Modelización matemática de la elasticidad de la cubierta volada del Hipódromo de la Zarzuela

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POLITÉCNICA

Seminario del Departamento de Matemática Aplicada

9 Octubre 2018

ETS DE ARQUITECTURA. UPM

1. Introducción.

Conferencia complicada:

- a mitad de camino entre divulgación y presentación de resultados nuevos en el campo (EDPs, Elasticidad, Matemática Aplicada),
- Seminario del Departamento de Matemática Aplicada (¿audiencia ?)

Indulgencia frente a una pequeña gimnasia mental: castellano / inglés

Shell structures are typically found in nature as well as in classical architecture.

There are two principal uses of shells in civil engineering:

•industrial structures: silos, tanks, cooling towers, reactor vessels etc.
•aesthetic and architectural special structures

(Potential very nice lecture of many pictures: a different lecture…)

Here, after recalling the mathematical modeling of the Shell deformation theory, we shall apply it to the Zarzuela Racecourse shells

HIPÓDROMO DE LA ZARZUELA. MADRID, 1935.

C. Arniches, L. Domínguez y Eduardo Torroja. Con la empresa constructora Agroman E.C.

!! Joya arquitectónica !!

Discusión Contribuciones Crear una cuenta Acceder

WIKIPEDIA
La enciclopedia libre

Artículo Discusión Leer Editar Ver historial Buscar en Wikipedia

Hipódromo de la Zarzuela

Coordenadas: 40°28′4.40″N 3°45′21.60″O (mapa)

Este artículo tiene referencias, pero necesita más para complementar su verificabilidad.
Puedes colaborar agregando referencias a fuentes fiables como se indica aquí. El material sin fuentes fiables podría ser cuestionado y eliminado.
Este aviso fue puesto el 9 de junio de 2015.

El **Hipódromo de la Zarzuela** está situado en las afueras de la ciudad de Madrid (España). Se encuentra enclavado en el monte de la Zarzuela, cerca de la localidad de El Pardo (Madrid), a la altura del kilómetro 7,8 de la Autovía del Noroeste (A-6). Fue diseñado por los arquitectos Carlos Arniches Moltó y Martín Domínguez con la colaboración del Ingeniero de Caminos, Canales y Puertos Eduardo Torroja. Sus tribunas fueron catalogadas como Monumento Histórico Artístico en el año 1980[1] y está participada en propiedad por la SEPI.

Carrera de caballos en el Hipódromo de la Zarzuela.

Índice [ocultar]

Historia y gestión [editar]

La construcción del Hipódromo de la Zarzuela comenzó en 1931 sobre unos terrenos propiedad del Patrimonio Nacional tras la expropiación del anterior Hipódromo de la Castellana para poder construir los Nuevos Ministerios.[1] En 1940, Francisco Franco dictó un decreto-ley para ceder los terrenos a la Sociedad de Fomento y Cría Caballar de España.[1]

Monumento [editar]

Se trata de un ejemplo del racionalismo madrileño y está considerado como la «última obra maestra de la arquitectura del tiempo de la República».[2] La belleza y amplitud del recinto hacen que se le considere un monumento. Uno de sus elementos más singulares son las tribunas, construidas por los arquitectos Carlos Arniches y Martín Domínguez junto al ingeniero de caminos Eduardo Torroja. La construcción es colindante con el monte de El Pardo.

En 2009 fue declarado Bien de Interés Cultural. En 2012, ganó el Primer Premio del Colegio de Arquitectos de Madrid (COAM) por su Proyecto de Restauración y Rehabilitación.[3]

Pruebas principales [editar]

Portada
Portal de la comunidad
Actualidad
Cambios recientes
Páginas nuevas
Página aleatoria
Ayuda
Donaciones
Notificar un error

Imprimir/exportar
Crear un libro
Descargar como PDF
Versión para imprimir

En otros proyectos
Wikimedia Commons

Herramientas
Lo que enlaza aquí
Cambios en enlazadas
Subir archivo
Páginas especiales
Enlace permanente
Información de la página
Elemento de Wikidata
Citar esta página

En otros idiomas
Deutsch
English
Euskara

- 5 m intervals and connected longitudinally by small reinforced concrete double curvature vaults.
- The cantilever roof, with a minimum thickness of 5 cm, overhangs to a distance of 12,8 m.

The race course is protected by a heritage listing *Bien de Interes Cultural* (2009).

2004 Remodelación: Jerónimo Junquera García del Diestro y Liliana Claudia Obal Díaz.

Impresionante listado de estudios, informes, exposiciones,…

i) Por el propio Eduardo Torroja:

- E. Torroja Miret: El nuevo hipódromo de Madrid. Antecedentes del concurso. *Hormigón y Acero*, nº 7, 1934, 287-288.
- E. Torroja Miret: Cubiertas laminares de hormigón armado, Hormigón y Acero nº3.; 1935/36.
- E. Torroja Miret: The Structures of Eduardo Torroja, F. W. Dodge Corporation, New York, 1958, Las estructuras de Eduardo Torroja CEDEX, Madrid, re-edición de1999.
- E. Torroja Miret: Estructura de las tribunas del nuevo hipódromo de Madrid. Revista de Obras Públicas, nº 2714, 1941, 213-222.

La solución:

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \qquad [1]$$

$$\sigma_z = -\frac{P}{I}(l-z)\,x \qquad [2]$$

$$\tau_{yz} = -\frac{\partial f(x\,y)}{\partial x} \qquad [3]$$

$$\tau_{xz} = \frac{\partial f(x\,y)}{\partial y} - \frac{P}{2I}x^2 + \frac{2(1+\upsilon}{\upsilon}\frac{P}{I}y^2, \qquad [4]$$

en la que P es la carga vertical total aplicada; I, el momento de inercia de la sección considerada; ν, el coeficiente de Poisson y en la que la función $f(x\,y)$ cumple, en todos los puntos de la sección, la condición:

$$\frac{\partial^2 f(x\,y)}{\partial x^2} + \frac{\partial^2 f(x\,y)}{\partial y^2} = 0, \qquad [5]$$

y en todos los puntos del contorno, la otra condición:

$$f(x\,y)_c = \frac{P}{2I}\int_c x^2\,dy - \frac{\upsilon}{2(1+\upsilon)}\frac{P}{I}y^3 + \text{Const.}, \qquad [6]$$

satisface el problema. (Indicamos con el subíndice c que se refiere al contorno.)

- E. Torroja Miret: Razon y ser de los tipos estructurales. CSIC, Madrid, 1957, re-edición de 1991.
- E. Torroja Miret: Hipódromo de la Zarzuela, Informes de la Construcción Vol. 14, nº 137, Enero, febrero de 1962.

ii) Análisis por otros autores (véase, *Hipódromo de la Zarzuela en la base de datos del Colegio de Arquitectos de Madrid* *http://212.145.146.10/biblioteca/fondos/ingra2014/index.htm#inm.F3.474* *)*

- F. Arredondo, C. Benito, G. Echegaray y J. Nadal: *La obra de Eduardo Torroja*. Artes Gráficas Soler, Valencia 1977.
- Varios autores: La modernidad en la obra de Eduardo Torroja. Catálogo de la exposición celebrada en el Colegio de Ingenieros de Caminos, Canales y Puertos, Madrid, junio de 1979), Ediciones Turner, 1979.
- J.A. Fernández Ordóñez y J.R. Navarro Vera: *Eduardo Torroja Ingeniero*. Pronaos, Madrid, 1999.
- C. García Reig: La geometría en la obra de Eduardo Torroja. *Revista de obras públicas*, nº 3393, 1999, 15-31.
- J. Antuña Bernardo: *Las estructuras de edificación de Eduardo Torroja Miret*. Tesis Doctoral. ETSAM, Madrid, 2002.
- P. Chías Navarro, y T. Abad Balboa: Eduardo Torroja. Obras y proyectos. Instituto de Ciencias de la Construcción Eduardo Torroja, Madrid, 2005.

iii) Véase también:

- M.C. Díez Pastor: *Carlos Arniches y Martín Domínguez, Arquitectos de la generación del 25.* Mairea. Madrid, 2005.
- P. Rabasco y M. Domínguez Ruiz: *Martín Domínguez Esteban.* Cornell AAP Publications. 2016.

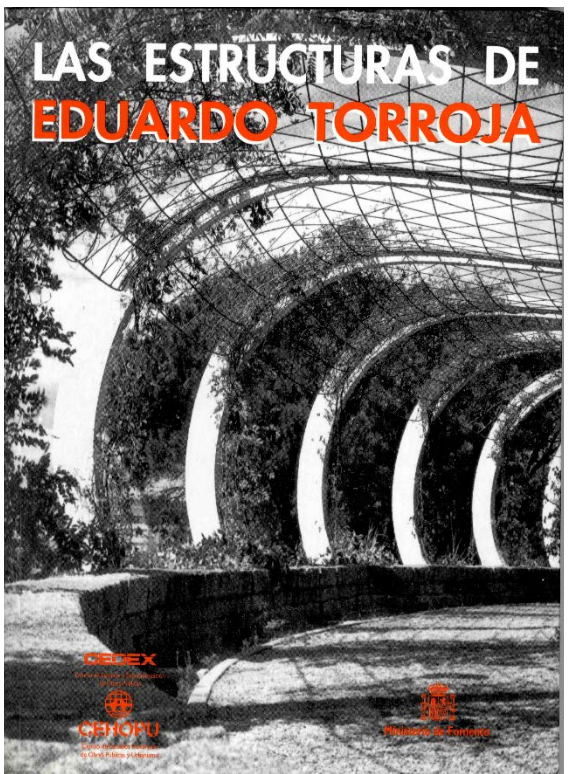
iv) Documentos de profesores de la Escuela de Arquitectura de la UPM: Rafael García García, Ana López Mozo, María Pastor, …

From the mathematical view point: geometrical considerations…. (a potential very nice lecture).

Main goal of this lecture: elastic modelling of this type of shells.

Personally, one (but not the single one !!) of the motivations to my interest :

(in his book, E. Torroja, The Structures of Eduardo Torroja, F. W. Dodge Corporation, New York, 1958 (Ministerio de Fomento, Madrid, 1999) he writes (p. 12):



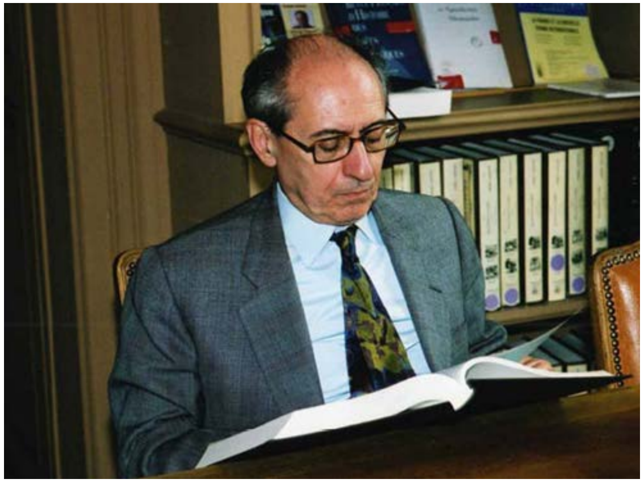
La teoría de la elasticidad no ha desarrollado aún procesos matemáticos adecuados para el análisis de esfuerzos en una estructura de este tipo, pero a pesar de ello y aunque no se cuente con un análisis de gran precisión, se sabe que poseen buenas propiedades estructurales en el espacio. Para el edificio que nos ocupa, se realizaron varias pruebas con el único propósito de estimar la dirección e intensidad de los esfuerzos más probables (véase el diagrama anterior).

The present talk, based on my joint papers with

Evariste Enrique Sanchez-Palencia,
Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, and Academie des Sciences, Paris.

