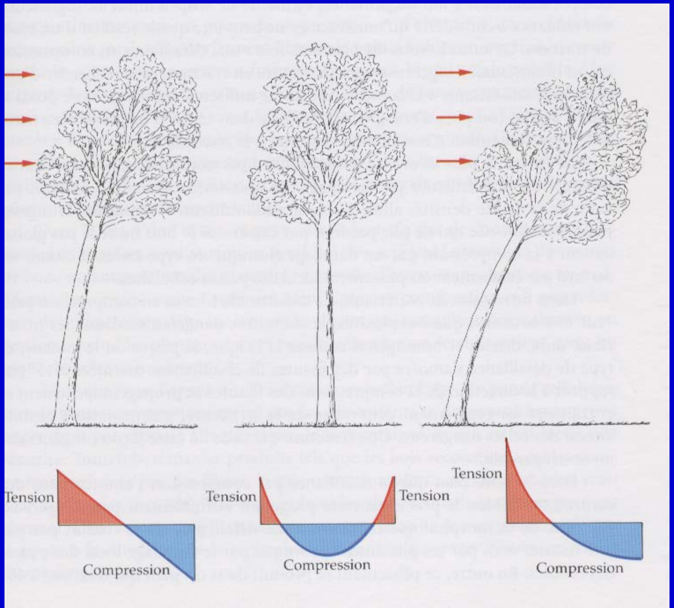


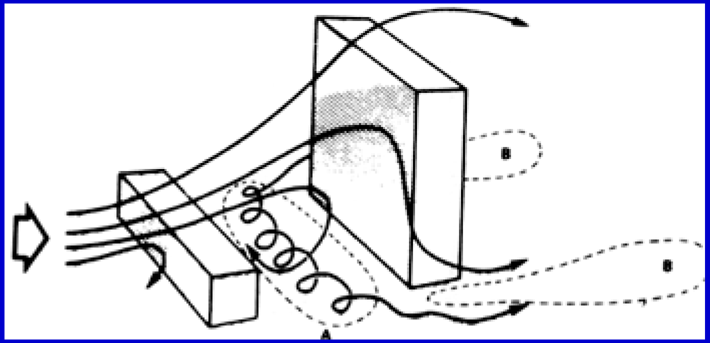
La Torre Sears de Chicago, 1974

Nuevas dificultades: estudios de viento

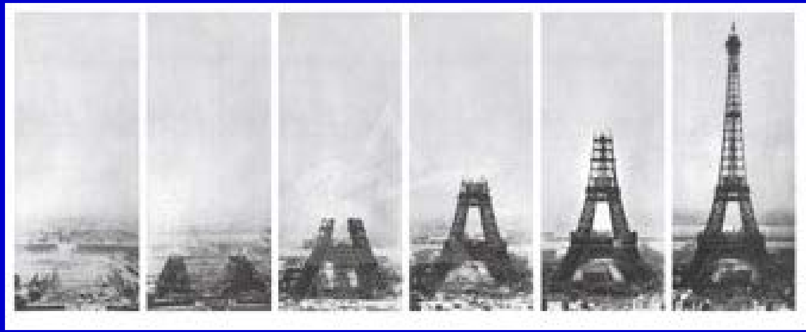






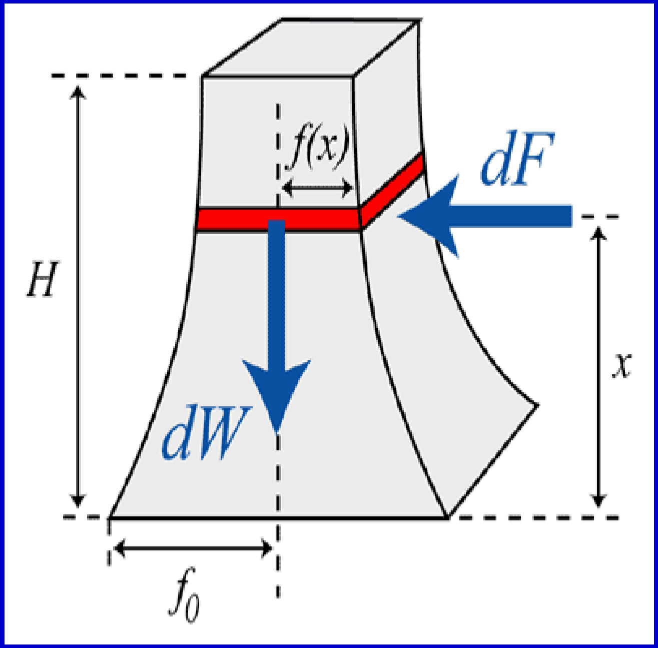


Diseño de la Torre Eiffel



“Considero que la curvatura de las cuatro aristas, consecuencia de los cálculos matemáticos, dará una gran impresión de resitencia y belleza” G. Eiffel /Le Tepms, 14 de febreo de 1887)

J. Gallant, Am. J. Phys. 2002



$$= 4\rho g f^2(x) dx$$

$$dF = 2Pf(x)dx,$$

$$\frac{1}{2}\int_x^H f^2(x)dx - \text{constant}\times(H-x) = \int_x^H x w(x) f(x) dx,$$

$$\tfrac{1}{2}f^2(x) - \text{constant} = xw(x)f(x).$$

$$\tfrac{1}{2}f^2(x) - \tfrac{1}{2}f_0^2 = xw(x)f(x).$$

$$f(x) = xw(x) - \sqrt{x^2w^2(x) + f_0^2},$$

$$w(x) = 0.690 - 1.53\times 10^{-3}x + 3.96\times 10^{-5}x^2 - 9.22 \times 10^{-8}x^3$$

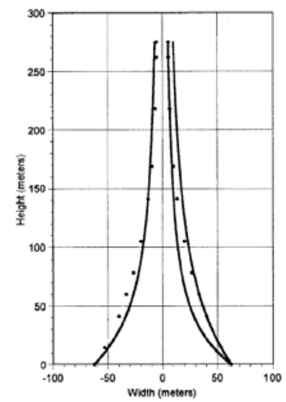


Fig. 2. The Eiffel function for three constant wind pressures ($w_0=0.700$ and $w_0=1.33$ on the right-hand side and $w_0=1.00$ on the left-hand side). The actual shape of the Tower are the data points.

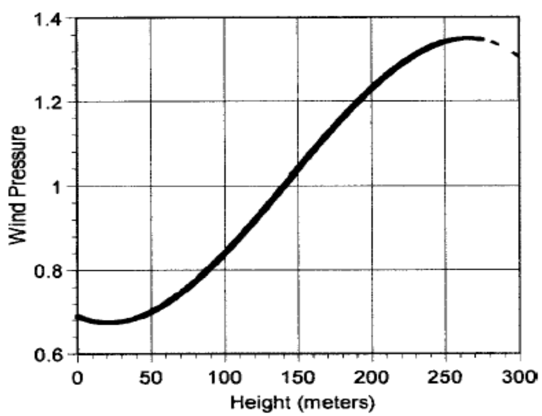


Fig. 3. The dimensionless maximum wind pressure $w(x)$ of Eq. (5) as a function of height.

