Special finite time extinction in nonlinear evolution systems:

dynamic boundary conditions and Coulomb friction type problems

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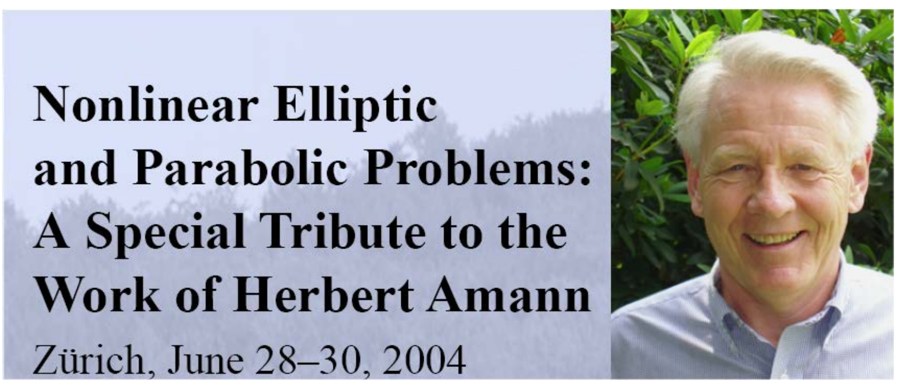


1. Introduction

nonlinear evolution problem

$$\begin{cases} u_t + A(u) = f(x,t) & \text{in } Q_\infty, \\ B(u) = g(x,t) & \text{on } \Sigma_\infty, \\ u(x,0) = u_0(x) & \text{on } \Omega \end{cases}$$

$$\Omega \subset \mathbb{R}^N, \ N \geq 1, \qquad Q_\infty = \Omega \times \mathbb{R}_+, \ \Sigma_\infty = \partial\Omega \times \mathbb{R}_+.$$

$$f(x,t) \to f_\infty(x) \quad \text{and} \quad g(x,t) \to g_\infty(x) \quad \text{as } t \to \infty$$

$$u(x,t) \to u_\infty(x) \quad \text{as } t \to \infty$$

$$A_\infty(u_\infty)(x) = f_\infty(x) \quad \text{in } \Omega,$$

$$B_\infty(u_\infty(x)) = g_\infty(x) \quad \text{on } \partial\Omega.$$

stronger property

let us assume that

$$\begin{cases} f(x,t) = f_\infty(x) & \text{for } t \geq T_f, \\ g(x,t) = g_\infty(x) & \text{for } t \geq T_g. \end{cases}$$

Definition 1.1. *Let $u(x,t)$ be a solution of the initial and boundary-value problem (1.1). We say that $u(x,t)$ stabilizes in a finite time to a stationary state $u_\infty(x)$ if there exists $t^* \in (0,\infty)$ such that*

$$\forall t \geq t^* \qquad u(x,t) \equiv u_\infty(x) \quad on\ \Omega.$$

Introducing the new unknown function $v(x,t) \equiv u(x,t) - u_\infty(x)$

$u(x,t)$ has the property of extinction in a finite time $\qquad u(x,t) \equiv 0 \quad \textit{for} \quad t \geq t^*.$

Main goal of the lecture: to present two results, in collaboration with H. Amann concerning special finite extinction time properties for some systems:

finite extinction time is not an universal property of all the solutions of the problem,

(a very different feature from the case of scalar dissipative equations).

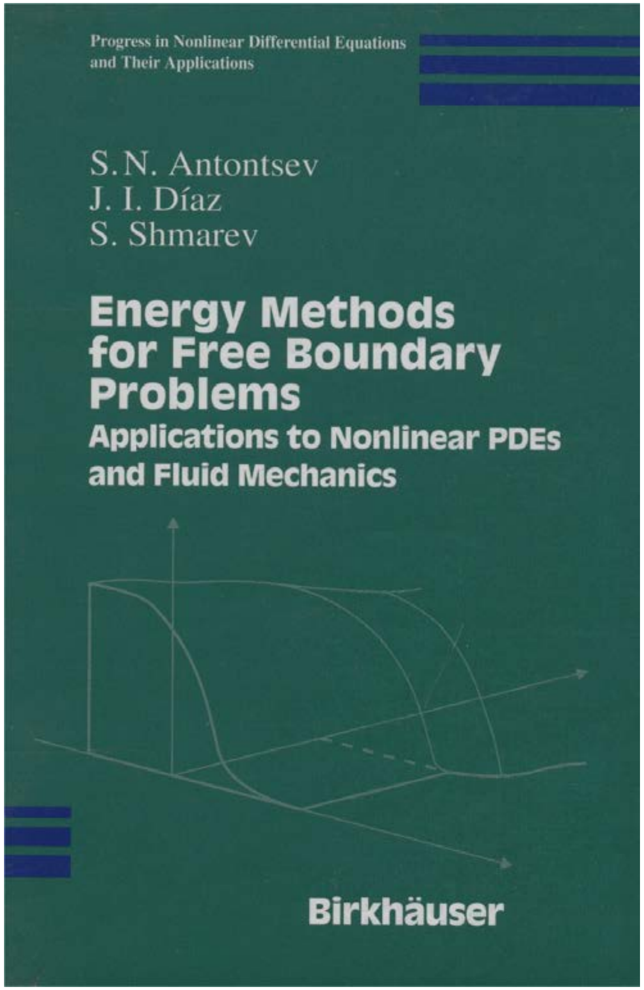
2. A (short) survey on methods to study extinction time properties

3. Extinction by components

4. Finite extinction time for a finite set of orbits.

5. Final conclusions

2. A (short) survey on methods to study extinction time properties



S.N. Antontsev, J.I. Díaz and S.I. Shmarev,

Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics,

Progress in Nonlinear Differential Equations and Their Applications, **48**, Birkhäuser, Boston, 2002.

2. 1. Abstract results on finite extinction time

H. Brezis, Monotone operators, nonlinear semigroups and applications. *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 2,* pp. 249--255. *Canad. Math. Congress, Montreal, Que.,* 1975.

X=H Hilbert space, A maximal monotone operator

$$\begin{cases} \frac{du}{dt}(t) + Au(t) \ni f(t) \quad \text{in } X \\ u(0) = u_0 \end{cases}$$

$$B(f(t), \epsilon) \subset A(0) \text{ for some } \epsilon > 0 \text{ and a.e. } t \geq t_0$$

X Banach space, A m-accretive operator

J.I.Díaz. Anulación de soluciones para operadores acretivos en espacios de Banach. Aplicaciones a ciertos problemas parabólicos no lineales. *Rev. Real. Acad. Ciencias Exactas, Físicas y Naturales de Madrid*, Tomo LXXIV, 865-880, 1980.