- J. I. Díaz, E. Sánchez-Palencia, On slender shells and related problems suggested by Torroja's structures, *Asymptotic Analysis*, 52, 2007, 259-297
- J. I. Díaz, E. Sánchez-Palencia, On a problem of slender slightly hyperbolic shells suggested by Torroja’s structures. *CRAS Mechanique*, 337 (2009) 1-7.
- J. I. Díaz, E. Sanchez-Palencia, A problem on slender nearly cylindrical shells suggested by Torroja's structures. *International Journal of Engineering Science*. 88 (2015) 83–98
- J. I. Díaz, E. Sanchez-Palencia, On periodic or pseudo-periodic slender nearly cylindrical shells modelling Torroja's structures (2018), To be submitted.

Évariste Enrique **Sanchez-Palencia** (Madrid 1941-).
Nacionalizado francés 1976 (residente en Paris desde 1964).



Directeur de Recherche de classe exceptionnelle, CNRS, emérito.

Ingeniero Aeronáutico (UPM) 1963-1964,
Doctor UPM, 1969.

- 1969-Médaille de bronze du C.N.R.S. (Mécanique.)
- 1974-Prix Henri de Parville de Mécanique, décerné par l'Académie des Sciences.
- 1981-Médaille d'argent du C.N.R.S. (Sciences Physiques pour l'Ingénieur)
- 1987-Elu Correspondant de l'Académie des Sciences, dans la section des Sciences Mécaniques, puis Membre en 2001.
- 1995-Prix de l'Institut Français du Pétrole, décerné par l'Académie des Sciences.

Ha publicado 5 libros, y más de 200 artículos de investigación. 20 Tesis doctorales dirigidas.

Su trabajo en los últimos años se centra en tres líneas:

- Homogeneización en medios heterogéneos (solidos, fluidos en medios porosos…)
- Vibración de sistemas acoplados
- Medios elásticos formados por capas finas

Próximo Dr. Honoris Causa por la Universidad Politécnica de Madrid, propuesto por la Escuela T.S. I. Aeronáuticos (Amable Liñán, …).

Some previous *naif* questions:

Q: What is a shell?

A: A three-dimensional elastic body occupying a thin neighborhood of a two-dimensional submanifold of $\mathbb{R}^3$. That is, a shell is a *physical object*. Our goal is to predict the displacement and stress (measurable physical quantities) arising in response to given loads and boundary conditions.

Q: What is a shell model?

A: A systems of equations (usually PDEs) which, when solved, yields a displacement (and stress) field approximating that of the physical shell. So the "reconstruction" of the 3D field is an essential part of the model. Shell models can involve PDEs in 3 variables (like the equations of 3D elasticity) or be *dimensionally reduced* and involve only PDEs in 2 variables.

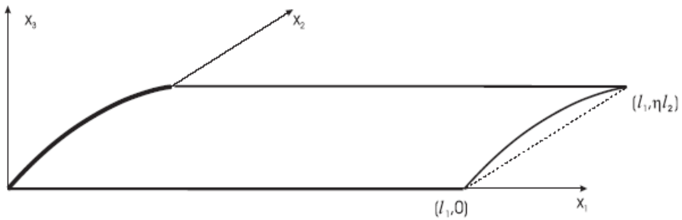
Q: What can we mathematicians do with shell models?

- Derive them
- Analyze the behavior of the shell model
- Determine the accuracy of the shell model in relation to the shell
- Solve the shell model (numerically)

for this last subject, please ask to some other colleagues !!!

Three kind of structures related to the Zarzuela shell roofs

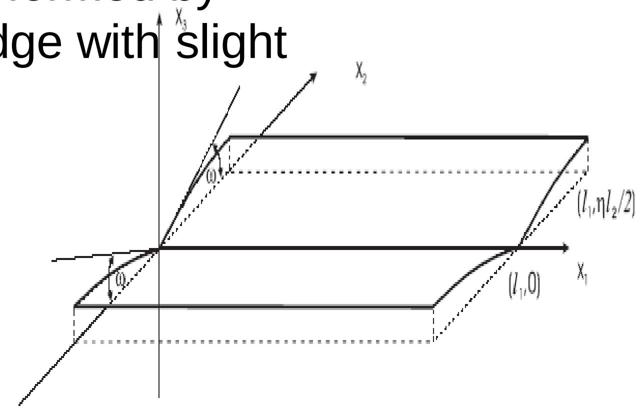
1. The basic element: We shall carry out the study of the asymptotic modelling of such kind of shell structures



¿Motivation?

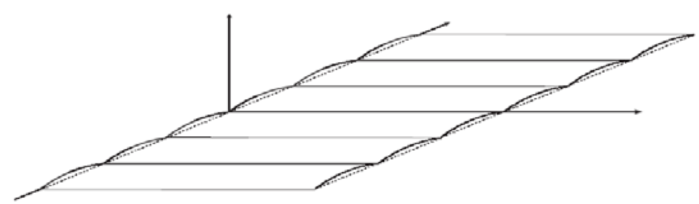
2. Two basic shells coupled by an edge with slight folding

We also will consider more sophisticated structures formed by coupling two of such basic shells by means of an edge with slight folding

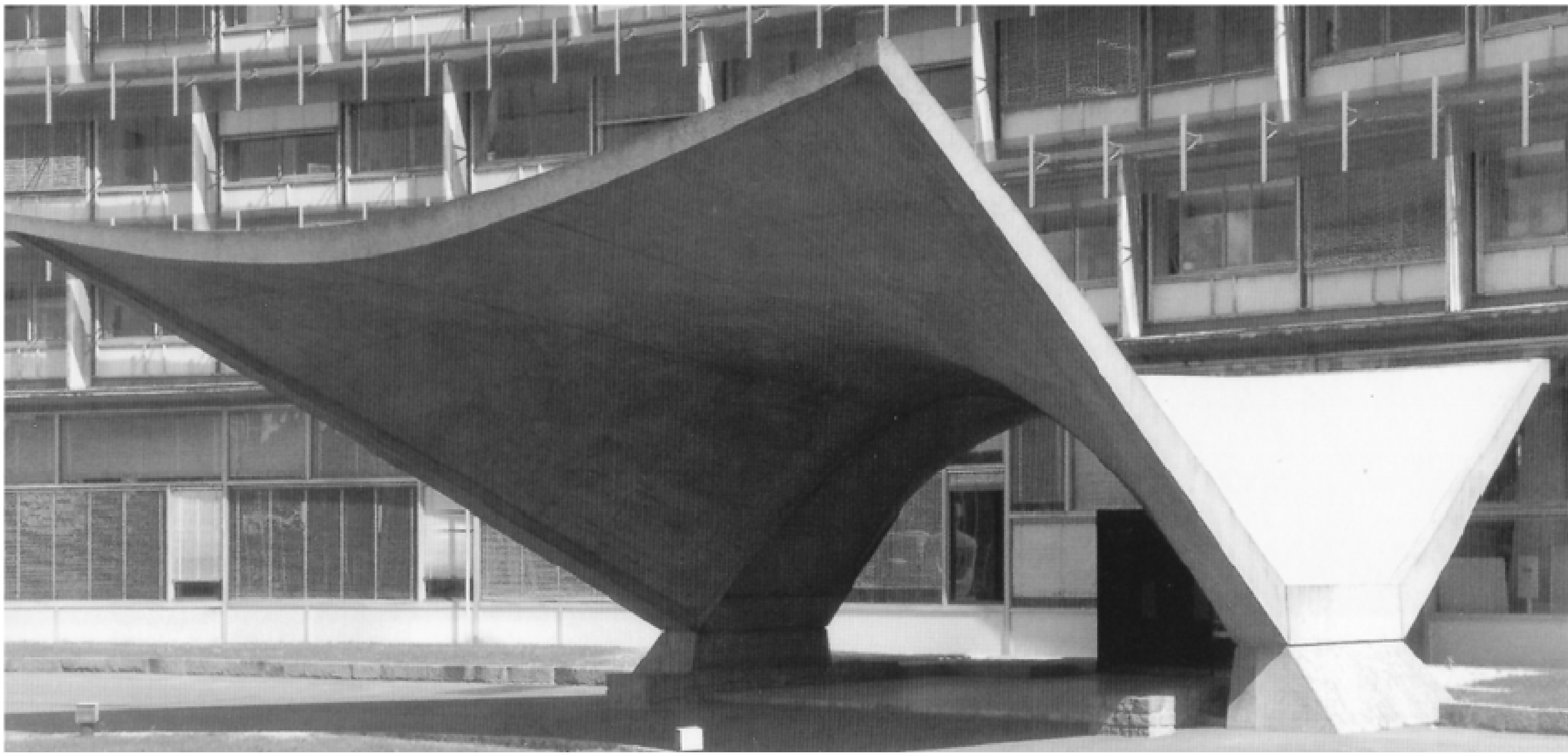


3. The periodic repetition of a basic shell

An *infinity set* of shells obtained by the periodic repetition of the basic structure



Another example is the "pedestrian access shell in the southwestern side of the UNESCO building (Paris, 1953-58) due to Marcel Breuer and Bernard Zehrfuss with the collaboration of Antonio and Pier Luigi Nervi



Many other similar Shells, …

Plan of the rest of the lecture:

2. Shells (coques minces, cáscaras,…): the difficult reduction from 3d to 2d

3. The basic problem.

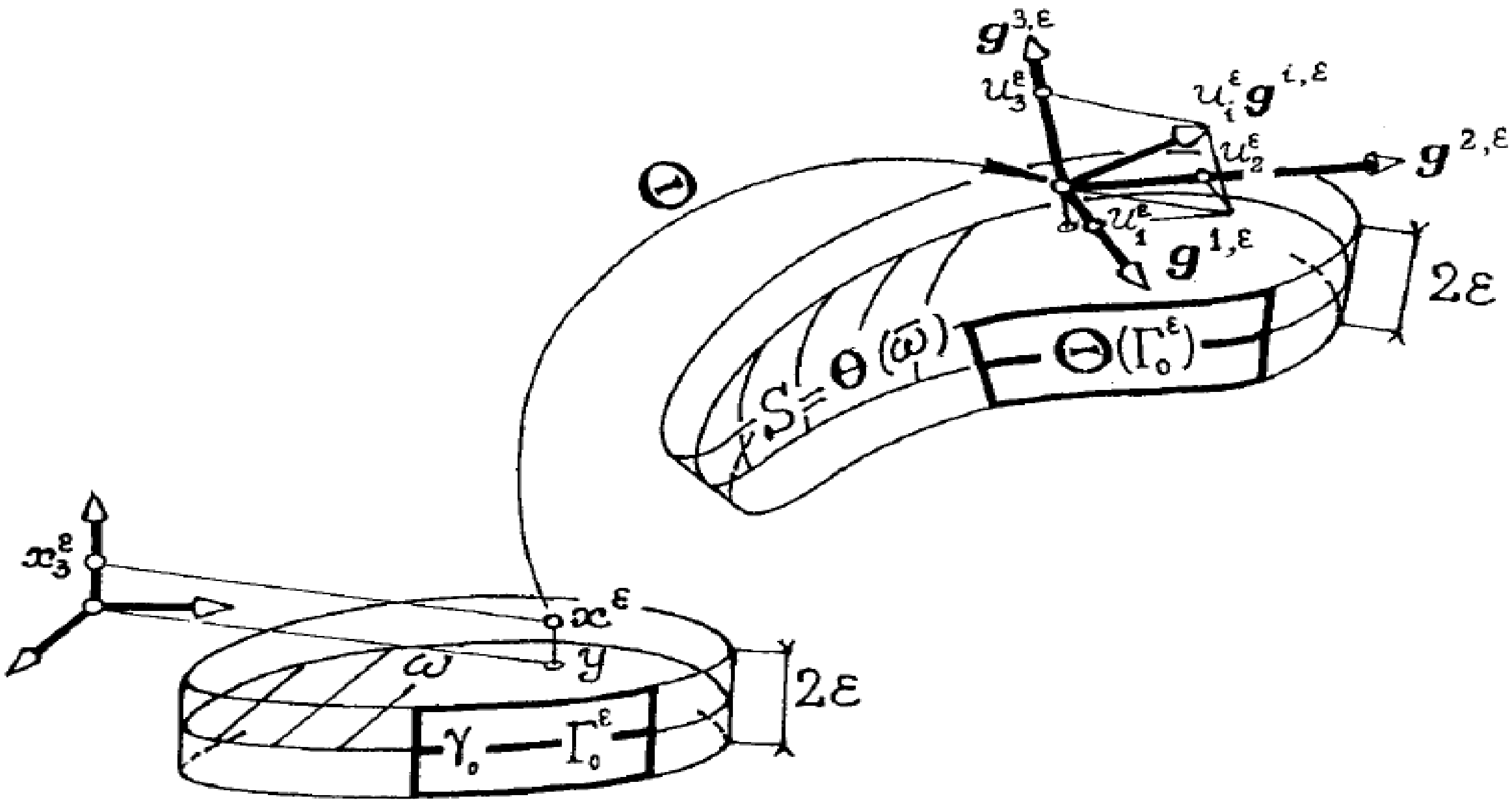
4. The shell with an edge with slight folding