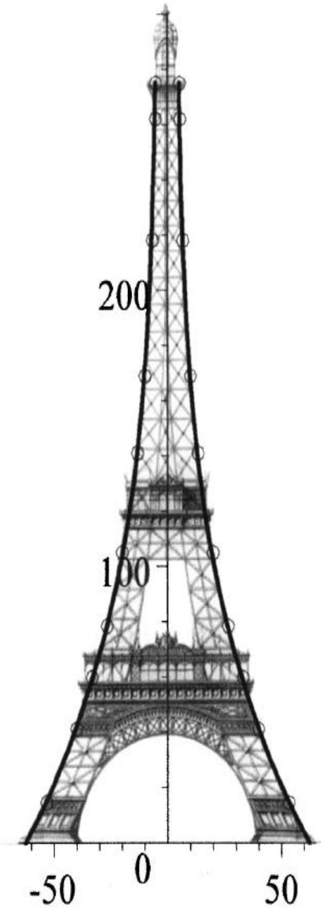
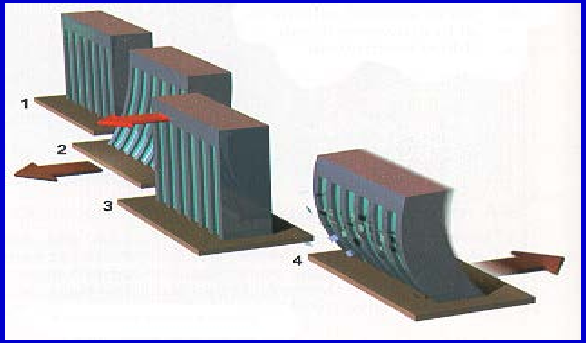
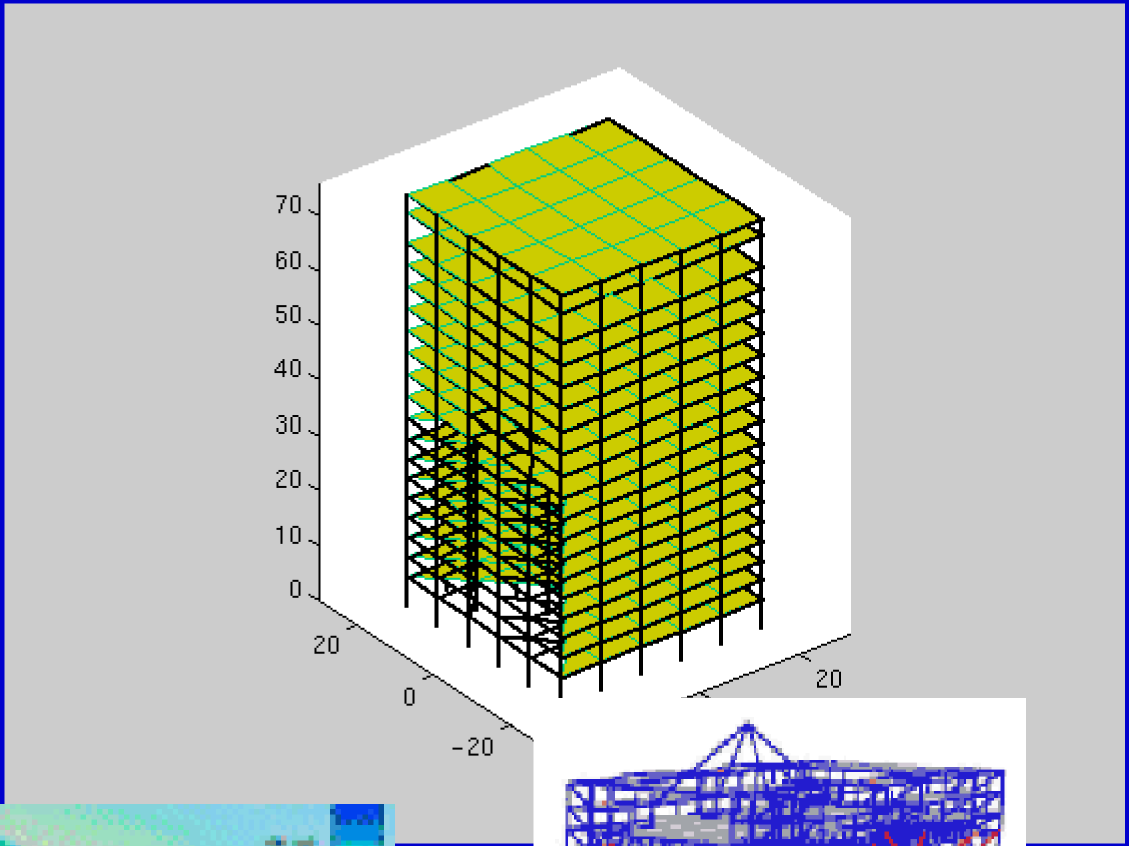
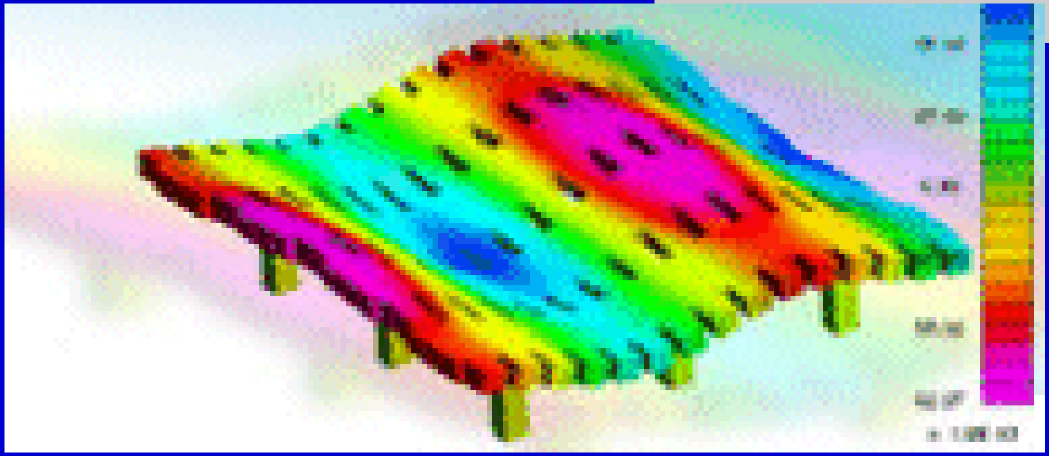


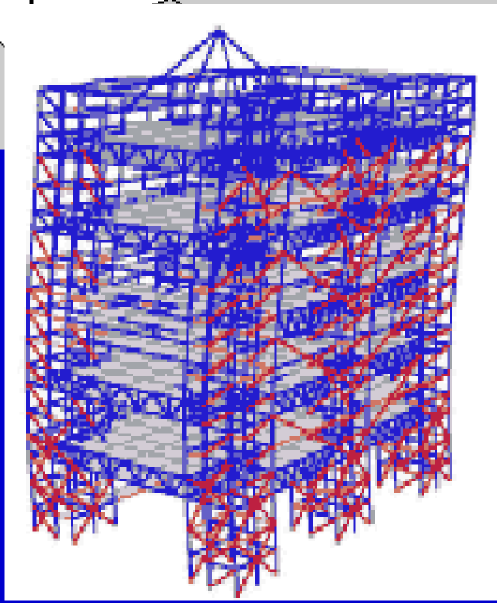
Fig. 4. The Eiffel function $f(x)$ with the cubic wind pressure, plotted with the data and an image of the Tower.