Applications to

Multivalued Parabolic perturbations: obstacle problems

$$v_t - a\Delta v + \beta(v) \ni 0.$$

Multivalued nonlinear diffusion equations

$$\begin{cases} u_t - \nu\Delta u - g\,\mathrm{div}\left(\frac{Du}{|Du|}\right) = c & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0,x) = u_0(x) & \text{on } \Omega, \end{cases}$$

E. C. Bingham, in 1922 (non-Newtonian fluids) [also in Image Processing [Chan et al, SIAM Journal on Scientific Computing, 20, 1999]

F.Andreu, V. Caselles, J.I. Díaz, J.M. Mazón, Some Qualitative properties for the Total Variation, *Journal of Functional Analysis,* **188,** 516-547, 2002

The abstract results do not apply to multivaled (second order) hyperbolic dry friction type problems

the damped string problem (DSP)

$$\begin{cases} u_{tt} - u_{xx} + \beta(u_t) \ni 0 & \text{in } (0,1) \times (0,+\infty), \\ u(t,0) = u(t,1) = 0 & t \geq 0, \\ u(0,.) = u_0(.) & \text{in } (0,1), \\ u_t(0,.) = v_0(.) & \text{in } (0,1), \end{cases}$$

where β is the maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ defined by

$$\beta(x) = \begin{cases} \{1\} & \text{if } x > 0, \\ [-1,1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases}$$

- Brezis (1972), Haraux (1978-1979):

$$u(t,x) \to \zeta(x) \quad \text{in } H_0^1(0,1) \quad \text{as } t \to +\infty,$$

with ζ verifying

$$-1 \leq \zeta_{xx} \leq 1.$$

- Brezis's conjecture : the equilibrium position is reached after a finite time, i.e. **"stabilization in finite time"**.

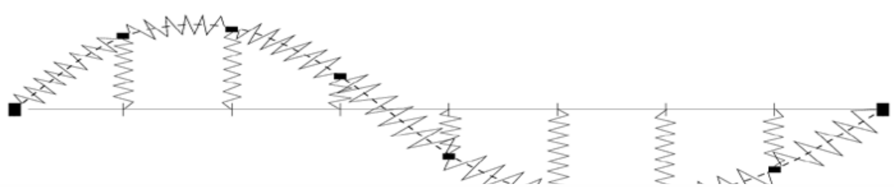
- Cabannes (1978) : for some special initial data u_0 and v_0.

Some easier formulations :

- The discretized vibrating string via a finite difference schema.

$$(DDSP)\quad \begin{cases} \dfrac{d^2u_i}{dt^2} - \dfrac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \beta\left(\dfrac{du_i}{dt}\right) \ni 0 \quad i = 1, \ldots, n, \\ u_0 = u_{n+1} = 0, \\ u_i(0) = a_i \quad \text{and} \quad \dfrac{du_i}{dt}(0) = b_i \qquad \text{for } i = 1, \ldots, n. \end{cases}$$

The motion of the discretized string describes the motion of N particles connected by springs like we can see :



Note : In the parabolic case, **"extinction in finite time"**. In (DSP), of hyperbolic type, the limit ζ may not be identically zero : **"stabilization in finite time"**.

An abstract result for (time) second order equations

$$(P)\begin{cases} \ddot{x}(t) + \partial\Psi(x(t)) + \partial\Phi(\dot{x}(t)) \ni 0, \text{ in } H, \ t > 0, \\ x(0) = x_0, \ \dot{x}(0) = v_0 \end{cases}$$

$$\Phi \geq \alpha \|\,\|, \text{ for some } \alpha > 0$$

$$0 \in int\partial\Phi(0), \ \partial\Psi(.) \text{ singelvalued}$$

$$\| \dot{\mathbf{x}}(t) \| + \| \mathbf{x}(t) - \mathbf{x}_\infty \| \to 0 \quad as\ t \to +\infty$$

Theorem. Assume that $-\partial\Psi(\mathbf{x}_\infty) \in int\partial\Phi(0)$. Then there exists t^* such that $\mathbf{x}(t) = \mathbf{x}_\infty$ for any $t \geq t^*$

Application to:

N-particles with friction (alternative to Bamberger-Cabannes)

Hyperbolic damped equation (Cabannes)

Viscoelastic Bingham materials

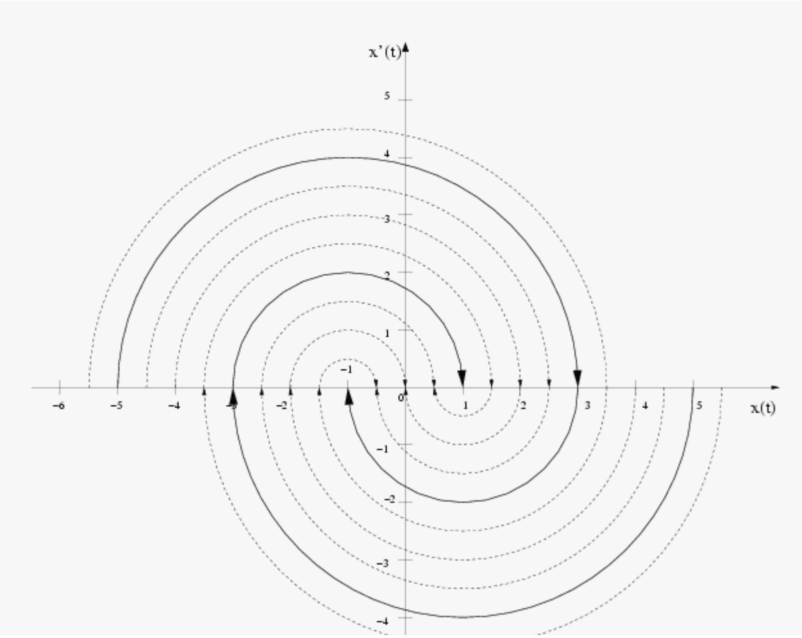
$$u_{tt} - \Delta u - \Delta\beta(u_t) \ni 0$$

2. 2. Finite extinction time via ODEs arguments

- The damped oscillator

$$(DO)\quad \begin{cases} \ddot{x}(t) + x(t) + \beta(\dot{x}(t)) \ni 0, \\ x(0) = x_0, \\ \dot{x}(0) = y_0. \end{cases}$$

By matching orbits of $\ddot{x} + x = \pm 1$, we get the phase plane

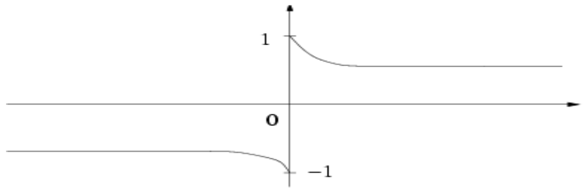


A particle take a time π to cross half a circle in the phase plane. Thus, a particle take a finite time to reach the equilibrium position.