5. On periodic or pseudo-periodic slender nearly cylindrical shells

6. Final remarks

2. Shells (*coques minces*, cáscaras,…): the difficult reduction from 3d to 2d

Planteamiento general (notación no universal,..)



F. Niordson, *Shell theory*, North Holland, Amsterdam, 1985

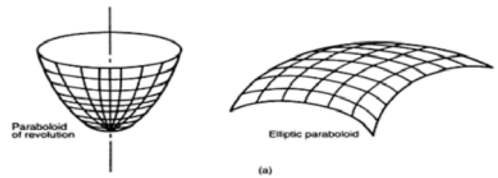
P. G. Ciarlet, *Mathematical Elasticity, Vol III, Theory of Shells*, North-Holland (2000)

J. Sanchez-Hubert and E. Sanchez Palencia*, Introduction aux méthods asymptotiques et à l'homogénéisation: application à la Mécanique des Milieux Continus*, Masson, Paris 1992.
J. Sanchez-Hubert and E. Sanchez Palencia, *Coques élastiques minces. Propriétés asymptotiques*, Masson, Paris 1997.

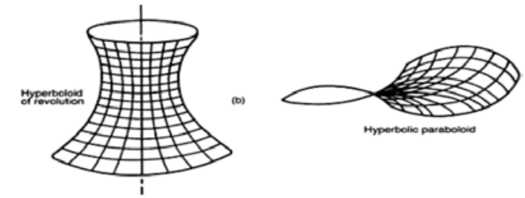
Many other books, surveys, expositions,…

Surface Rigidity !!! (previous considerations to the mechanics model)

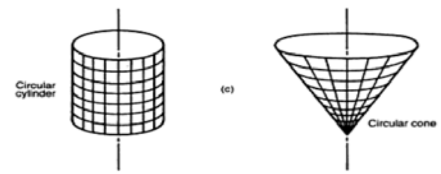
Structural typologies



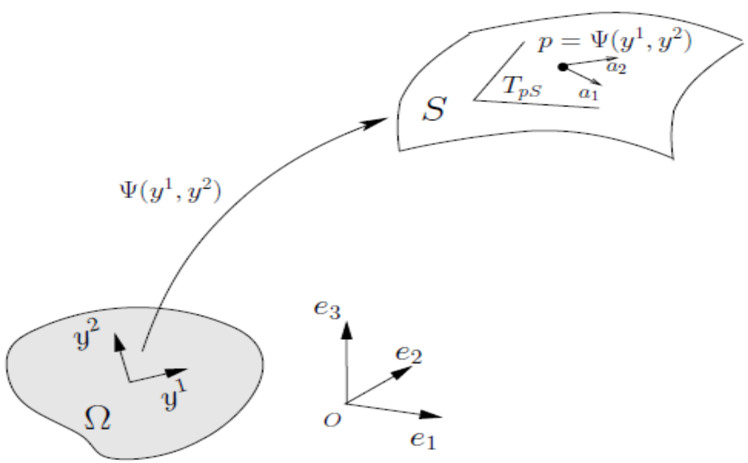
Eliptic paraboloid



Hyperbolic paraboloid



Circular cylinder/cone



$$\Psi : \ \Omega \subset \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^3$$
$$(y^1, y^2) \longmapsto \Psi(y^1, y^2)$$

$$a_\alpha = \frac{\partial \Psi(y^1, y^2)}{\partial y^\alpha} \qquad \alpha \in \{1, 2\}$$

$$N = \frac{a_1 \wedge a_2}{\|a_1 \wedge a_2\|}$$

The set (a_1, a_2, N) *is called the covariant basis*

the *metric tensor* $(a_{\alpha\beta})$ $\qquad a_{\alpha\beta} = a_\alpha \cdot a_\beta$

Considering a regular curve Γ of the surface S, defined in a parametric form $(y^1 = y^1(t), y^2 = y^2(t))$, the infinitesimal length ds of an element of Γ is given by:

$$ds^2 = a_{11}dy^1dy^1 + 2a_{12}dy^1dy^2 + a_{22}dy^2dy^2$$

first fundamental form

The infinitesimal area element dS of S is related to the infinitesimal area element dy^1dy^2 of Ω by:

$$dS = \sqrt{a}dy^1dy^2 \quad with \;\; a = det(a_{\alpha\beta}) = a_{11}a_{22} - (a_{12})^2$$

Because the covariant basis is generally not orthonormal, one may associate a *contravariant basis* (a^1, a^2, a^3) such that:

$$a^\beta \cdot a_\alpha = \delta_\alpha^\beta \quad and \quad a^3 = N$$

δ_α^β is the Kronecker symbol

$$a_\alpha = a_{\alpha\beta}\; a^\beta \qquad a^\alpha = a^{\alpha\beta}\; a_\beta$$

$(a^{\alpha\beta})$ is the metric tensor associated with the contravariant basis

$$a^{\alpha\beta} = a^\alpha \cdot a^\beta = (a_{\alpha\beta})^{-1} = \frac{1}{a}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix}$$

The *second fundamental form* of the surface S is directly linked to its curvature.

$$b(dy^1, dy^2) = b_{11}dy^1dy^1 + 2b_{12}dy^1dy^2 + b_{22}dy^2dy^2$$

$b_{\alpha\beta}$ are the covariant components of the curvature tensor defined by:

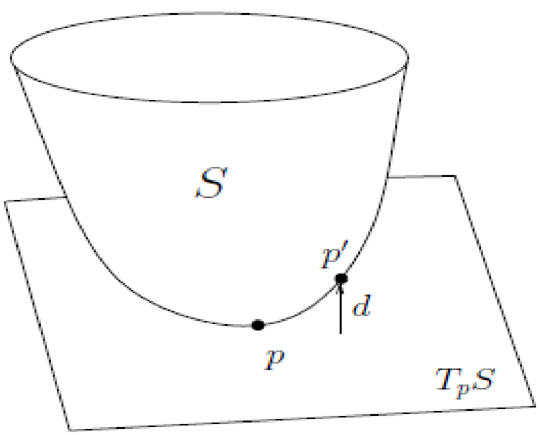
$$b_{\alpha\beta} = b_{\beta\alpha} = -a_\alpha \cdot N_{,\beta} = N \cdot a_{\alpha,\beta}$$

It can also be written under a contravariant form:

$$b^{\alpha\beta} = a^{\alpha\mu}a^{\lambda\beta}b_{\mu\lambda}$$

Classification of Surfaces

the distance d



$$d = \left[\Psi(y^1 + dy^1, y^2 + dy^2) - \Psi(y^1, y^2)\right] \cdot a_3$$

The directions (dy^1, dy^2) that cancel

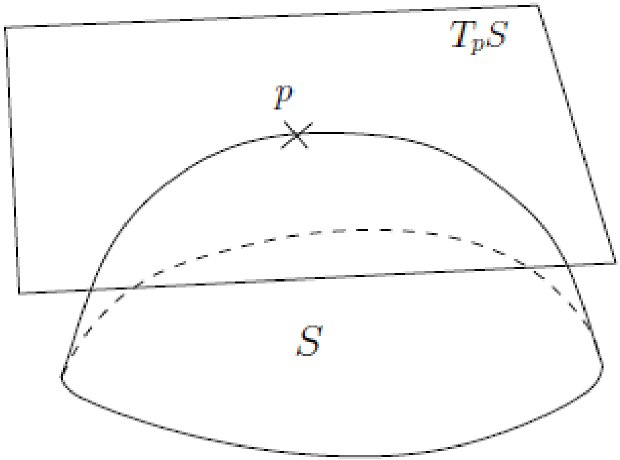
$$d = \frac{1}{2}\left[b_{11}dy^1dy^1 + 2b_{12}dy^1dy^2 + b_{22}dy^2dy^2\right] + \dots$$

are called *asymptotic directions*

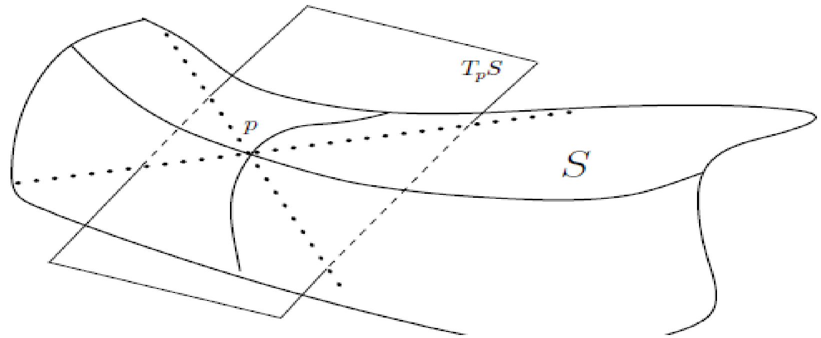
The nature of the middle surface at the point p depends on the sign of d which is the same as the sign of the determinant of the second fundamental form:

Definition. A surface S is said to be **uniformly elliptic**, **parabolic** or **hyperbolic** when the determinant of the second fundamental form is respectively positive, null, or negative at each point of S.