Dinámica: riesgos sísmicos

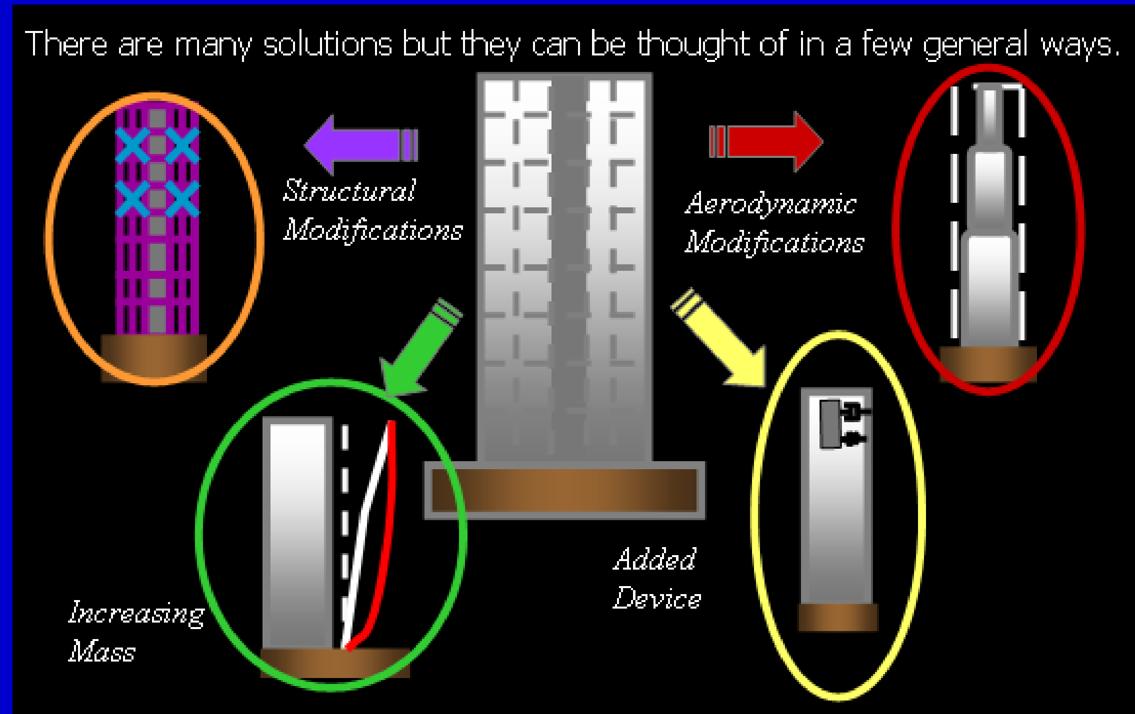


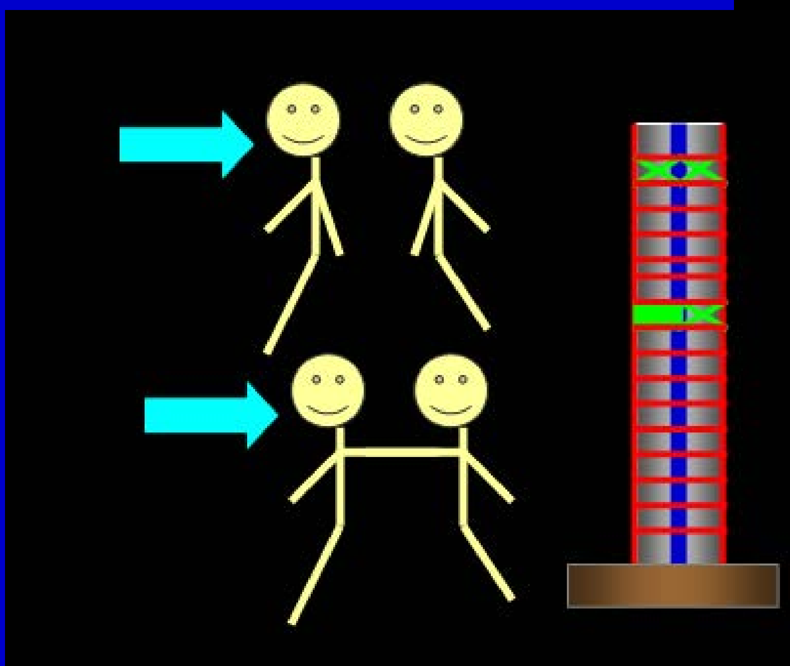




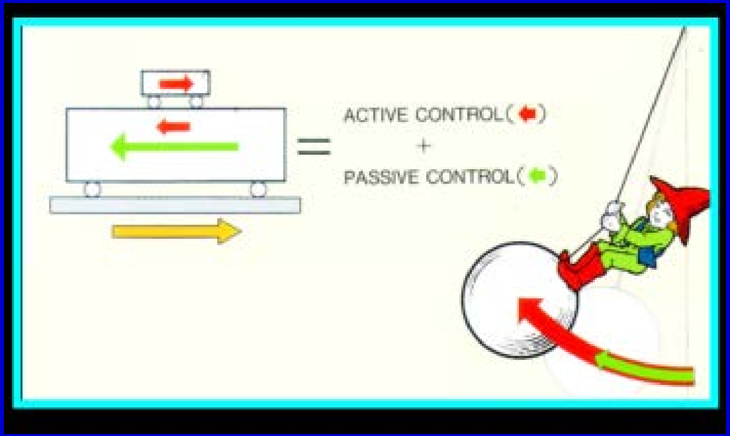


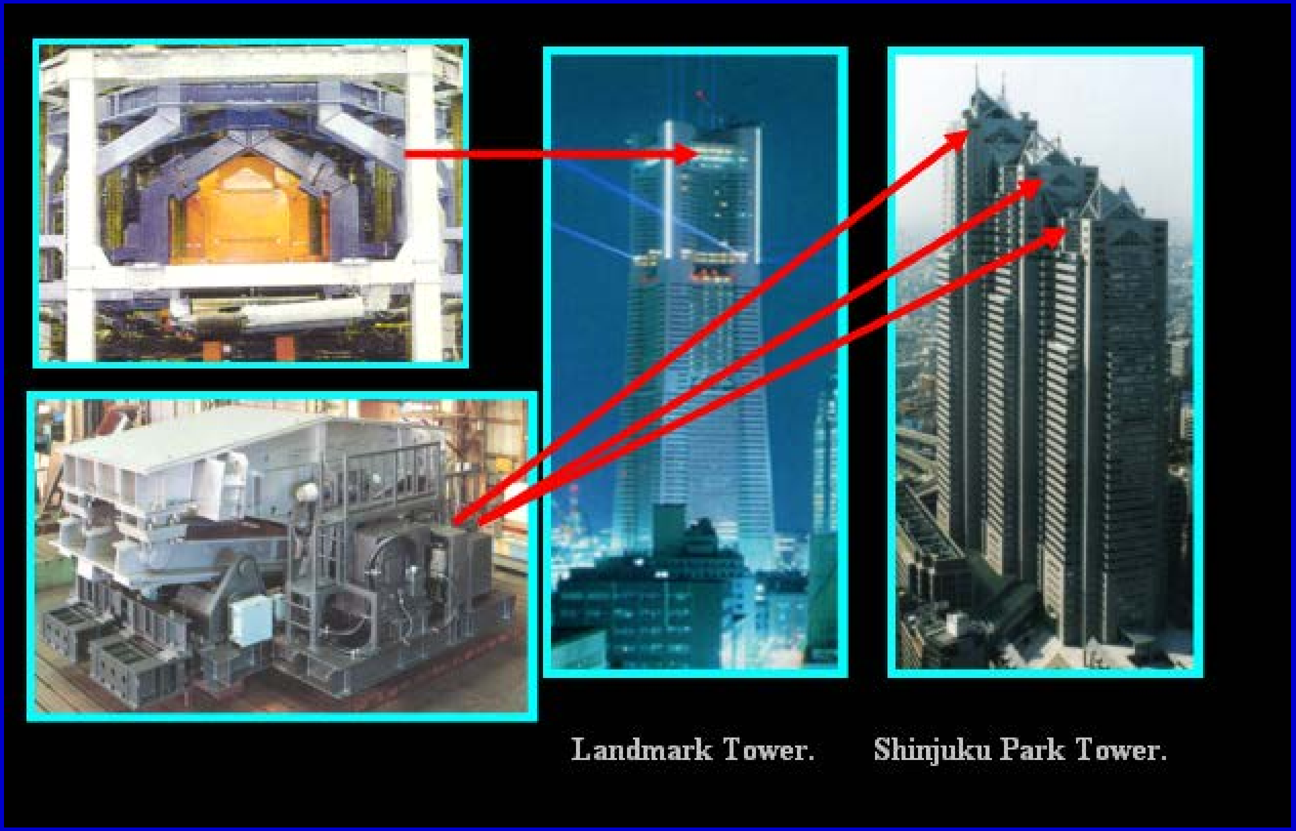
Optimización en el diseño





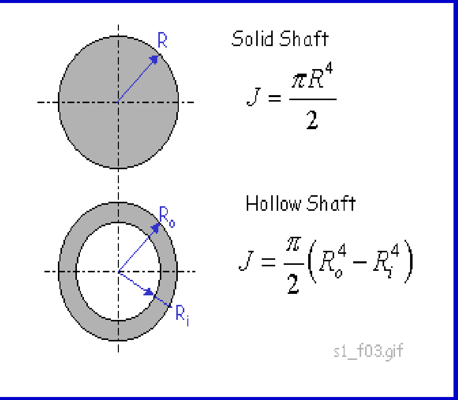