Now, we will consider some generalization :

$$(DO)' \quad \ddot{x}(t) + \omega^2 x(t) + \beta(\dot{x}(t)) + g(\dot{x}(t)) \ni 0.$$

Here, $\omega > 0$ and g is Lipschitz continuous and verifies $g(0) = 0$.
The functiom g represents a Coulomb-Rayleigh friction (see Rayleigh (1877)).



D-Millot (2003)

Proposition 3 *Assume that there exists $\alpha > 0$ such that*

$$|v| + g(v)v \geq \alpha|v| \quad \forall v \in \mathbb{R}.$$

Then, for all initial data $(x_0, y_0) \in \mathbb{R}^2$, there exists a unique solution $x(t)$ of (DO)' and

$$x(t) \to \zeta \in [-1/\omega^2, 1/\omega^2] \quad , \qquad \textit{as } t \to +\infty \ .$$

Now, what about stabilization in finite time for (DO)' ?

1) If $g(v)v \geq 0$, we increase the friction, then **one can think** that $x(t)$ reaches the limit after a finite time smaller than in the case $g = 0$. This phenomenon appears in the equation

$$\dot{v}(t) + +\beta(v(t)) + g(v(t)) \ni 0.$$

2) If $g(v)v < 0$, we decrease the friction, then **one can think** that we increase the time to reach the limit.

1) If g increase sufficiently near 0, **we loose** the stabilization in finite time.

2) If $g(v)v < 0$, we **always have** stabilization in finite time.

More precisely :

Theorem 4 *If the limit* $\zeta \in]-1/\omega^2, 1/\omega^2[$*, then there exists* $T \geq 0$ *such that*

$$\forall t \geq T, \quad x(t) = \zeta.$$

Theorem 5 *If* $g(v)v \leq 0$ *near 0, then for all initial data, there exists* $T \geq 0$ *such that*

$$\forall t \geq T, \quad x(t) = \zeta.$$

Theorem 6 *Suppose that there exists* $\lambda \geq 0$ *such that*

$$g(v) = \lambda v + \rho(v) \qquad \textit{near zero,}$$

where ρ *is a continuous function such that*

$$\frac{|\rho(v)|}{|v|} \to 0 \quad , \quad v \to 0.$$

Then

1) *if* $\lambda < 2\omega$, *for all initial data, there exists* $T \geq 0$ *such that*

$$\forall t \geq T, \quad x(t) = \zeta.$$

2) *if* $\lambda \geq 2\omega$ *and* $\int_{\mathcal{V}(0)} r^{-2} \sup_{|v| \leq r} (|\rho(v)|) dr < \infty$, *there exist solutions such that*

$$\forall t \geq 0, \quad x(t) > 1/\omega^2 \quad (\textit{ resp. } x(t) < -1/\omega^2 \text{ }).$$

2. 3. Energy methods on finite extinction time: numerical approximation

We consider the quasilinear problem associated to the nonlinear heat equation with absorption

$$(\mathcal{P})\begin{cases} \frac{\partial}{\partial t}\left(u\,|u|^{\gamma-1}\right)-\mathrm{div}\,\left(|\nabla u|^{p-2}\nabla u\right)+|u|^{\sigma-1}u=f+\mathrm{div}\,\mathbf{g}. & \text{in } Q:=\Omega\times(0,+\infty),\\ u=0 & \text{on } \sum_{T}=\Gamma\times(0,+\infty)\\ u(x,0)=u_0(x) & \text{in } \Omega. \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded of boundary Γ

$$\gamma>0,\qquad\sigma>0,\qquad 1\leq p<\infty,\qquad\lambda>0. \tag{1}$$

(if $p=1,\nabla u$ represents the *total variation*). For N=1 the equation becomes

$$\left(u|u|^{\gamma-1}\right)_t-\left(|u_x|^{p-2}u_x\right)_x+\lambda u|u|^{\sigma-1}=f(x,t)+g_x(x,t)$$

We shall assume that

$$\exists T_0\geq 0 \text{ such that } f(t,.)=0,\mathbf{g}(t,.)=\mathbf{0} \text{ in } \Omega,\text{ if } t>T_0.$$

As we shall see, under suitable conditions, the "solution" (notion to be made precise) satisfy the integral energy inequality

Theorem 1 *Let* $u \in L^1_{loc}(T_0, +\infty : W^{1,p}_0(\Omega))$ *for some* $p > 1$ *(or* $u \in L^1_{loc}(T_0, +\infty$ $BV_0(\Omega))$, *if* $p = 1$*) such that* $\exists\ \gamma, k, c > 0, \lambda \geq 0, \sigma > k - 1$ *for which*

$$|u|^{\gamma+k}, |u|^{\sigma+k}, |Du|^p\,|u|^{k-1} \in L^1_{loc}(T_0, +\infty : L^1(\Omega))$$

and

$$y(t) + c\int_s^t \int_\Omega |Du|^p\,|u|^{k-1} + \lambda \int_{t_f}^t \int_\Omega |u|^{\sigma+k} \leq y(s) \text{ a.e. } s, t \in (T_0, +\infty),$$

where

$$y(t) = \int_\Omega |u|^{\gamma+k}\, dx.$$

Assume that

$$1 \leq p < \gamma + 1 \quad \text{and} \quad \lambda = 0$$

or

$$1 \leq p, \quad \sigma < \gamma \quad \text{and} \quad 0 < \lambda,$$

and let

$$k = \begin{cases} 1 & \text{if } N \leq p \text{ or } (\gamma+1) \leq \dfrac{Np}{N-p}, \\ \dfrac{N-p}{p}\left(1 + \gamma - \dfrac{p(N-1)}{N-p}\right) > 1 & \text{if } 1 < p < N \text{ and } \gamma + 1 > \dfrac{Np}{(N-p)}. \end{cases}$$

Then $u \in C^{0,\alpha}_{loc}(T_0, +\infty : L^{\gamma+k}(\Omega))$ *for some* $\alpha \in (0,1)$ *and there exists a* $T_e \in (T_0, +\infty)$ *such that* $u(t, .) \equiv 0$ *in* Ω $\forall t \geq T_e$.