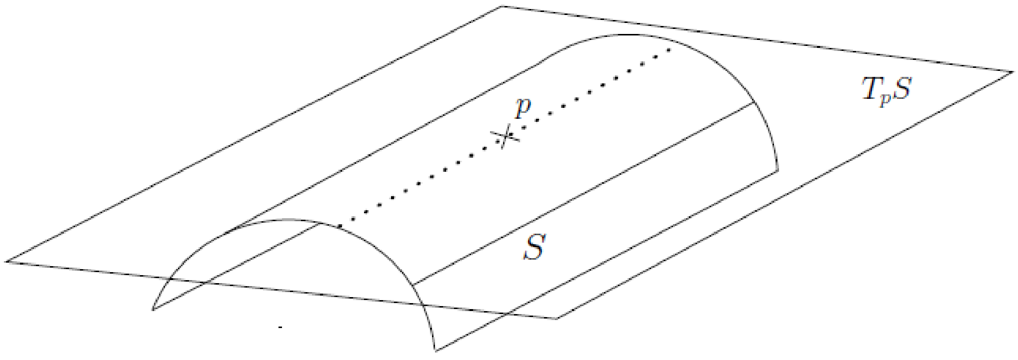
$$b_{11}dy^1dy^1 + 2b_{12}dy^1dy^2 + b_{22}dy^2dy^2 = 0$$



Example of an elliptic point - No real asymptotic line

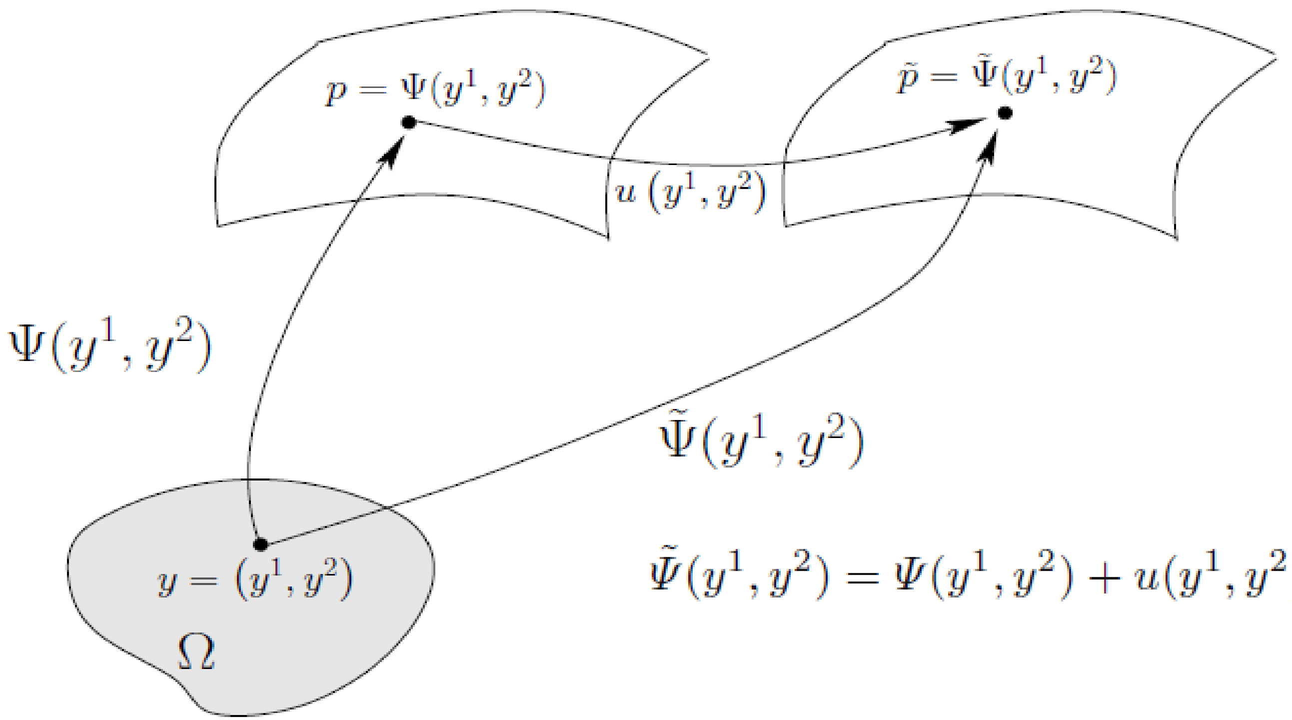


Example of a hyperbolic point - Two families of simple asymptotic lines



Example of a parabolic point - One family of double asymptotic lines

Deformation of a Surface



We can now compute the new metric tensor $\tilde{a}_{\alpha\beta}$ and the new tensor of curvature $\tilde{b}_{\alpha\beta}$ associated with the deformed surface $\tilde{S}$. To characterize the deformation of the surface subjected to the displacement u, we define the tensors of *metric variation* $\gamma_{\alpha\beta}(u)$ and the tensor of *curvature variation* $\rho_{\alpha\beta}(u)$ as follows:

$$\gamma_{\alpha\beta} = \frac{1}{2}(\tilde{a}^*_{\alpha\beta} - a_{\alpha\beta})$$

Because the displacements are supposed to be small, the linearization

$$\rho_{\alpha\beta} = \tilde{b}^*_{\alpha\beta} - b_{\alpha\beta}$$

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}(D_\alpha u_\beta + D_\beta u_\alpha) - b_{\alpha\beta}u_3$$

$$\rho_{\alpha\beta}(u) = -\partial_\alpha\partial_\beta u_3 + \Gamma^\lambda_{\alpha\beta}\partial_\lambda u_3 + b^\lambda_\alpha b_{\lambda\beta}u_3 - D_\alpha\left(b^\lambda_\beta u_\lambda\right) - b^\lambda_\alpha D_\beta u_\lambda$$

with $D_\alpha u_\beta = u_\beta|_\alpha$ $\qquad$ $\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha} = a^\lambda \cdot a_{\alpha,\beta}$

Christoffel symbols associated with the first fundamental form

Theorem (W.T. Koiter, 1959) *Let $u(p)$ be a displacement field applied to a surface S, whose points are defined by $p = \Psi(y^1, y^2)$. If $\gamma_{\alpha\beta}(u) = \rho_{\alpha\beta}(u) = 0 \;\forall\; \alpha, \beta \in \{1, 2\}$, u is then a rigid body displacement.*

The Rigidity System and its Characteristic Curves

The rigidity of a surface is directly linked to the existence of non-trivial displacements which can deform the surface without modifying its metrics called "inextensional displacements" and such inextensional displacements are solution of the the rigidity system:

$$\gamma_{\alpha\beta}(u) = 0$$

$$\begin{cases} D_1 u_1 - b_{11} u_3 = 0 \\ D_2 u_2 - b_{22} u_3 = 0 \\ \frac{1}{2}(D_1 u_2 + D_2 u_1) - b_{12} u_3 = 0 \end{cases}$$

u is a pure bending

covariant derivatives D_α $\boxed{D_\alpha u_\beta = u_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta} u_\lambda}$ $u_{\alpha,\beta} = \partial_\beta u_\alpha$

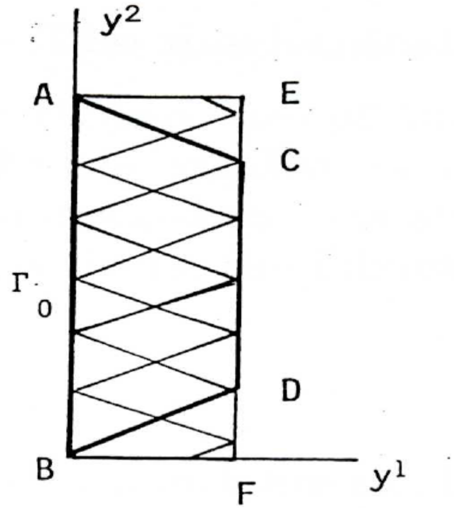
Caution !!

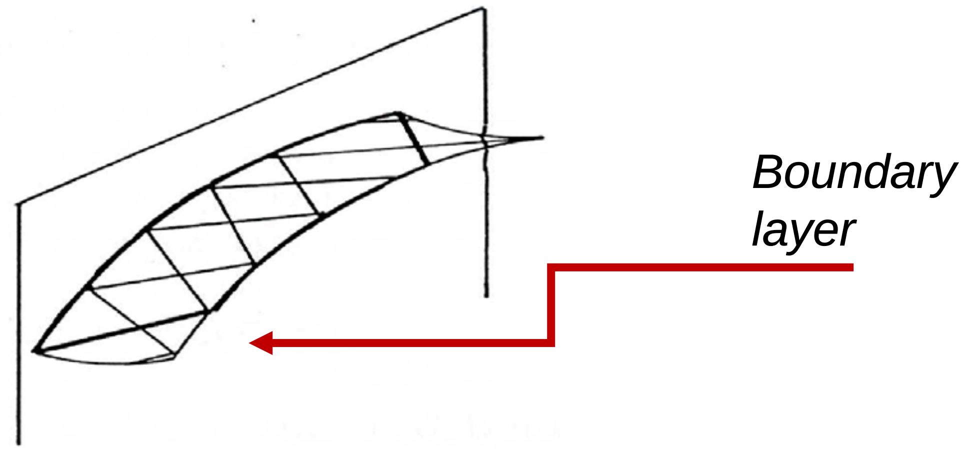
This system, called ***system of rigidity***, is essentially a system of two equations of first order in u_1 u_2 . It is elliptic or hyperbolic at elliptic or hyperbolic points of S. The asymptotic lines of S are the characteristics of this system.

The surface *S,* fixed by a part of its boundary Γ_0, is said to be **rigid** when the above system with the boundary conditions u=0 on Γ_0 has only the solution u = 0

Let us first consider the case of an *elliptic surface.* The local uniqueness theorem for the Cauchy problem implies that u vanishes in the vicinity of Γ0. Moreover, u vanishes identically on *S* provided a unique continuation theorem hold true, for instance if S is analytic. The surface is then *rigid.*

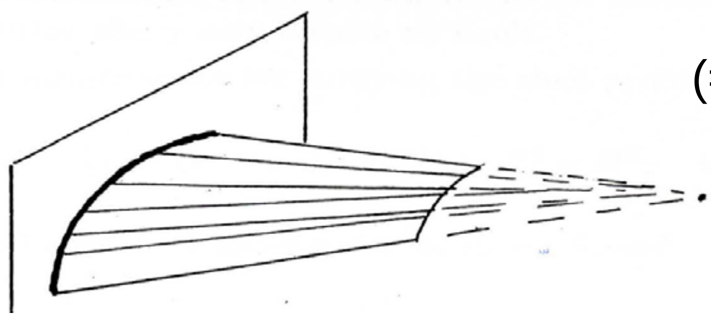
In the case when the surface is hyperbolic, uniqueness for the Cauchy problem only holds true in the *determinacy domain* bounded by the convergent characteristic issued form the extremities of Γ0





Hyperbolic surface. The region spanned by two characteristics issued from the fixation is rigid

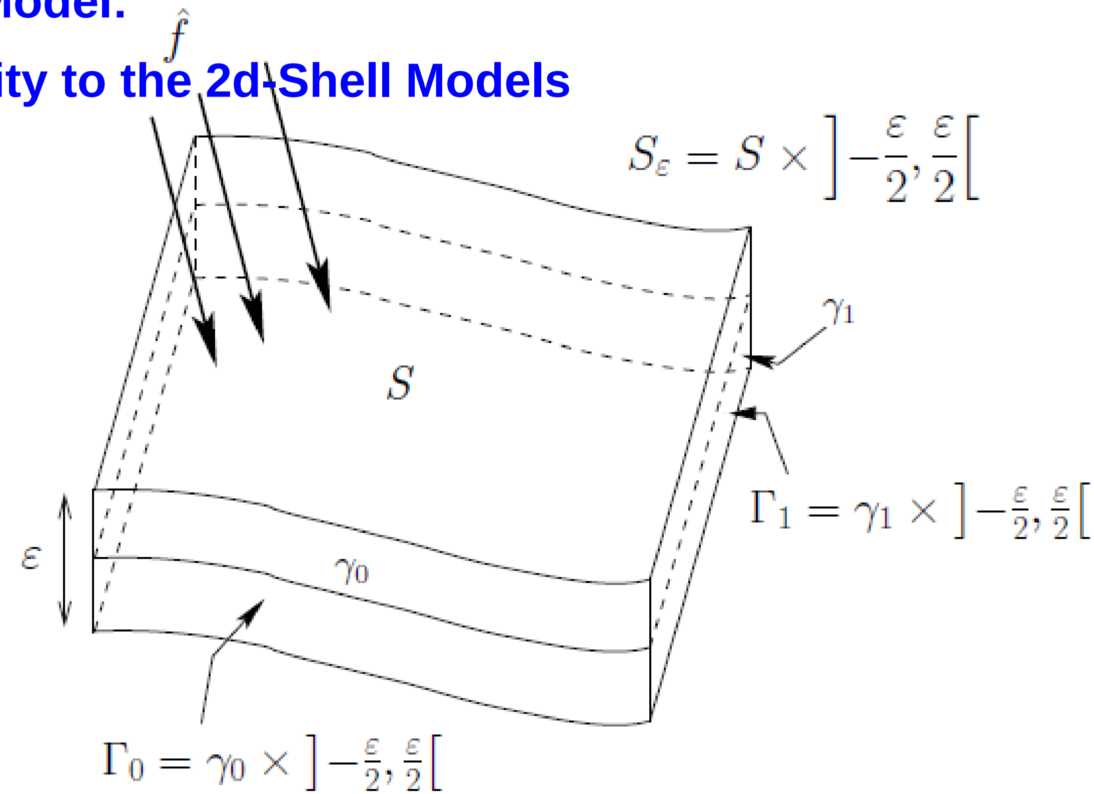
PARABOLIC SURFACES
(=DEVELOPPABLE SURFACES)