Control de movimientos sísmicos

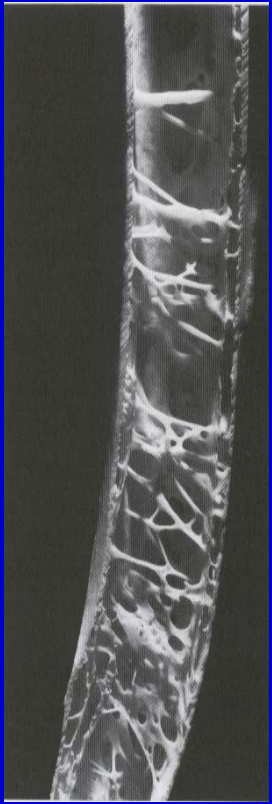


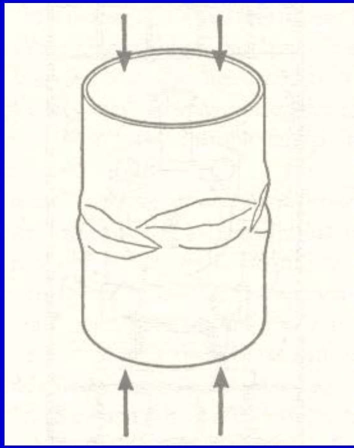


Regreso al estudio de la columna:

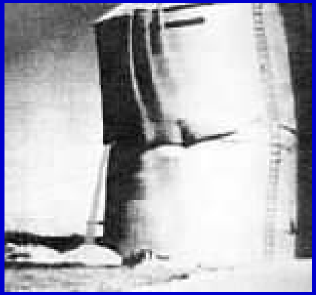
consideraciones sobre la distribución de masa