Remark. Some relevant choices of the parameters γ, p, σ which provide the fulfillment of the above conditions are:

1. $p = 2$, $\gamma = 1$; (the principal part of the equation is linear). Then the condition holds if $\sigma < 1$.

2. $\sigma = 1$, $p = 2$; (the diffusion and absorption are linear). Then $\gamma > 1$ implies the assumption.

3. $\sigma = 1$, $\gamma = 1$; (only the diffusion is not linear). Then the condition is guaranteed by the

Several notions of solutions are possible (for simplicity, we assume now $p > 1$). The "variational theory" search for solutions in the "energy space" $u \in L^{p'}(0,T;W_0^{1,p}(\Omega))$, and use that (if $p \geq \frac{2N}{N+2}$)

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega).$$

At least for $k = 1$, $u \in L^p(0,T;W_0^{1,p}(\Omega)), \forall T > 0 \Longrightarrow |Du|^p\,|u|^{k-1} \in L^1_{loc}(0,+\infty : L^1(\Omega))$. A first problem arises with the zero order term $|u|^{\sigma-1}u$ since $u \in L^{p'}(0,T;W_0^{1,p}(\Omega)) \nRightarrow |u|^{\sigma+k} \in L^1_{loc}(0,+\infty : L^1(\Omega))$. Then, if the equation takes place in $D'(\Omega)$ the natural regularity for u_t is

$$|u_t|^{\gamma-1}\,u_t \in L^{p'}_{loc}(0,+\infty : W^{-1,p'}(\Omega)) + L^1_{loc}(0,+\infty : L^1(\Omega)).$$

In that case the test functions must be taken in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T : L^\infty(\Omega))$.

The existence of solutions in the above framework is due to many authors (Dubinsky, J.L. Lions, Raviart, Bamberger, Grange-Mignot, Benilan, Alt-Luckhaus, Díaz-de Thelin,....) assumed

$$|u_0|^{(\gamma-1)}u_0 \in L^2(\Omega) \text{ and } f, \mathbf{g} \in L^{p'}_{loc}(0,+\infty : L^{p'}(\Omega))$$

and the (so called *weak solution*) satisfies that $u \in C([0,+\infty) : L^2(\Omega))$. The regularity $|u|^{\gamma+k}, |u|^{\sigma+k}, |Du|^p |u|^{k-1} \in L^1_{loc}(T_0,+\infty : L^1(\Omega))$ can be obtained by asking some extra regularity to the data (see, e.g. Bocardo, Díaz, Giacheti, Murat 1988). A nontrivial fact is the justification of the time integration by parts formula

$$\left\langle (|u|^{\gamma-1} u)_t, |u|^{k-1} u \right\rangle = \frac{\gamma}{\gamma+k} \int_0^T \left[\frac{d}{dt} \int_\Omega |u(t,.)|^{\gamma+k} dx \right] dt.$$

It could be easily justified for the case of *strong solutions* (i.e. $\frac{\partial}{\partial t}\left(u\,|u|^{\gamma-1}\right) \in L^1(Q)$) but it is know that this class of solutions are quite exceptional. More in general, but for the easier case of $\gamma = k = 1, \lambda = 0$ this was proved in a pioneering paper by J.L. Lions (1963). For $\gamma \neq 1$ and $k = 1$ the result was proved (under different assumptions) by Bamberger, Grange-Mignot, Alt-Luckhaus, Bernis, Otto, Carrillo, Carrillo-Wittbold,...The case $k \neq 1$ is due to Benilan

But the notion of solution can be found out of the energy space W. Among the several types of solutions leaving out side the energy space we could mention, specially, the so called *mild solutions* motivated by the numerical analysis and the abstract Semigroup Theory.

Given $\epsilon > 0$ and a time discretization $t_0 = 0 < t_1 < \cdots < t_n \leq T, t_i - t_{i-1} < \epsilon, T - t_n < \epsilon$, and given $f_i \in L^\infty(\Omega)$, $w_0 \in L^\infty(\Omega)$ we consider the implicit time-discretization,

$$(DP) \left\{ \frac{b(w_i) - b(w_{i-1})}{t_i - t_{i-1}} - \operatorname{div}\left(|\nabla w_i|^{p-2} \nabla w_i\right) + \lambda |w_i|^{\sigma-1} w_i = f_i \text{ in } D'(\Omega), \right.$$

where $b(u) = |u|^{\gamma-1} u$. Notice that $w_i \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Now, let

$$u_0 \in L^\gamma(\Omega), f \in L^1(0,T : L^1(\Omega)), \mathbf{g} = \mathbf{0}.$$

Definition. A *mild solution of (P)* is a function u such that $b(u) \in C([0,+\infty) : L^1(\Omega)), u(0,.) = u_0(.)$, and, for any $\epsilon > 0$ there exists $(t_0, t_1, \cdots, t_n, f_0, f_1, \cdots, f_n, w_0, w_1, \cdots, w_n)$ satisfying (DP) with

$$\|b(u_0) - b(w_0)\|_1 \leq \epsilon, \; \sum_i \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|_1 \, dt \leq \epsilon$$

and $\|b(u(t)) - b(w_0)\|_1 \leq \epsilon$ for any $t \in (t_{i-1}, t_i], i = 1, \ldots, n.$ ■

The existence of a mild solution was due to Benilan (1976). Moreover, in Benilan-Wittbold (1996) it is proven that if in addition

$$u_0 \in L^{\gamma+1}(\Omega), f \in L^{p'}(0,T : W') + L^{(\sigma+1)'}((0,T) \times \Omega)$$

then the mild solution is also a energy solution.