Portion of a cone. The entire zone spanned by the generators issued from the fixation is rigid

The general Koiter Shell Model:

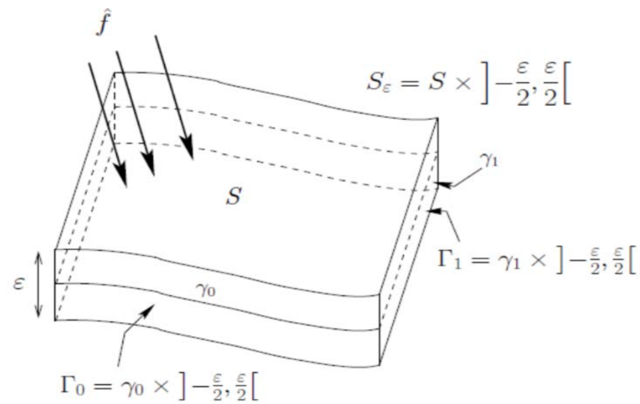
From the linear 3d-elasticity to the 2d-Shell Models



Let us consider a shell having a middle surface S and a thickness ε. It occupies the domain

$$S_\varepsilon = S \times \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[$$

of $\mathbb{R}^3$ in its initial configuration. It is made up of an isotropic linear elastic material. The aim of the mechanical problem is to find the displacement field u^ε of the shell subjected to a loading $\hat{f}$ and satisfying the boundary conditions. Obviously, the solution u^ε of the Koiter model depends on the thickness ε of the shell. In all the problems considered, we suppose that the forces are small enough to stay in the framework of the linear elasticity.



The two-dimensional Koiter model is established from the linear elastic three-dimensional problem due to several hypothesis:

- the Reissner–Mindlin hypothesis: the normal strain vanishes in the direction normal to the shell (there is no thickness variation during the deformation).
- the compression stress in the direction normal to the shell vanishes.

These two hypothesis led to a Nagdhi type model which includes transversal shear. If the shell is thin, transversal shear can be neglected, which constitutes the Kirchhoff–Love kinematical hypothesis:

- a point situated on a normal (at a point p) to the initial middle surface remains on the normal to the deformed surface.

These three hypothesis allow us to establish the two-dimensional model of Koiter from the three-dimensional equations of the linear elasticity and after an integration over the thickness.

$$Find\ u^{\varepsilon} \in V,\ such\ that,\ \forall\ v \in V:$$
$$\varepsilon \int_S A^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}(u^{\varepsilon}) \gamma_{\alpha\beta}(v) dS + \frac{\varepsilon^3}{12} \int_S A^{\alpha\beta\lambda\mu} \rho_{\lambda\mu}(u^{\varepsilon}) \rho_{\alpha\beta}(v) dS = \int_S \hat{f}^i v_i dS$$

with $V = \Big\{ v = (v_1, v_2, v_3) \in H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$; *v satisfying the kinematic boundary conditions* $\Big\}$.

The coefficients $A^{\alpha\beta\lambda\mu}$ represent the coefficients of the linear elastic isotropic constitutive law. They are fourth-order tensors. Their expressions are given by

$$A^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \left[a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\mu} \right]$$

where ν and E are, respectively, the Poisson's ratio and the Young's modulus.

These coefficients allow to link the membrane stress tensor to the membrane strain tensor, and the bending moments tensor to the curvature variation tensor with the following relations:

$$T^{\alpha\beta}(u^{\varepsilon}) = A^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}(u^{\varepsilon})$$

$$M^{\alpha\beta}(u^{\varepsilon}) = \frac{1}{12} A^{\alpha\beta\lambda\mu} \rho_{\lambda\mu}(u^{\varepsilon})$$

Why bending moments ??

3D-Linearized Elasticity System

$\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^3$ Deformation

$\boldsymbol{u} := \boldsymbol{\Phi} - \mathbf{id},$ Displacement

$\tilde{\Omega} := \boldsymbol{\Phi}(\Omega)$

Stress principle of Euler and Cauchy,

$$
\begin{aligned}
-\mathbf{div}\,\tilde{\boldsymbol{T}}(\tilde{x}) &= \tilde{\boldsymbol{f}}(\tilde{x}) \quad \text{for all} \quad \tilde{x} \in \tilde{\Omega}, \\
\tilde{\boldsymbol{T}}(\tilde{x})\tilde{\boldsymbol{n}}(\tilde{x}) &= \tilde{\boldsymbol{g}}(\tilde{x}) \quad \text{for all} \quad \tilde{x} \in \tilde{\Gamma}_1, \\
\tilde{\boldsymbol{T}}(\tilde{x}) &\in \mathbb{S}^3 \quad \text{for all} \quad \tilde{x} \in \tilde{\Omega}.
\end{aligned}
$$

equations of equilibrium in the deformed configuration:

$$\boldsymbol{T}(x) := \tilde{\boldsymbol{T}}(\boldsymbol{\Phi}(x))\mathbf{Cof}\,\boldsymbol{\nabla\Phi}(x) \quad \text{for all } x \in \overline{\Omega},$$

first Piola-Kirchhoff stress tensor field.

$$-\mathbf{div}\,\boldsymbol{T}(x) = \boldsymbol{f}(x) \quad \text{for all} \quad x \in \Omega.$$

$\boldsymbol{\Sigma}(x) := \boldsymbol{\nabla\Phi}(x)^{-1}\boldsymbol{T}(x)$ **second Piola-Kirchhoff stress tensor field** $\boldsymbol{\Sigma}$

symmetric

$$
\begin{aligned}
-\mathbf{div}\,(\boldsymbol{\nabla\Phi}(x)\boldsymbol{\Sigma}(x)) &= \boldsymbol{f}(x) \quad \text{for all} \quad x \in \Omega, \\
(\boldsymbol{\nabla\Phi}(x)\boldsymbol{\Sigma}(x))\boldsymbol{n}(x) &= \boldsymbol{g}(x) \quad \text{for all} \quad x \in \Gamma_1,
\end{aligned}
$$

Constitutive equations of elastic materials

$$C(x) = I + 2E(x), \qquad E(x) := \frac{1}{2}(\nabla u^T(x) + \nabla u(x) + \nabla u^T(x)\nabla u(x))$$

Green-St Venant strain tensor

$$\Sigma(x) = \lambda(\operatorname{tr} E(x))I + 2\mu E(x) \text{ for all } x \in \overline{\Omega},$$

Lamé constants

$$E(x) = \frac{1}{2\mu}\Sigma(x) - \frac{\nu}{E}(\operatorname{tr}\Sigma(x))I \text{ for all } x \in \overline{\Omega}.$$

Poisson coefficient ν and *Young modulus* E

equations of linearized three-dimensional elasticity

$$E(x) = \frac{1}{2}(\nabla\Phi^T(x)\nabla\Phi(x) - I) = \frac{1}{2}(\nabla u^T(x) + \nabla u(x)) + o_x(|\nabla u(x)|),$$

$$\begin{aligned} T(x) &= \nabla\Phi(x)\Sigma(x) = (I + \nabla u(x))\Big(\lambda(\operatorname{tr} E(x))I + 2\mu E(x)\Big) \\ &= \frac{\lambda}{2}\operatorname{tr}(\nabla u^T(x) + \nabla u(x)) + \mu(\nabla u^T(x) + \nabla u(x)) + o_x(|\nabla u(x)|). \end{aligned}$$

$$\begin{aligned} -\operatorname{div}\sigma(x) &= f(x), & x &\in \Omega, \\ u(x) &= 0, & x &\in \Gamma_0, \\ \sigma(x)n(x) &= g(x), & x &\in \Gamma_1, \end{aligned}$$

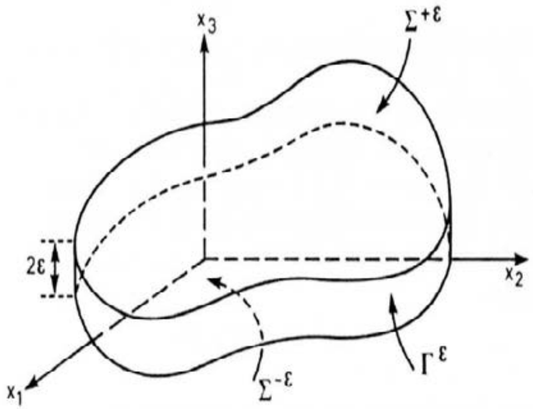
$$\sigma(x) = \lambda(\operatorname{tr} e(x))I + 2\mu e(x) \text{ and } e(x) = \frac{1}{2}(\nabla u^T(x) + \nabla u(x))$$

This linear system has a unique solution in appropriate function spaces !!!

Main idea of the passing to the 2d-problema by means of asymptotic analysis:

$$u^{\varepsilon}(x) = u_3^0(z)\,e_3 + \varepsilon u^1(z,y) + \varepsilon^2 u^2(z,y) + \dots$$

$$\left.\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \end{aligned}\right| \quad \left. y_3 = \frac{x_3}{\varepsilon} \right| \qquad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial z_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}$$



$$\Omega^{\varepsilon} = \{x \;;\; (x_1, x_2) \in \omega \;,\; -\varepsilon < x_3 < \varepsilon\}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j}(u^{\varepsilon}) = 0 \qquad \text{dans } \Omega^{\varepsilon}$$

$$\sigma_{ij}(u^{\varepsilon}) = a_{ijkl}\left(\frac{x_3}{\varepsilon}\right) e_{kl}(u^{\varepsilon})$$

$$\sigma_{ij}(u^{\varepsilon})\,n_j \equiv \pm\sigma_{i3}(u^{\varepsilon}) = 0 \qquad \text{sur } \Sigma^{\pm\varepsilon}$$

$$(u^{\varepsilon}) = \psi(x_1, x_2)\,e_3 \qquad \text{sur } \Gamma^{\varepsilon}$$

$$e_{ij}(u^k) = \frac{1}{2}\left(\frac{\partial u_i^k}{\partial x_j} + \frac{\partial u_j^k}{\partial x_i}\right) = e_{ijz}(u^k) + \frac{1}{\varepsilon} e_{ijy}(u^k)$$

$$e_{ij}(u^\varepsilon) = e_{ij}^0 + \varepsilon e_{ij}^1 + \ldots$$

$$e_{ij}^k = e_{ijz}(u^k) + e_{ijy}(u^{k+1})$$

$$e_{\alpha\beta}^k = e_{\alpha\beta z}(u^k)$$

$$e_{\alpha 3}^k = \frac{1}{2}\frac{\partial u_3^k}{\partial z_\alpha} + \frac{1}{2}\frac{\partial u_\alpha^{k+1}}{\partial y_3}$$

$$e_{33}^k = \frac{\partial u_3^{k+1}}{\partial y_3}$$

$$\sigma_{ij}(u^\varepsilon) = \sigma_{ij}^0 + \varepsilon \sigma_{ij}^1 + \ldots$$

$$\sigma_{ij}^l = a_{ijkh} e_{kh}^l$$

3d-Variational Formulation

$$\mathcal{V}^\varepsilon = \{ v \in H^1(\Omega^\varepsilon) \quad ; \quad v|_{\Gamma^\varepsilon} = 0 \}$$

$$\mathcal{U}^\varepsilon_{\mathrm{adm}} = \{ v \in H^1(\Omega^\varepsilon) \quad ; \quad v|_{\Gamma^\varepsilon} = \psi(x_1, x_2)\, e_3 \}$$

$$u^\varepsilon \in \mathcal{U}^\varepsilon_{\mathrm{adm}} \qquad \int_{\Omega^\varepsilon} a_{ijkl}(y_3)\, e_{kl}(u^\varepsilon)\, e_{ij}(v)\, \mathrm{d}x = 0 \qquad \forall v \in \mathcal{V}^\varepsilon$$