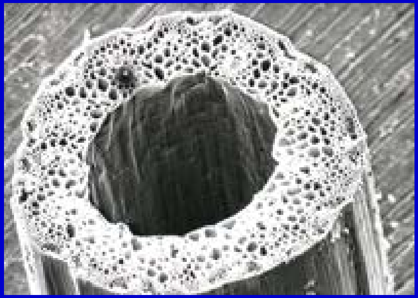






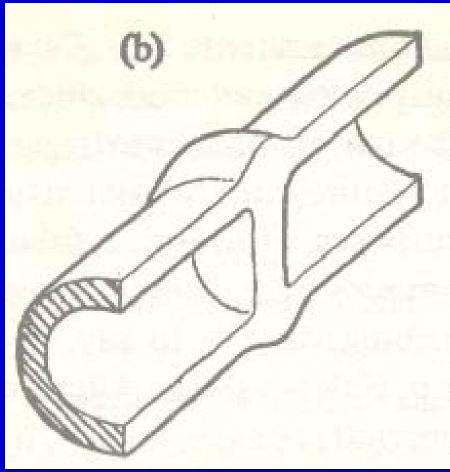
Fractura de Brazier



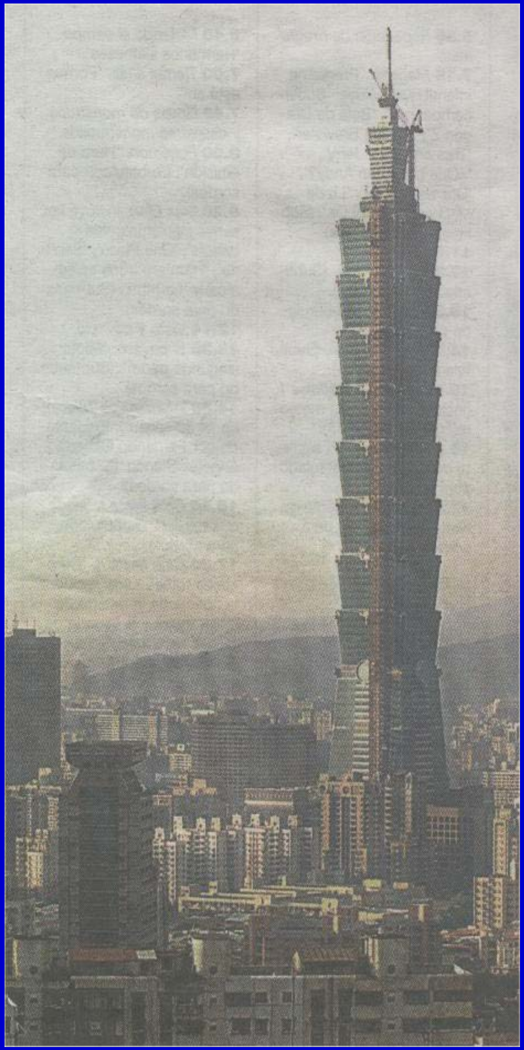


Una primera solución

de reforzamiento



Torre Jinmao de Shanghai

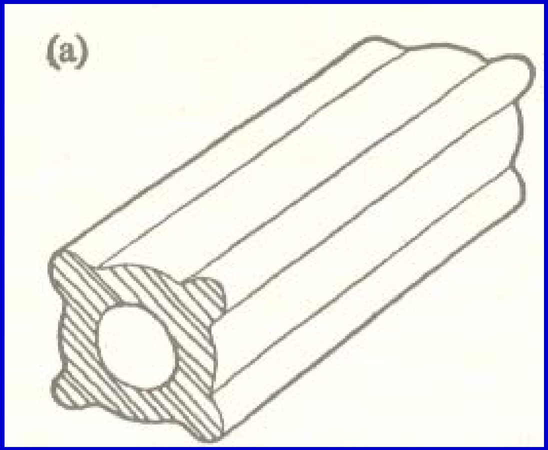


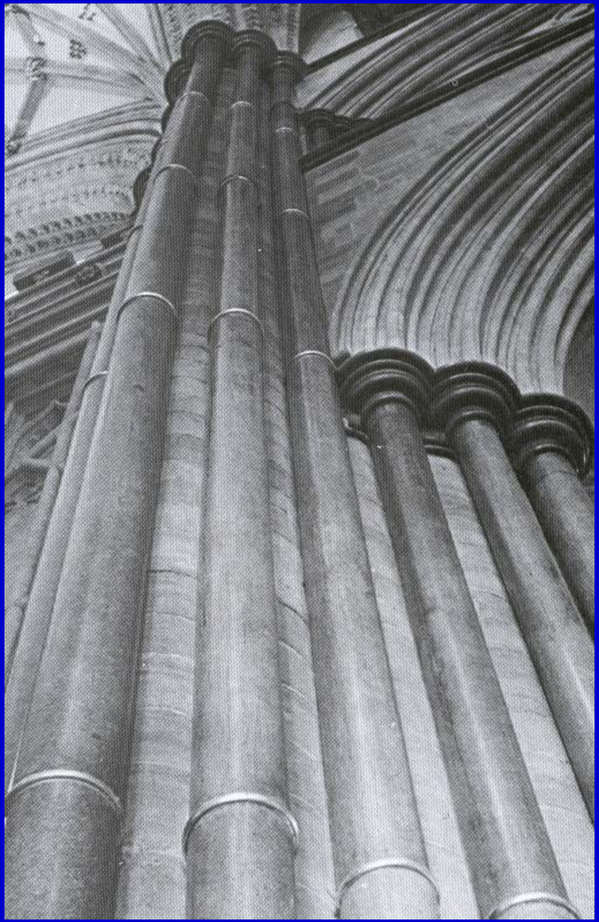
Taipei 101 C.Y. Lee & Partners

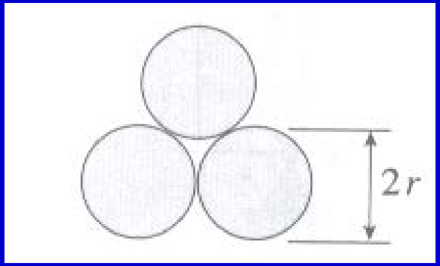