Now we can study the finite extinction time for the step function

$$w_\epsilon(t) := w_i \text{ if } t \in (t_{i-1}, t_i], \; i = 1, \ldots, n.$$

Definition. *We say that $w_\epsilon(t)$ extincts in a finite time if there exists $T_{\epsilon,e} = t_j$, for some $j \leq n$ such that $\|w_\epsilon(t)\|_\infty > 0$ for $t \in [0, T_{\epsilon,e})$ and $\|w_\epsilon(t)\|_\infty = 0$ for $t \in [T_{\epsilon,e}, T]$.*

Corollary 1. *Assume that $\exists T_0 = t_m$, $m \leq n$ such that $f_\epsilon(t,.) = 0$, in Ω, if $t > T_0$ ($f_\epsilon(t,.)$ defined as done for $w_\epsilon(t)$). Then, under the assumption of the Theorem on γ, p, k, and σ, function $w_\epsilon(t)$ extincts in a finite time $T_{\epsilon,e}$.*

Corollary 2 (Benilan-Crandall (1981)). *Let u be a mild solution and assume that $\exists T_0 \geq 0$ such that $f(t,.) = 0, g(t,.) = 0$ in Ω,if $t > T_0$ and that*

$$u(T_0) \in L^{\gamma+k}(\Omega)).$$

Then $u(t)$ extincts in a finite time T_e(only dependent on $\|u(T_0)\|_{\gamma+k}$). ■

Remark. It is possible to have a finite energy at time T_0 ($u(T_0) \in L^{\gamma+k}(\Omega)$)) even if $u_0 \in L^{\gamma}(\Omega)$ (this is the case when there are smoothing effects: Benilan(1978), Veron (1981),....). ■

Remark. A unpleasant fact of mild solutions is the lake of an easy characterization in terms of test functions and the lake of information on the spatial regularity. A different notion of solutions corresponds to the so called *renormalized solutions* (see, Bocardo, Díaz, Giacheti, Murat 1988, 1993) which has its origins in the study of the Boltzman equation and the works by Di Perna and P.L. Lions. The finite extinction time phenomenon can be obtained for such solutions assumed, again, $u(T_0) \in L^{\gamma+k}(\Omega)$). ■

Remark. The assumption $u(T_0) \in L^{\gamma+k}(\Omega)$ is, in some sense, necessary. A counterexample can be built in other case: take $\gamma = 1, \lambda = 0, p = 1$. Assume that $0 \in \Omega$ and $u(0,.) = \delta_0$ (the Dirac Delta distribution at the origin). Then, it is shown in Andreu, Caselles, Mazón that there is no regularizing effect and $u(t,.) = C(t)\delta_0$ with $C(t) > 0$.

The extinction time also exists for other time-discretizations (now of type semi-implicit). We write (assuming now $w \geq 0$)

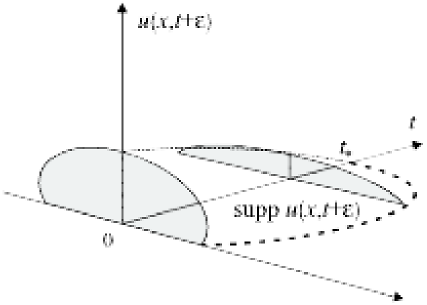
$$(w^\gamma)_t = \frac{\gamma}{\gamma+1-p}(w^{p-1})(w^{\gamma+1-p})_t \approx \frac{\gamma}{\gamma+1-p}(w_i^{p-1})\frac{(w_i^{\gamma+1-p} - w_{i-1}^{\gamma+1-p})}{t_i - t_{i-1}}$$

Remark. For another type of semi-implicit time-discretization,

$$(DP)\left\{\frac{b(w_i)-b(w_{i-1})}{t_i - t_{i-1}} - \text{div }\left(|\nabla w_{i-1}|^{p-2}\nabla w_i\right) + \lambda |w_i|^{\sigma-1} w_i = f_i \text{ in } D'(\Omega),\right.$$

see Kacur (1994) or Bermejo-Diaz-Tello (2002). The existence of a finite extinction time can be also proved.■

2. 3. Other methods



Finite extinction time via comparison principle

Philippe Souplet lecture

Finite extinction time via comparison of symmetrical rearrengements

J.I.Díaz, J.Mossino, Isoperimetric inequalities in the parabolic obstacle problems. *Journal de Mathematiques Pures et Appliqueés*, Vol. 71, 233-266. 1992.

Finite extinction time by spectral arguments

Y. Bellout-D (2004)

Finite extinction and control theory,...

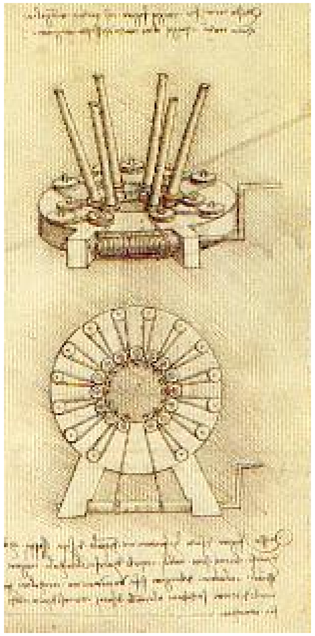
3. Extinction by components

Unpublished result by H. Amann and the author (Madrid, October 1988),

Madrid Leonardo's manuscript

After Leonardo's death in 1519,.... " little books by Leonardo about the anatomy, and many other interesting things",

Mentioned also by an early sixteenth century source,



1630, Pompeo Leoni, a sculptor at the court of the King of Spain,

Remained undiscovered until 1966, when they were found quite by chance in the archives of the National Library of Madrid.

Consider the model problem

$$\begin{cases} u_t - \Delta u = f(t,x) & \text{in } Q_\infty \\ u_t + u_\nu + \beta(u) \ni g(t,x) & \text{on } \Sigma_\infty \\ u(0) = u_0^\Omega & \text{in } \Omega \\ u(0) = u_0^\Gamma & \text{in } \Gamma \end{cases}$$

Here Ω is a **convex** bounded domain in $\mathbb{R}^n$

$\beta : D(\beta) \to \mathcal{P}(\mathbb{R})$ is the Signorini maximal monotone graph

$$\beta(r) = \begin{cases} \phi & \text{if } r < 0 \\ (-\infty, 0] & \text{if } r = 0 \\ 0 & \text{if } r > 0 \end{cases}$$

$u_0^\Omega \in L^\infty(\Omega)$, $u_0^\Gamma \in L^\infty(\Gamma)$

Theorem. *Assume that* $f(t,x) \leq -\varepsilon^2$ near $\partial\Omega$ for $t \geq t_f$, $g(t,x) \leq -\varepsilon^2$ on $\partial\Omega \times (t_g, +\infty)$. Then

Then the trace $u(t,\cdot)$ *on* Γ *vanishes after a finite time.*

Remark. Notice that we can reformulate the problem in terms of a vectorial abstract problem: take the Hilbert spaces V and H, with $V = \{(u,v) \in H^1(\Omega) \times H^{1/2}(\Gamma);\ u_{|\Gamma} = v\}$ which is a real separable Hilbert space isomorphic to $H^1(\Omega)$

$$\|u\|_{H^1(\Omega)} = \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u_{|\Gamma}\|^2_{L^2(\Gamma)}\right)^{1/2},$$