Take as test function

$$v = \theta(z)\, w(y) \qquad \theta \in \mathcal{D}(\omega),\ w \in H^1(-1, 1)$$

Then

$$e_{ij}(v) = \frac{1}{\varepsilon} \theta(z)\, e_{ijy}(w) + 0(1)$$

$$0 = \int_{\Omega^\varepsilon} (\sigma^0_{ij} + \varepsilon \sigma^1_{ij} + \ldots) \left[\frac{1}{\varepsilon} \theta(z)\, e_{ijy}(w) + 0(1) \right] \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

$$x_3 = \varepsilon\, y_3 \qquad 0 = \int_\omega \theta(z)\, \mathrm{d}z_1 \mathrm{d}z_2 \int_{-1}^{1} \sigma^0_{ij} e_{ijy}(w)\, \mathrm{d}y_3 \qquad \int_{-1}^{1} \sigma^0_{ij} e_{ijy}(w)\, \mathrm{d}y_3 = 0$$

$$\int_{-1}^{1} a_{ijkl}(y_3)\,[e_{klz}(\boldsymbol{u}^0) + e_{kly}(\boldsymbol{u}^1)]\,e_{ijy}(\boldsymbol{w}(y))\,\mathrm{d}y_3 = 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{H}^1(-1,1)$$

$$E^1_{3\alpha} = E^1_{\alpha 3} = \frac{1}{2}\frac{\partial u^0_3}{\partial z_\alpha}$$

$$\boldsymbol{u}^1 = -\frac{\partial u^0_3}{\partial z_\alpha} y_3 \boldsymbol{e}_\alpha$$

$$\boldsymbol{u}^1 = -\frac{\partial u^0_3}{\partial z_\alpha} y_3 \boldsymbol{e}_\alpha + \hat{\boldsymbol{u}}^1(z)$$

This term justifies the presence of Momentum terms

Coming back to the Shell variational formulation

(already said)

Find $u^\varepsilon \in V$, *such that*, $\forall\ v \in V$:

$$\varepsilon \int_S A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u^\varepsilon)\gamma_{\alpha\beta}(v)dS + \frac{\varepsilon^3}{12}\int_S A^{\alpha\beta\lambda\mu}\rho_{\lambda\mu}(u^\varepsilon)\rho_{\alpha\beta}(v)dS = \int_S \hat{f}^i v_i dS$$

with $V = \Big\{ v = (v_1, v_2, v_3) \in H^1(\Omega)\times H^1(\Omega)\times H^2(\Omega)$; *v satisfying the kinematic boundary conditions* $\Big\}$.

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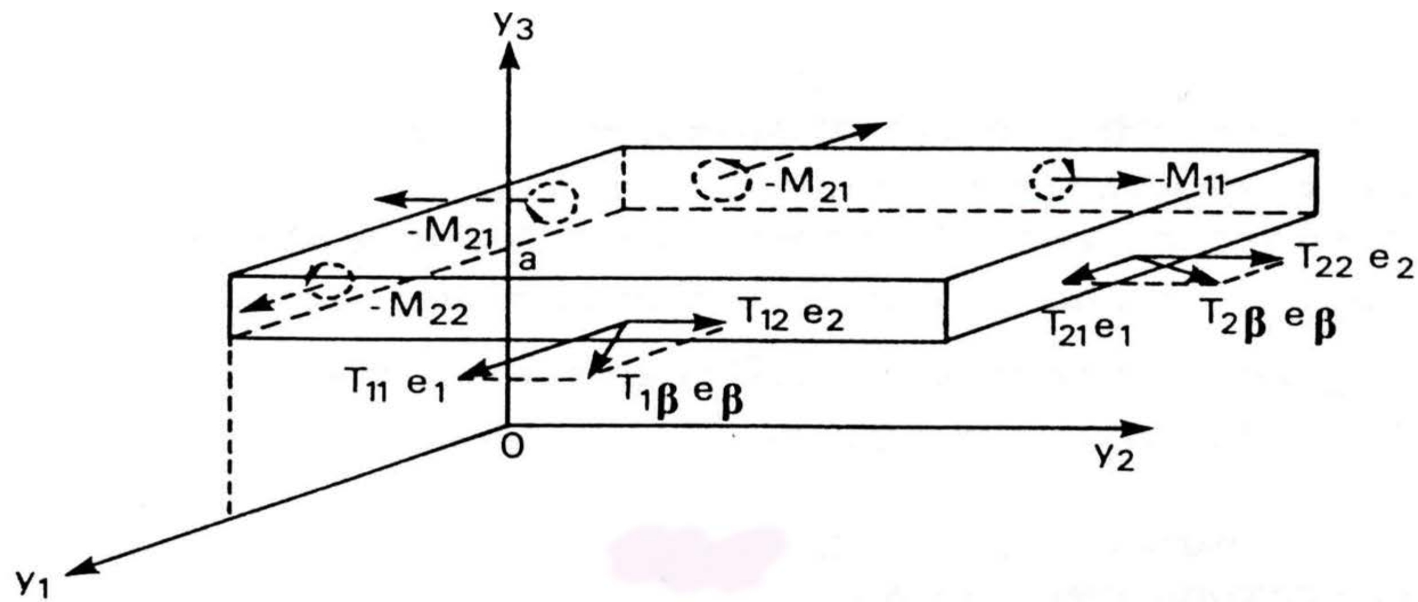
$$A^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left[a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right]$$

where ν and E are, respectively, the Poisson's ratio and the Young's modulus.

These coefficients allow to link the membrane stress tensor to the membrane strain tensor, and the bending moments tensor to the curvature variation tensor with the following relations:

$$T^{\alpha\beta}(u^\varepsilon) = A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u^\varepsilon)$$

$$M^{\alpha\beta}(u^\varepsilon) = \frac{1}{12}\,A^{\alpha\beta\lambda\mu}\rho_{\lambda\mu}(u^\varepsilon)$$



They satisfy the usual conditions of symmetry

$$A^{\alpha\beta\lambda\mu} = A^{\beta\alpha\lambda\mu} = A^{\lambda\mu\alpha\beta}$$

and positivity

$$\exists C > 0 \quad \textit{such that} \quad A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}\gamma_{\alpha\beta} \geq C \sum \gamma_{\alpha\beta}^2$$

We can also define the compliance coefficients $B_{\alpha\beta\lambda\mu}$ which satisfy the inverse relations:

$$\gamma_{\alpha\beta} = B_{\alpha\beta\lambda\mu}T^{\lambda\mu}$$

$$\rho_{\alpha\beta} = 12 B_{\alpha\beta\lambda\mu}M^{\lambda\mu}$$

The left-hand side of the above variational formulation comprises two parts. The first one is the bilinear form of membrane energy, proportional to the thickness ε, whereas the second one, the bilinear form of bending energy, proportional to the cube of the thickness ε^3. The bending rigidity is consequently much weaker than the membrane one when ε is small.

To address the asymptotic process of the Koiter model when $\varepsilon \searrow 0$, we rewrite it under the following form $\mathcal{P}(\varepsilon)$:

$$\mathcal{P}(\varepsilon)\begin{cases} \textit{Find } u^\varepsilon \textit{ in } V \textit{ such that} \\ \\ a_m(u^\varepsilon, v) + \varepsilon^2 a_b(u^\varepsilon, v) = \langle f, v\rangle \quad \forall v \in V \end{cases}$$

where

$$a_m(u^\varepsilon, v) = \int_S A^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u^\varepsilon)\gamma_{\alpha\beta}(v)dS$$

denotes the bilinear form of membrane energy and

$$a_b(u^\varepsilon, v) = \frac{1}{12}\int_S A^{\alpha\beta\lambda\mu}\rho_{\lambda\mu}(u^\varepsilon)\rho_{\alpha\beta}(v)dS$$

the bilinear form of bending energy. Moreover, we set

$$\langle f, v\rangle = \int_S f^i v_i dS$$

with

$$\hat{f} = \varepsilon f$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between V' and V.

For ε>0, the existence and uniqueness of the solution of P(ε) is showed by applying the **Lax–Milgram theorem**
(Bernadou, M., Ciarlet, P.G.: Sur l'ellipcité du modele lineaire des coques de W.T. Koiter. In: Glowinski, R., Lions, J.-L. (eds.) Computing Methods in Sciences and Engineering. Lectures notes in Economics and Math. Systems, vol. 134, pp. 89–136. Springer, Heidelberg (1976)

This theorem relies on the continuity and the coerciveness of the left hand side of bilinear forms. The coerciveness of this bilinear form was proved in the above reference by using **Korn's inequalities** on a surface.

The problem $\mathcal{P}(\varepsilon)$ can be written under the following local form

$$\begin{cases} -D_\alpha T^{\alpha\beta} - \varepsilon^2 \left[b_\gamma^\beta D_\alpha M^{\alpha\gamma} + D_\gamma (b_\alpha^\gamma M^{\alpha\beta}) \right] = f^\beta \quad for \quad \beta = 1, 2 \\ -b_{\alpha\beta} T^{\alpha\beta} + \varepsilon^2 \left[D_\alpha D_\beta M^{\alpha\beta} - b_\delta^\alpha b_{\beta\delta} M^{\alpha\beta} \right] = f^3 \end{cases}$$

An interesting application: Rigidity of narrow layers (of a developable surface, along a generator)

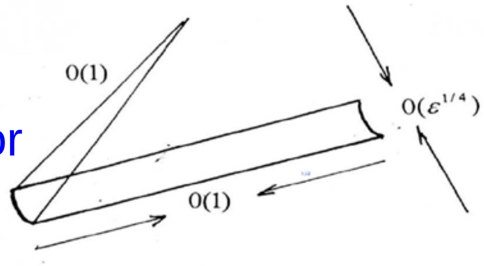
¿Motivation?

In the previous scheme:

-The limit solution holds true whatever the value of c.

-The solution with small $\varepsilon > 0$ holds true (up to exponentially small terms) for c large with respect to $\varepsilon^{1/4}$. Even better: after appropriate modification of the layer on account of the boundary conditions, it holds true for $c = 0(\varepsilon^{1/4})$.

As a consequence, this kind of solutions hold true for very narrow pieces along a generator.