(amortiguamiento en planta 88),

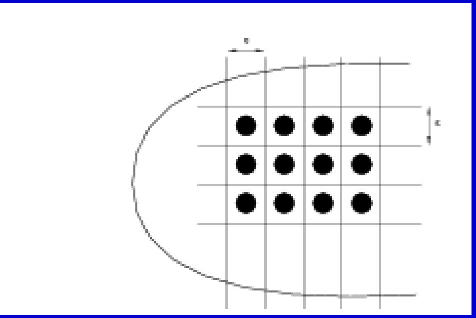
Una segunda solución de reforzamiento

Medios compuestos

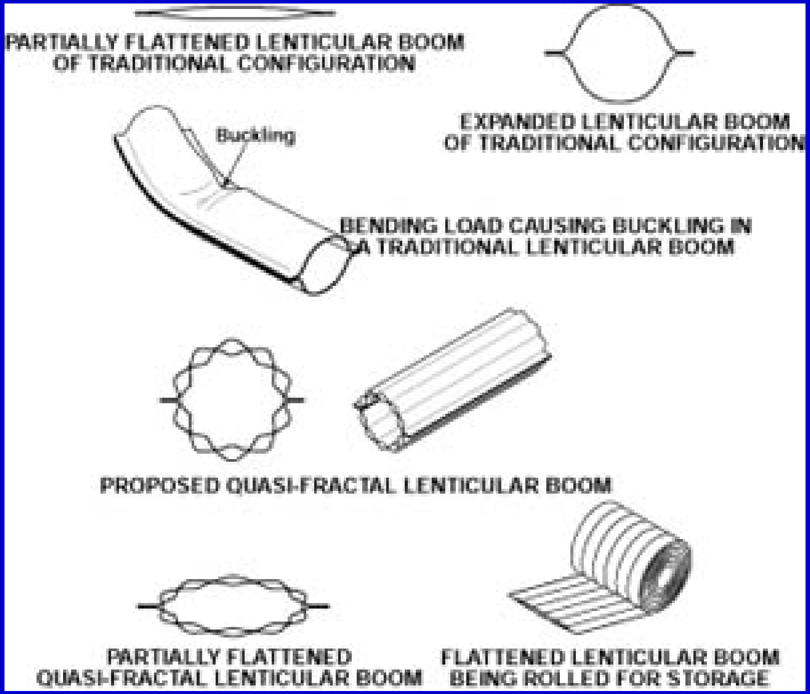






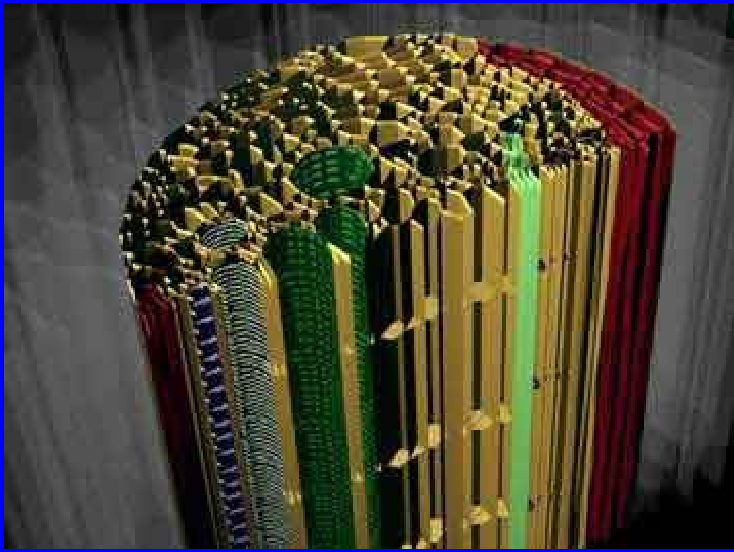


Teoría de la Homogeneization

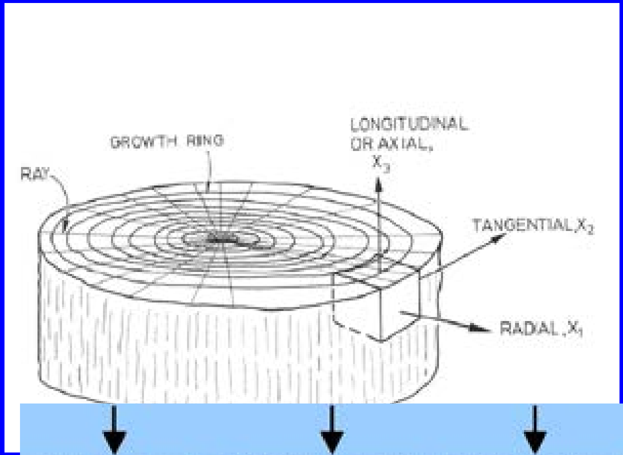


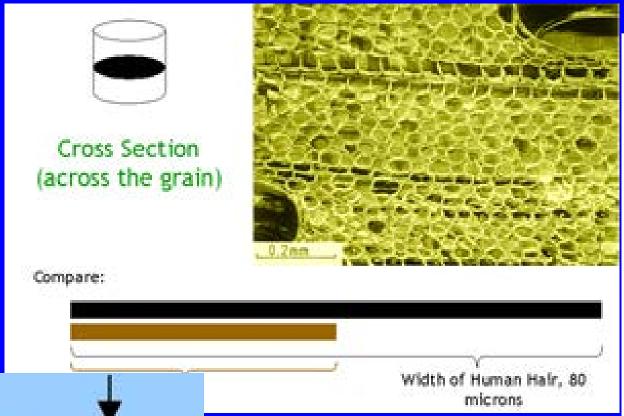
Sección Quasifractal (NASA)