$H = L^2(\Omega) \times L^2(\Gamma)$ which endowed with the usual inner product

$$\langle (u,v),(\tilde{u},\tilde{v})\rangle = \langle u,\tilde{u}\rangle_{L^2(\Omega)} + \langle v,\tilde{v}\rangle_{L^2(\Gamma)},$$

Let us define $A : V \to V^*$ by

$$(A(u,v),(\varphi,\psi)) = \int_\Omega \nabla u \cdot \nabla \varphi \, dx,$$

We define the restriction

$A_H : D(A_H) \subset H \to H$ of A to H by $D(A_H) = \{(u,v) \in V;\ A(u,v) \in H\}$ and

$$A_H(u,v) = A(u,v), \text{ for each } (u,v) \in D(A_H).$$

It is easy to see that

$$D(A_H) = \{(u,v) \in L^2(\Omega) \times L^2(\Gamma);\ \Delta u \in L^2(\Omega),\ u_\nu \in L^2(\Gamma),\ u_{|\Gamma} = v\}$$

and $A_H(u,v) = (-\Delta u, u_\nu)$.

Then the solution is associated to the vector U(t,.)=$(u(t,\cdot), u(t,\cdot)_{|\Gamma}) \in L^2(\Omega) \times L^2(\Gamma)$ can be formulated as

$$\frac{d}{dt}\mathrm{U}(t,.) + \mathrm{AU}(t,.) + \mathrm{BU}(t,.) \ni \mathrm{F}(t,.)$$
$$\mathrm{U}(0,.) = \mathrm{U}_0$$

with $\mathrm{BU}(.) = (0, \beta(u))$ and the above theorem shows that the component $u(t,\cdot)_{|\Gamma}$ vanishes in a finite time although, $u(t,\cdot) \neq 0$ in Ω for any time.

4. Finite extinction time for a finite set of orbits.

$$m x_{tt} + \mu |x_t|^{\alpha-1} x_t + kx = 0,$$

$\alpha \in (0,1)$ and $\mu, k > 0.$

$$x_{tt} + |x_t|^{\alpha-1} x_t + x = 0$$

rescaling $\tilde{x}(\tilde{t}) = \beta^{1/(\alpha-1)} x(\lambda \tilde{t})$

$\lambda = \frac{\sqrt{m}}{\sqrt{k}}$ and $\beta = \frac{\mu}{k^{(2-\alpha)/2} m^{\alpha/2}}.$

$\alpha \to 0$ corresponds to the Coulomb friction equation $\quad x_{tt} + sign(x_t) + x \ni 0$

$\alpha \to 1$ linear damping equation $\quad x_{tt} + x_t + x = 0.$

$$P_\alpha \begin{cases} x_{tt} + |x_t|^{\alpha-1} x_t + x = 0 & t > 0, \\ x(0) = x_0, \ x_t(0) = v_0 \end{cases}$$

The asymptotic behavior, for $t \to \infty$, of solutions of the limit problems P_0 and P_1 is well known (see, for instance, Jordan and Smith [49]). In the first case the decay is exponential. In the second one it is easy to see that "given x_0 and v_0 there exist a finite time $T = T(x_0, v_0)$ and a number $\zeta \in [-1, 1]$ such that $x(t) \equiv \zeta$ for any $t \geq T(x_0, v_0)$". For problem P_α it is well-known that $(x(t), x_t(t)) \to (0, 0)$ as $t \to \infty$ (see, e.g. Haraux [41]).

Main goal of D-Liñán (2001,2002), D-Amann (2003): the generic asymptotic behavior above described for the limit case P_0 is only exceptional for the sublinear case $\alpha \in (0,1)$ since the generic orbits $(x(t), x_t(t))$ decay to $(0,0)$ in a infinite time and only two uniparametric families of them decay to $(0,0)$ in a finite time: in other words, when $\alpha \to 0$ the exceptional behavior becomes generic.

planar system $\begin{cases} x_t = y \\ y_t = -x - |y|^{\alpha-1} y \end{cases}$

$$y_x = \frac{-x - |y|^{\alpha-1} y}{y}$$

the plane phase is antisymmetric

$$(x^2 + y^2)_t = 2\,|y|^{\alpha+1}.$$

$(1/x, 1/y)$ satisfy a system which has the point $(0,0)$ as a spiral unstable critical point.

For $\alpha = 1$ character of the trajectories close to the origin depends on the parameter β

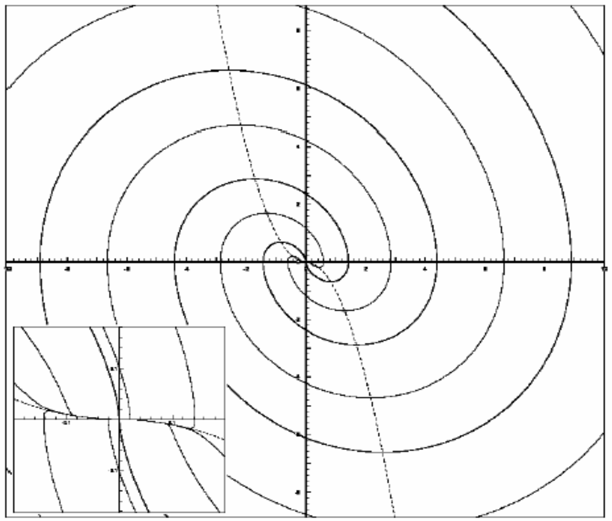
$\beta > \beta_c := 2$ the origin is a stable mode $\beta < \beta_c$ is a stable spiral

We proved that there are two modes of approach to the origin and so that the origin $(0,0)$ is a node for the system

The lines of zero slope $-x = |y|^{\alpha-1} y$

convergence to $(0,0)$ is only possible through the regions

$\{(x,y) : x > 0, y < -x^{1/\alpha}\} \cup \{(x,y) : x < 0,\ y > (-x)^{1/\alpha}\}$



"ordinary" mode small effects of the inertia

Let $-y = \tilde{y} > 0$. $\boxed{\tilde{y}\tilde{y}_x = -x + \tilde{y}^{\alpha}.}$ line of zero slope is $\tilde{y} = x^{1/\alpha}$

we search for orbits obeying, for $0 < x << 1$, $\tilde{y} = x^{1/\alpha} + z(x)$ for some function $z(x)$.