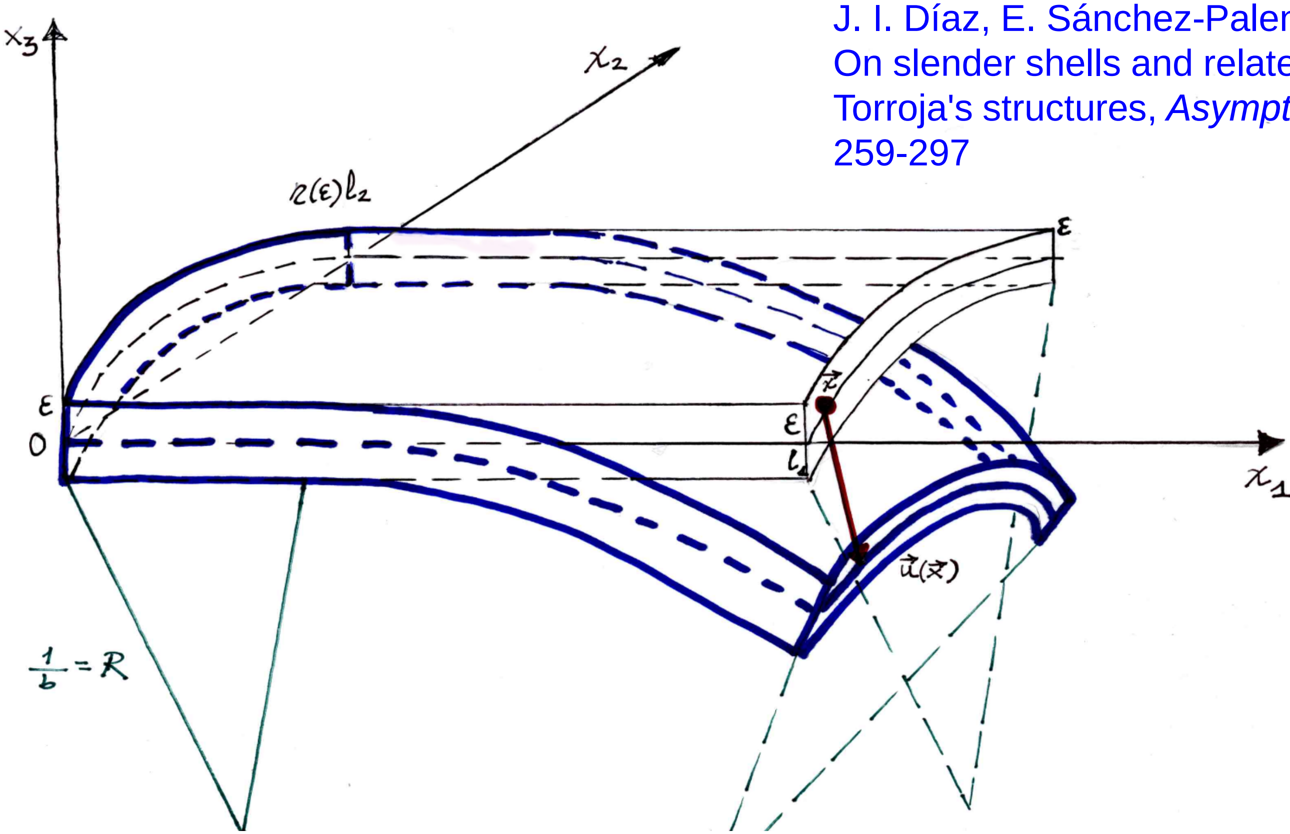


Thickness 0(ε)

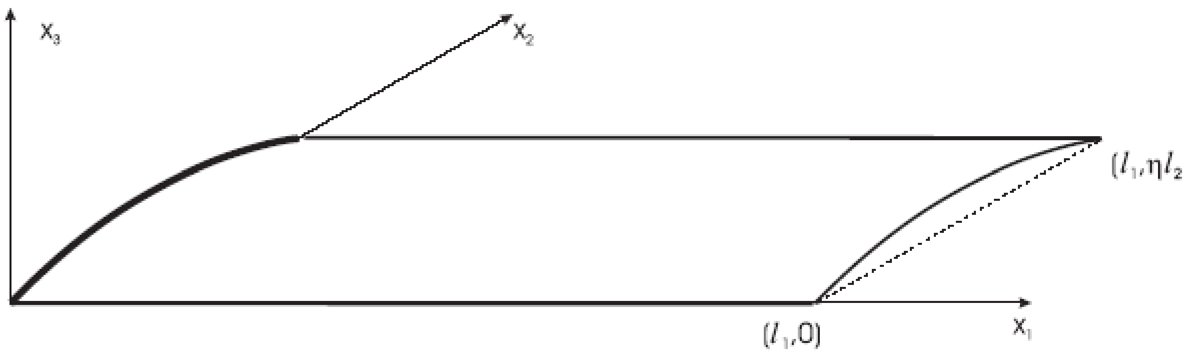
3. The basic problem.

We consider a slender cylindrical shell

J. I. Díaz, E. Sánchez-Palencia,
On slender shells and related problems suggested by Torroja's structures, *Asymptotic Analysis*, 52, 2007, 259-297



According to standard notations in cylindrical shell theory the "plane of parameters x_1, x_2" is merely the middle surface (cylinder) of the shell developed into a plane. We chose x_1 in the direction of the generators and x_2 normal to them, so that the principal curvatures are zero in the direction x_1 and $b = 1/R$ in the direction x_2, where R denotes the radius of the cross section



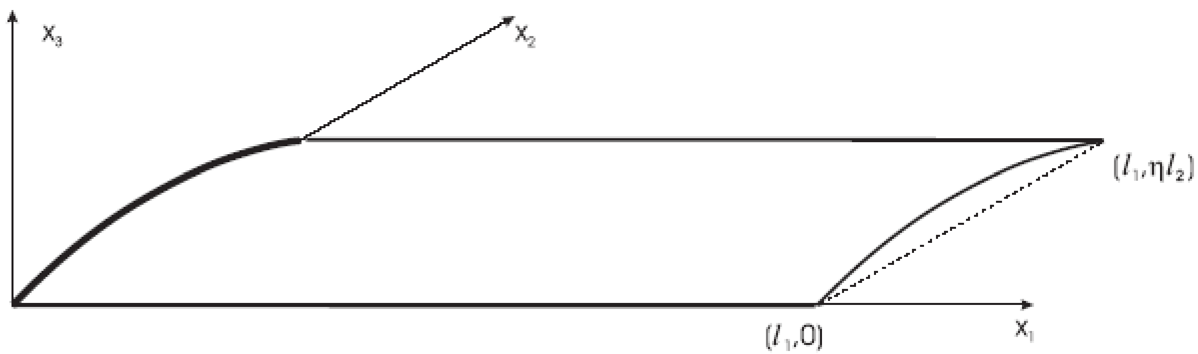
Accordingly, the second fundamental form of the surface has components $b_{11} = b_{12} = 0$ and $b_{22} = b$, Moreover, the Christoffel symbols of the surface vanish identically, so that covariant and classical differentiation coincide. Since $b_{12}^2 - b_{11}b_{22} = 0$ the surface is parabolic, i.e. the directions of the principal curvatures coincide

Remark *As a matter of fact, the Torroja's structure mentioned at the introduction was not composed by cylindrical elements but by slightly hyperbolic ones. Nevertheless, the curvature in the longitudinal direction was much smaller (and even it vanished in early projects by Torroja: see [36] Chapter 1) than in the transversal direction, so that our model with zero longitudinal curvature may be considered as a first approximation.*

Let ε be a small parameter, the relative thickness of the plate. Let $\eta = \eta(\varepsilon)$ be a new small parameter satisfying

$$\varepsilon^{1/3} \leq \eta \leq 1.$$

the typical example will be $\eta = \varepsilon^{1/4}$, Let us denote the shell domain by $\Omega_\varepsilon = (0, l_1) \times (0, \eta l_2)$, with $\eta l_2 \leq 2R$.



The corresponding tangential displacements are $\tilde{u}_1, \tilde{u}_2$, whereas $\tilde{u}_3$ is the displacement normal to the shell. Some times we shall use the notation $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}^\varepsilon$ to indicate explicitly the ε-dependence. We shall admit, in this section, that the shell is clamped by the "small curved boundary" $(\{0\} \times [0, \eta l_2]$ and free by the rest This implies the kinematic boundary conditions:

$$0 = \tilde{u}_1 = \tilde{u}_2 = \tilde{u}_3 = \tilde{\partial}_1 \tilde{u}_3 \text{ on } \{0\} \times [0, \eta l_2],$$

where

$$\tilde{\partial}_\alpha = \frac{\partial}{\partial x_\alpha}.$$

The space of configuration will be denoted by V_ε. It is the subspace of $H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \times H^2(\Omega_\varepsilon)$ formed by the functions satisfying the kinematic boundary conditions

Remark *In the special case when the curvature b vanishes, there is uncoupling between the membrane and flection problems; the "normal" loading only produces flection. Moreover, as the width of the shell (the plate, in that case) is* $0(\eta)$*, the global rigidity is* $0(\eta\varepsilon^3)$*, of the same order as the total applied force so that, in that case, the solutions (*u_3^ε *in fact) have a non zero limit.*

We shall see that in our case (i. e., with non zero b) the displacements are very small and only converge to a non zero limit after an appropriate scaling. This amounts to a very high rigidity produced by the curvature as commented at the Introduction.

Problem P_ε. *Find* $\tilde{\mathbf{u}}^\varepsilon \in \mathbf{V}_\varepsilon$ *satisfying* $\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \tilde{\mathbf{v}} \in \mathbf{V}_\epsilon.$

Remark *Since the bilinear forms* $a(\mathbf{u}, \mathbf{v})$ *and* $b(\mathbf{u}, \mathbf{v})$ *are symmetric, from well known results we deduce that, in fact,* $\tilde{\mathbf{u}}^\varepsilon$ *is the unique solution of the minimization problem*

$$Min_{\mathbf{V}} \widetilde{J}_\varepsilon(\mathbf{v})$$

where

$$\widetilde{J}_\varepsilon(\mathbf{v}) = \frac{\varepsilon}{2} a(\mathbf{v}, \mathbf{v}) + \frac{\varepsilon^3}{2} b(\mathbf{v}, \mathbf{v}) - \langle \mathbf{f}, \mathbf{v} \rangle.$$

The objective of the rest of the section is to study its asymptotic behavior as $\varepsilon \downarrow 0$.

Scaling and a priori estimates in the basic problem.

Let us perform the change of variables :

$$\begin{cases} \mathbf{x} = (x_1, x_2) \Rightarrow \mathbf{y} = (y_1, y_2), \\ y_1 = x_1, \qquad y_2 = \eta^{-1} x_2 \end{cases}$$

so, the domain Ω_ε is transformed into Ω and

$$\partial_1 = \tilde{\partial}_1, \qquad \partial_2 = \eta \tilde{\partial}_2; \qquad \partial_\alpha = \frac{\partial}{\partial y_\alpha}.$$

Moreover, we shall perform the change of unknowns

$$\begin{cases} \tilde{u}_1(\mathbf{x}) = \eta^\theta u_1(\mathbf{y}), \\ \tilde{u}_2(\mathbf{x}) = \eta^{\theta-1} u_2(\mathbf{y}), \\ \tilde{u}_3(\mathbf{x}) = \eta^{\theta-2} b^{-1} u_3(\mathbf{y}), \end{cases}$$

As θ is not defined, the total level of the scaling is not specified, only the mutual ratios of dilatation of the three components are fixed. They are chosen in analogy with layers in parabolic shells. Specifically, the ratio between the components 1 and 2 is fixed in order that the new form of the shear membrane strain $\tilde{e}_{12}$ be formed by two terms of the same order (which, on the other hand, are asymptotically large, forming a constraint for the limit problem). The ratio between the components 2 and 3 is also fixed in such a way that the new form of the membrane strain $\tilde{e}_{22}$ be formed by two terms of the same order.