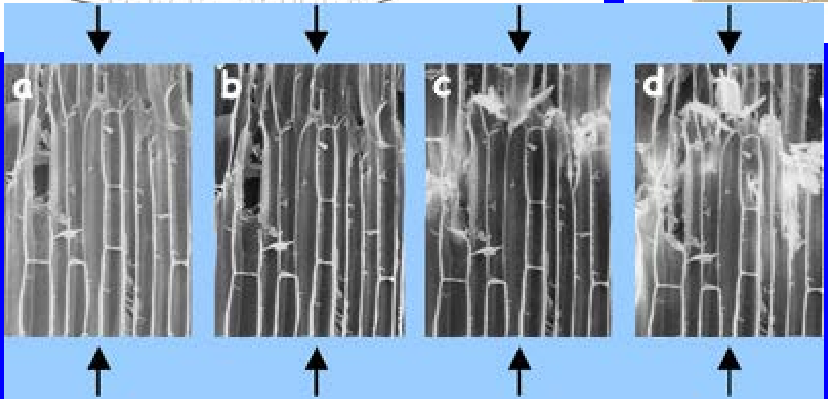
Sobre el proyecto
Shangai bionic tower

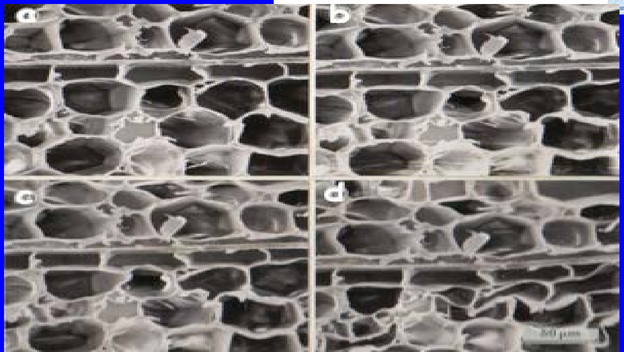


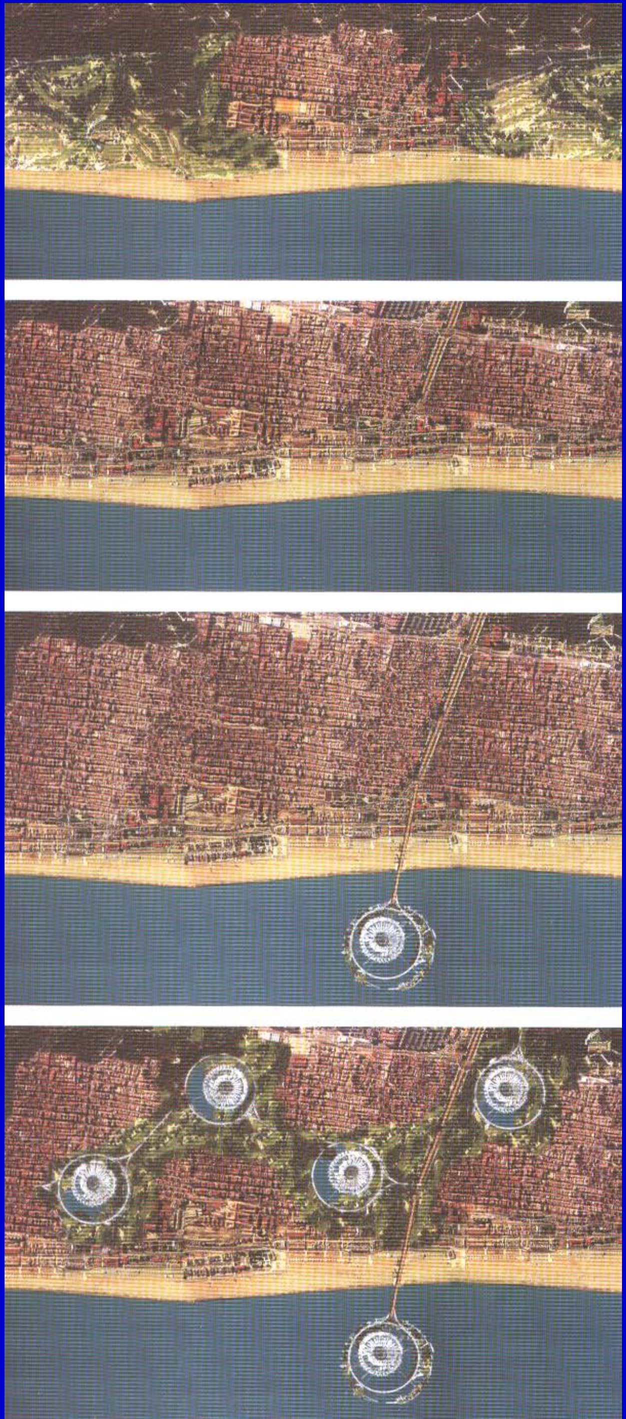
Inspiraciones naturales



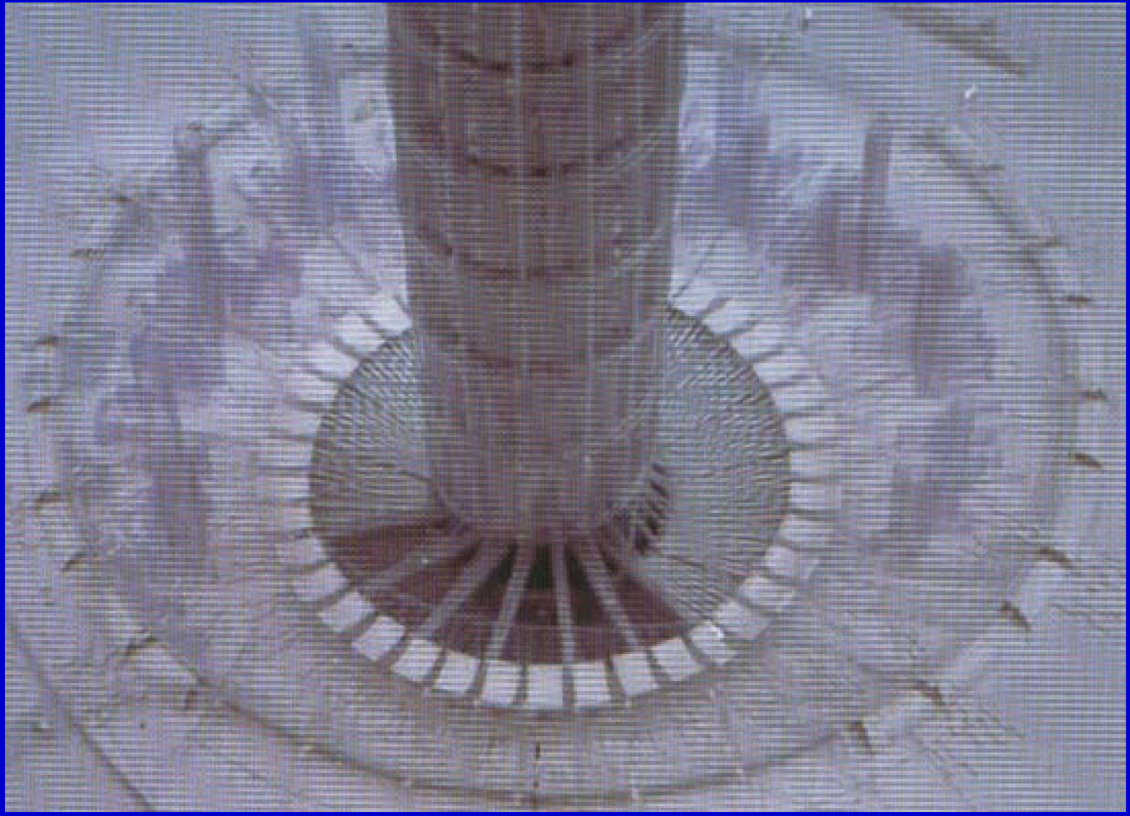


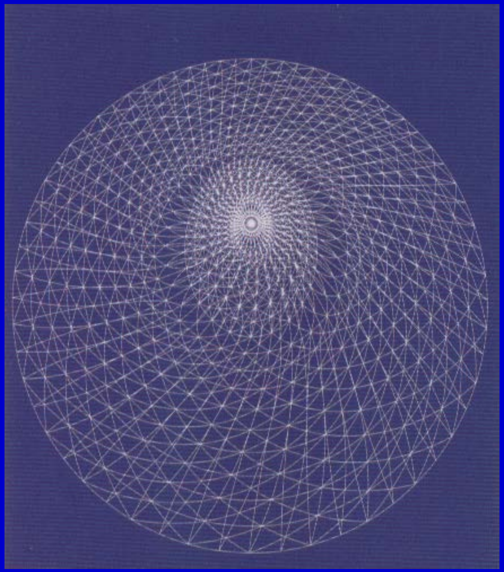
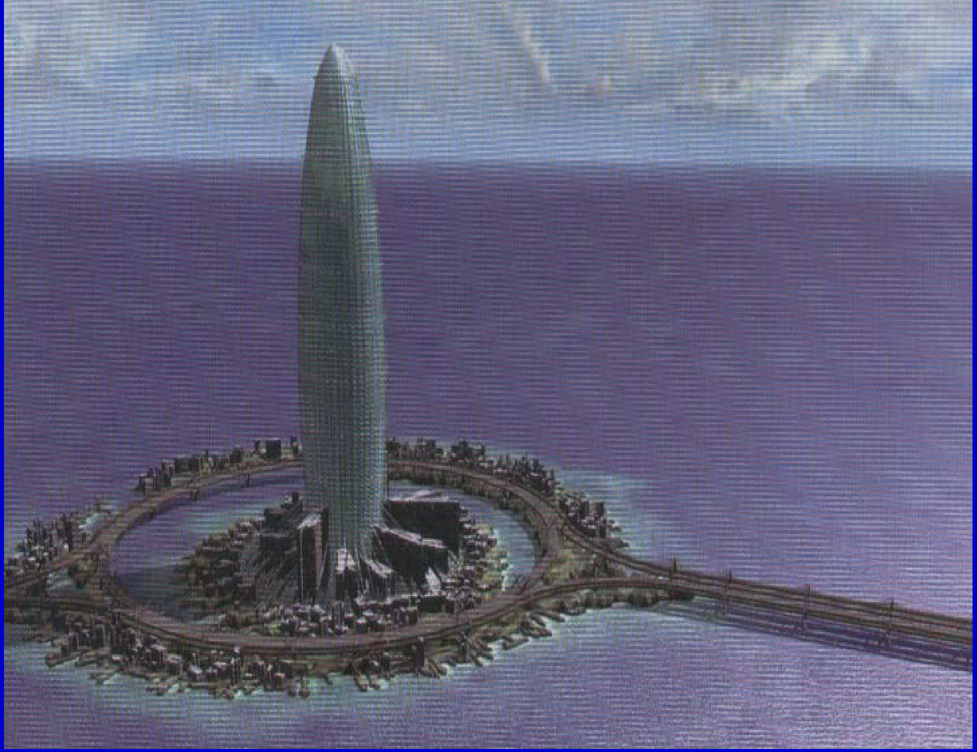
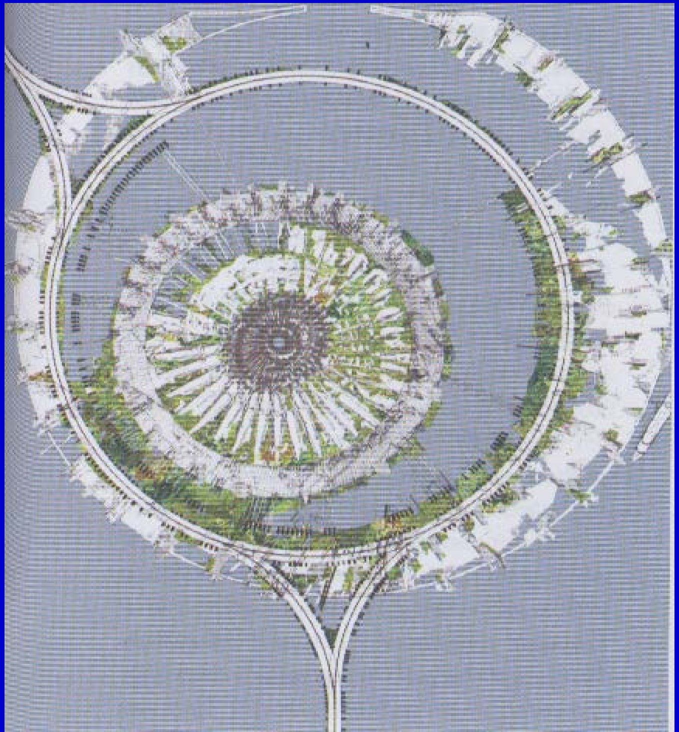
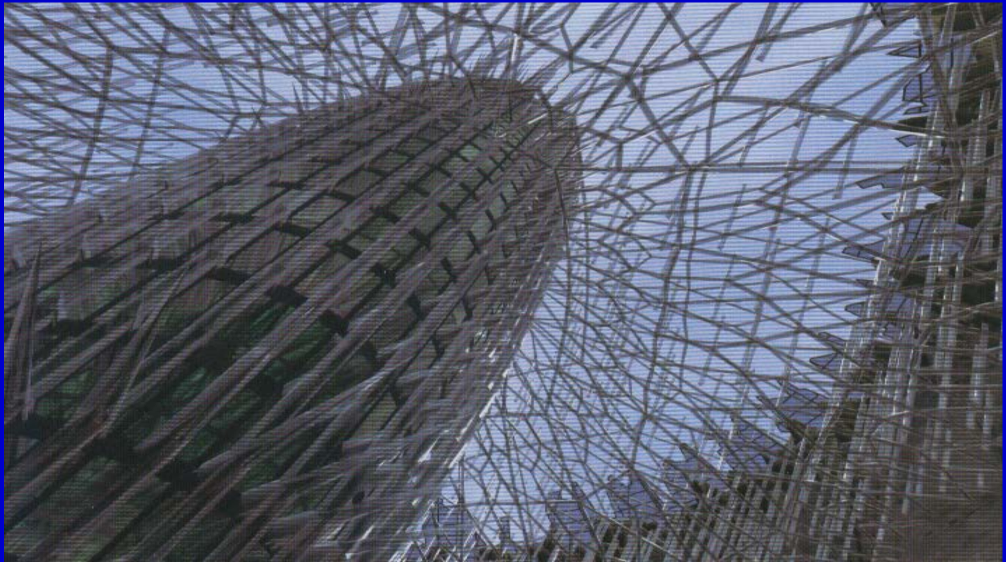