condition $0 < z(x) << x^{1/\alpha}$, "linearized form" $\frac{1}{\alpha}x^{(\frac{1}{\alpha}-1)}z + x^{\frac{1}{\alpha}}z_x - \alpha x^{(1-\frac{1}{\alpha})}z = 0.$

$$z(x) \sim C\exp\{-[\alpha^2/2(1-\alpha)]x^{-\frac{2(1-\alpha)}{\alpha}}\}$$

C an arbitrary constant (which explain the name of "ordinary" orbits).

close to the origin, $\tilde{y} \sim x^{1/\alpha} \sim -\frac{dx}{dt}$ $\quad \frac{dx}{dt} = -x^{1/\alpha} \quad x(t) \sim [\frac{\alpha}{(1-\alpha)(t+t_1)}]^{\alpha/(1-\alpha)}$

Some different orbits approaching the origin can be found

large values of $|y|$ compared with $|x|^{1/\alpha}$.

$$\boxed{\tilde{y}\tilde{y}_x = \tilde{y}^\alpha}$$

$$\tilde{y}(x) = -\{(2-\alpha)x\}^{1/(2-\alpha)}.$$

it involves no arbitrary constant. which justifies the term of "extraordinary" orbit.

$$-\frac{dx}{dt} = [(2-\alpha)x]^{1/(2-\alpha)}$$

$$x(t) = \frac{1}{(2-\alpha)}[\frac{(2-\alpha)(1-\alpha)}{2\alpha}(t_0-t)_+]^{(2-\alpha)/(1-\alpha)}$$

$$v_0 \sim \pm[(2-\alpha)\,|x_0|]^{1/(2-\alpha)}$$

exceptional orbits separatrix curve in the phase plane.

A rigorous proof of the existence of the extraordinary orbits

Theorem 1 *There exists a, b, with $0 < a < (1-\alpha)^{1/(1-\alpha)} < b$, $R > 0$ and $t_0 > 0$ such that for some initial data (x_0, v_0) satisfying*

$$-R{t_0}^{(2-\alpha)/(1-\alpha)} \leq x_0 < 0$$

and

$$a{t_0}^{1/(1-\alpha)} \leq v_0 \leq b{t_0}^{1/(1-\alpha)}$$

the associate solution $x(t)$ vanishes identically for any $t \geq t_0$.

it is useful to work backwards in time,

i.e. we search $X : [-t_0, 0] \to \mathbb{R}$ such that

$$X(-t) = x(t_0 - t), \text{ if } t \in [0, t_0]. \qquad X(-t_0) = x_0 \text{ and } X(0) = 0.$$

$$\begin{cases} X_s = Y \\ Y_s = -X - |Y|^{\alpha-1} Y \end{cases} \qquad s = -t \in [-t_0, 0]$$

We define the Banach spaces

$$E = \{X \in C[-t_0, 0] : X(0) = 0, \|X\| < \infty\} \qquad \|X\| := \sup_{s\in[-t_0,0]} \frac{|X(s)|}{|s|^{(2-\alpha)/(1-\alpha)}}$$

$$V = \{Y \in C[-t_0, 0] : Y(0) = 0, |\|Y\|| < \infty\} \qquad |\|Y\|| := \sup_{s\in[-t_0,0]} \frac{|Y(s)|}{|s|^{1/(1-\alpha)}}$$

We also define the operator $\mathcal{T}$: $E \times V \to E \times V$

$$[\mathcal{T}(X,Y)](s) = (-\int^0 Y(r)dr, \int^0 (|Y(r)|^{\alpha-1} Y(r) + X(r))dr).$$

if (X, Y) is a fixed point of $\mathcal{T}$ then (X, Y) is the searched solution

We introduce the closed and convex sets

$$K_R := \{X \in E : -R|s|^{(2-\alpha)/(1-\alpha)} \le X(s) \le 0, \ \forall s \in [-t_0, 0]\}$$

$$S_{a,b} := \{Y \in V : a|s|^{1/(1-\alpha)} \le Y(s) \le b|s|^{1/(1-\alpha)}, \ \forall s \in [-t_0, 0]\}$$

$$\|(X,Y)\| := \max(\|X\|, |||Y|||)$$

$$b\frac{(1-\alpha)}{(2-\alpha)} \leq R.$$

$$\mathcal{T}(K_R \times S_{a,b}) \subset K_R \times S_{a,b}$$

$$a^{\alpha}(1-\alpha) - R\frac{(1-\alpha)}{(3-2\alpha)}t_0^2 \quad \geq \quad a$$

To see that $\mathcal{T}$ is a contraction it is enough

$$\|D\mathcal{T}(X,Y)\| < 1$$

$$\alpha(1-\alpha)a^{\alpha-1} + \frac{(1-\alpha)}{(3-2\alpha)}|t_0|^2 < 1$$

satisfied if we take

$$[\alpha(1-\alpha)]^{1/(1-\alpha)} < a < (1-\alpha)^{1/(1-\alpha)} \leq b, \qquad R \geq b\frac{(1-\alpha)}{(2-\alpha)}$$

$$t_0 \leq \min[\frac{(3-2\alpha)}{R(1-\alpha)}((1-\alpha)a^{\alpha} - a), \frac{(3-2\alpha)}{(1-\alpha)}(1-\alpha(1-\alpha)a^{\alpha-1})]^{1/2}$$

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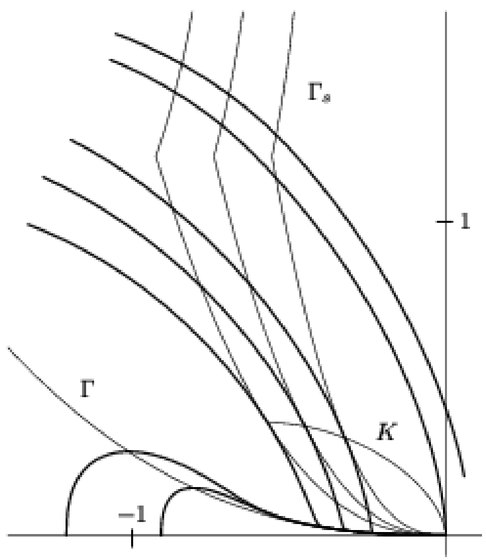

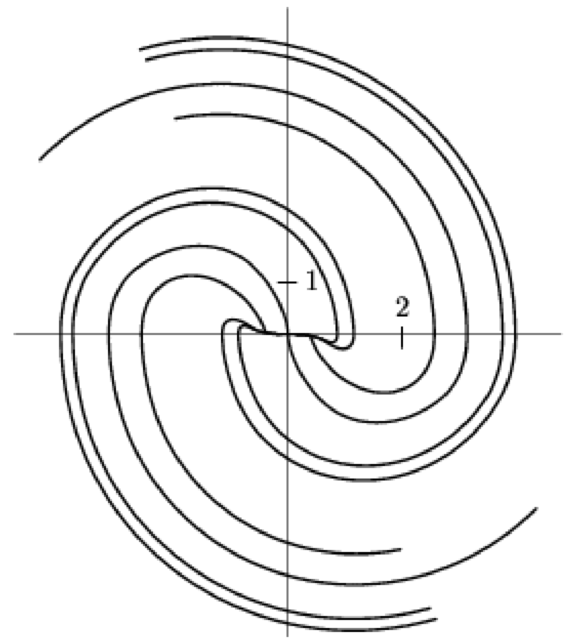

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Work in progress (D-G. Hetzer)

Dry friction and impulsive forces

$$m\ddot{\mathbf{x}}(t) + k\mathbf{A}\mathbf{x}(t) + \mu_\beta \mathbf{B}(\dot{\mathbf{x}}(t)) + \mu_\beta \mathbf{G}(\dot{\mathbf{x}}(t)) \ni \mathbf{f}(t)$$

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5. Final conclusions

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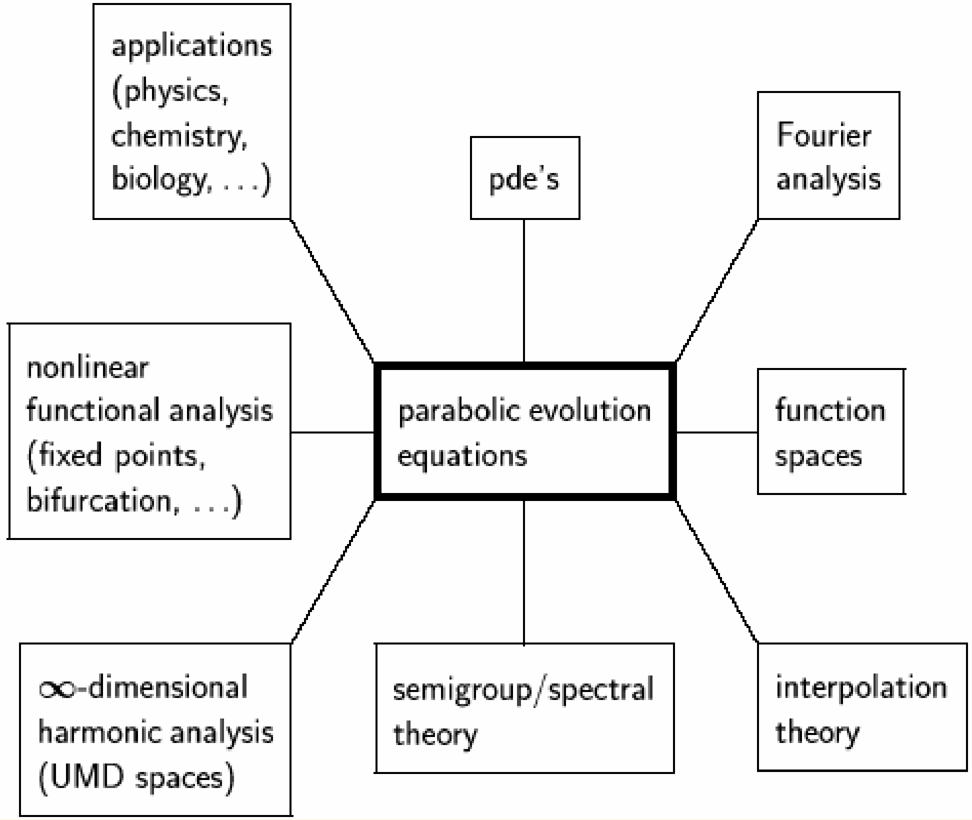
November 7th, 2001

In the following, I shall try to give an idea of some of my research interests of the last twenty years. During that period I was predominantly concerned with a functional-analytical approach to parabolic evolution equations. In my opinion, functional analysis, combined with so-called "hard analysis" and many other mathematical subjects, is particularly well suited for providing an abstract, powerful, and sufficiently general framework for the study of nonlinear partial differential equations.

Of course, it is well-known that there are intimate connections between functional analysis and partial differential equations. In fact, large parts of linear functional analysis have been developed in order to provide the abstract tools for an efficient and unified study of linear partial differential equations. The point I want to make is that functional analysis is also very useful for the investigation of nonlinear differential equations.

9. A broader view

In order to summarize and to give a somewhat broader view I discuss now some of the interrelationships of the theory of parabolic evolution equations with other fields of analysis, as indicated in the following diagram



I have put parabolic evolution equations in the middle, since they are in the center of my present interest, and have grouped around them several other subjects. I did not put arrows on the connecting lines since in many cases the interaction is bilateral.

Let us start at the left upper corner. It is well-known — and I have taken reaction-diffusion systems as an example — that many concrete models for the understanding of phenomena in science lead to parabolic evolution equations and, vice versa, results on parabolic evolution equations have immediate interpretations and consequences for those models.

Partial differential equations are, of course, intimately connected with parabolic evolution equations. However, neither forms a subfield of the other. For example, parabolic evolution equations encompass also other systems like integro-differential equations or infinite systems of reaction-diffusion equations involving even uncountably many unknowns, as they occur in statistical physics (see [9], [17]).

The connection between parabolic evolution equations and Fourier analysis lies on a more technical level and can be described adequately by more detailed explanations only.

As pointed out earlier, the choice of the correct state space is fundamental when studying partial differential equations, parabolic evolution equations in particular. The well-developed theory of function spaces provides us with a wide variety of possibilities. Spaces more refined than integer order Sobolev spaces like Besov and Bessel potential spaces have become increasingly important during the last years. This is true, in particular, in the study of the Navier-Stokes equations (cf. [10], [13], [14], and the references therein).

Finally, methods from nonlinear functional analysis, fixed point theorems, bifurcation theory, etc., play an important rôle in the difficult and fascinating investigation of qualitative properties of the semiflows generated by parabolic evolution equations.

I hope that this enumeration of subjects, which is far from being complete, shows that the field of parabolic evolution equations is a fascinating one, invoking a lot of deep and beautiful mathematics.

RACSAM, 97 (1), 2003, 89-105

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