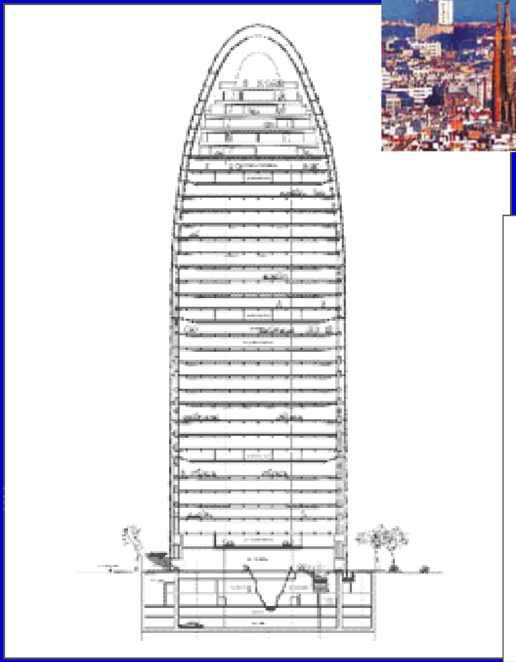
Metas ecológicas y sociales

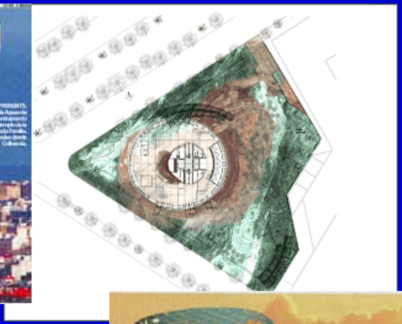


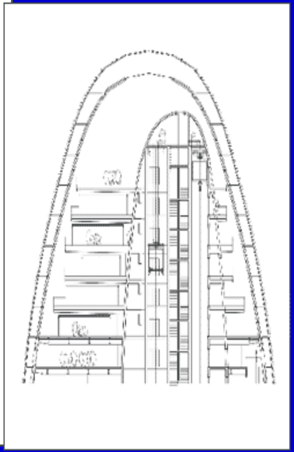





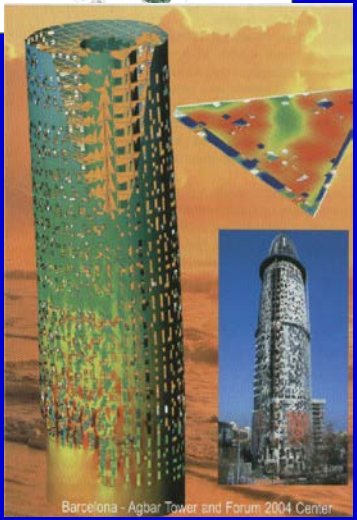
Torre Agbar, Barcelona 2004

Jean Nouvel









Swiss Re Tower, Norman Foster

La parte positiva de los sueños

y la ciencia-ficción

Gracias por vuestra atención