We then perform the previous change for $\tilde{\mathbf{u}}^\varepsilon$ as well as for $\tilde{\mathbf{v}}$ in P_ε and we have

$$\tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \eta^\theta \partial_1 v_1$$

$$\tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta-1}\frac{1}{2}(\partial_2 v_1 + \partial_1 v_2),$$
$$\tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta-2}(\partial_2 v_2 + v_3),$$
$$\tilde{\rho}_{11}(\tilde{\mathbf{v}}) = \eta^{\theta-2} b^{-1}\partial_1^2 v_3,$$
$$\tilde{\rho}_{12}(\tilde{\mathbf{v}}) = \tilde{\rho}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta-3} b^{-1}\partial_1\partial_2 v_3, \qquad \tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta-4} b^{-1}\partial_2^2 v_3.$$

It will prove useful to define

$$\gamma_{11}^\varepsilon(\mathbf{v}) = \partial_1 v_1$$
$$\gamma_{12}^\varepsilon(\mathbf{v}) = \gamma_{21}^\varepsilon(\mathbf{v}) = \eta^{-1}\frac{1}{2}(\partial_2 v_1 + \partial_1 v_2),$$
$$\gamma_{22}^\varepsilon(\mathbf{v}) = \eta^{-2}(\partial_2 v_2 + v_3);$$
$$\rho_{11}^\varepsilon(\mathbf{v}) = \eta^2\partial_1^2 v_3,$$
$$\rho_{12}^\varepsilon(\mathbf{v}) = \rho_{21}(\mathbf{v}) = \eta\partial_1\partial_2 v_3,$$
$$\rho_{22}^\varepsilon(\tilde{\mathbf{v}}) = \partial_2^2 v_3.$$

so that:

$$\tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \eta^\theta\gamma_{11}^\varepsilon(v)$$
$$\tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \eta^\theta\gamma_{12}^\varepsilon(\mathbf{v})$$
$$\tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \eta^\theta\gamma_{22}^\varepsilon(\mathbf{v})$$
$$\tilde{\rho}_{11}(\tilde{\mathbf{v}}) = \eta^{\theta-4} b^{-1}\rho_{11}^\varepsilon(\mathbf{v})$$
$$\tilde{\rho}_{12}(\tilde{\mathbf{v}}) = \tilde{\rho}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta-4} b^{-1}\rho_{12}^\varepsilon(\mathbf{v}) \qquad \tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta-4} b^{-1}\rho_2^\varepsilon(\mathbf{v}).$$

We recall that the spatial domain is now $\Omega = (0, l_1) \times (0, l_2)$. The space of configuration, after scaling will be denoted by $\mathbf{V}$. It is the subspace of

$$H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$$

formed by the functions satisfying the kinematic boundary conditions

$$0 = u_1 = u_2 = u_3 \text{ on } \{0\} \times [0, l_2].$$

The expression $\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle$ then becomes:

$$P \int_\Omega A^{\alpha\beta\lambda\mu} \gamma^\varepsilon_{\alpha\beta}(\mathbf{u}_\varepsilon) \gamma^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} + Q \int_\Omega B^{\alpha\beta\lambda\mu} \rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \rho^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} = R \int_\Omega F_3(y_1, y_2) v_3(y_1, y_2) d\mathbf{y},$$

with

$$P = \varepsilon \eta^{2\theta+1}$$

$$Q = \varepsilon^3 \eta^{2\theta-7} b^{-2}$$

we shall determine the $b(\varepsilon)$ and θ as functions of ε and the function $\eta(\varepsilon)$ using the two equations

$$P = Q = R.$$

This gives $b = \varepsilon / \eta^4$ and $\eta^{\theta-2} = \varepsilon$.

b is always small with respect to η^{-1}, and equal to 1 (or rather 0(1)) in the "typical example" $\eta = \varepsilon^{1/4}$ we have $\theta = 6$.

Once θ is determined, the scaling $\tilde{u}_3(\mathbf{x}) = \eta^{\theta-2} b^{-1} u_3(\mathbf{y}),$ is perfectly defined. We then observe that the factor $\eta^{\theta-2} b^{-1}$ takes the form: η^4 which is always small. It means that the scaling of the component u_3^ε is such that, after scaling, it is asymptotically large with respect to the case before scaling. As we shall prove in the sequel, the scaled unknown u_3^ε has a non zero limit; it follows that the initial unknown $\tilde{u}_3^\varepsilon$ tends to 0 at the ratio η^4. We shall come again on this property, which amounts to the rigidification of the plate with respect to the plane case.

Summing up, the problem P_ε becomes after scaling:

Problem Π_ε. *Find* $\mathbf{u}^\varepsilon \in \mathbf{V}$ *satisfying*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) = \int_\Omega F_3(y_1, y_2) v_3(y_1, y_2) d\mathbf{y}.$$

$\forall v \in \mathbf{V}$, *where*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) \stackrel{def}{=} \int_\Omega A^{\alpha\beta\lambda\mu} \gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \gamma^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu} \rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \rho^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y}.$$

It should be emphasized that, by virtue of the definitions the coefficients involve various powers of η, running from -4 to $+4$. The terms in η^{-4} to η^{-1} are "penalty terms", whereas those in η^1 to η^4 are "singular perturbation terms". Only the terms of order 1 will remain in the limit expression.

Remark $\mathbf{u}^\varepsilon$ *is the unique solution of the minimization problem*

Formulation in terms of Calculus of Variations

$$Min_{\mathbf{V}} J_\varepsilon(\mathbf{v})$$

where

$$J_\varepsilon(\mathbf{v}) = \frac{1}{2} a^\varepsilon(\mathbf{v}, \mathbf{v}) - \langle \mathbf{f}, \mathbf{v} \rangle.$$

Let us proceed to the a priori estimates. From the expression of $a^\varepsilon(\mathbf{v},\mathbf{v})$ with $\mathbf{u}^\varepsilon = \mathbf{v}$,

Lemma *The estimates:*

$$\|\partial_1 v_1\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\eta^{-1}\frac{1}{2}(\partial_2 v_1 + \partial_1 v_2)\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\eta^{-2}(\partial_2 v_2 + v_3)\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\partial_2^2 v_3\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\eta\partial_1\partial_2 v_3\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\eta^2\partial_1^2 v_3\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v},\mathbf{v})$$

hold true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

Now, in order to prove that the functional in the right hand side is bounded independently of ε, we need an estimate on u_3 itself.

Lemma *The estimate:*

$$\|v_3\|^2_{L^2((0,\mathrm{l}_1);H^2(0,\mathrm{l}_2))} \le c a^\varepsilon(\mathbf{v},\mathbf{v})$$

holds true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

PROOF. Discarding the factors in η and differentiating we have:

$$\|\partial_2^2 v_1 + \partial_2\partial_1 v_2)\|^2_{L^2((0,l_1);H^{-1}(0,l_2))} \le c a^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\partial_1\partial_2 v_2 + \partial_1 v_3)\|^2_{H^{-1}((0,l_1);L^2(0,l_2))} \le c a^\varepsilon(\mathbf{v},\mathbf{v}).$$

using the fact that v_1 vanishes on $\{0\} \times [0, l_2]$, by using the Poincaré's inequality we obtain:

$$\|v_1\|^2_{H^1((0,l_1);L^2(0,l_2))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v})$$

and differentiating,

$$\|\partial_2^2 v_1\|^2_{H^1((0,l_1);H^{-2}(0,l_2)))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}).$$

taking the weaker norm, it follows that

$$\|\partial_2\partial_1 v_2\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v})$$

$$\|\partial_1 v_3\|^2_{H^{-1}((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v})$$

or even (integrating with respect to y_1 on account of the vanishing of the trace on $\{0\} \times [0, l_2]$):

$$\|v_3\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}).$$

The conclusion follows.

Lemma *The estimate*

$$|\int_\Omega F_3 v_3 d\mathbf{y}| \leq ca^\varepsilon(\mathbf{v}, \mathbf{v})^{1/2}$$

holds true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

Now, taking $\mathbf{v} = \mathbf{u}^\varepsilon$ we get the energy estimate:

Lemma *Let $\mathbf{u}^\varepsilon$ be the solution of Π_ε. The estimates*

$$\|\gamma^\varepsilon_{\alpha\beta}(u^\varepsilon)\| \leq C \qquad \alpha, \beta = 1, 2 \qquad \|\partial_1 u_1^\varepsilon\|^2_{L^2(\Omega)} \leq C$$

$$\|\eta^{-1}\frac{1}{2}(\partial_2 u_1^\varepsilon + \partial_1 u_2^\varepsilon)\|^2_{L^2(\Omega)} \leq C \qquad \|\eta^{-2}(\partial_2 u_2^\varepsilon + u_3^\varepsilon)\|^2_{L^2(\Omega)} \leq C$$

$$\|\partial_2^2 u_3^\varepsilon\|^2_{L^2(\Omega)} \leq C \qquad \|\eta\partial_1\partial_2 u_3^\varepsilon\|^2_{L^2(\Omega)} \leq C \qquad \|\eta^2\partial_1^2 u_3^\varepsilon\|^2_{L^2(\Omega)} \leq C$$

hold true for a certain $C > 0$ independent of ε.

We shall need an estimate on u_2^ε itself. We shall obtain it by differentiating with respect to y_2 and integrating in y_1.

Lemma *Let $\mathbf{u}^\varepsilon$ be the solution of Π_ε. The estimates*

$$\|u_1^\varepsilon\|_{H^1((0,l_1);L^2(0,l_2))} \leq C \qquad \|u_2^\varepsilon\|_{\widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2))} \leq C \qquad \|u_3\|^2_{L^2((0,l_1);H^2(0,l_2))} \leq C,$$

holds true for a certain $C > 0$ independent of ε, where

$$\widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2)) = \{w \in H^1((0,l_1);H^{-1}(0,l_2)) \text{ such that } w(0,\cdot) = 0\}.$$

A first result of convergence is

Lemma *Let $\mathbf{u}^\varepsilon$ be the solution of Π_ε. The following convergences (as $\varepsilon \to 0$) hold true (in the sense of subsequences, the limits being not necessarily unique):*

$$u_1^\varepsilon \to u_1^* \quad \text{weakly in } \widetilde{H}_0^1((0,l_1);L^2(0,l_2)) \qquad u_2^\varepsilon \to u_2^* \quad \text{weakly in } \widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2))$$

$$u_3^\varepsilon \to u_3^* \quad \text{weakly in } L^2((0,l_1);H^2(0,l_2))$$

where $\mathbf{u}^* = (u_1^*, u_2^*, u_3^*)$ are distributions on Ω, belonging to the spaces specified

$$\partial_2 u_1^* + \partial_1 u_2^* = 0$$

$$\partial_2 u_2^* + u_3^* = 0.$$

Finally,

$$\gamma^{\varepsilon}_{\alpha\beta}(\mathbf{u}^{\varepsilon}) \to \gamma^{*}_{\alpha\beta} \qquad \textit{weakly in } L^2(\Omega), \ \ \alpha,\beta = 1,2,$$

for some $\gamma^{*}_{\alpha\beta} \in L^2(\Omega)$.

Limit problem and convergence in the basic problem.

Let us define the space $\mathbf{G}$ for the definition of the limit problem:

$$\mathbf{G} = \{\mathbf{v} = (v_1, v_2, v_3) \in \widetilde{H}^1_0((0,l_1); L^2(0,l_2)) \times \widetilde{H}^1_0((0,l_1); H^{-1}(0,l_2)) \times L^2((0,l_1); H^2(0,l_2)),$$

$$\partial_2 v_1 + \partial_1 v_2 = 0, \quad \partial_2 v_2 + b v_3 = 0\},$$

where we observe that v_1 defines completely v_2 and then v_3.

Clearly, $\mathbf{G}$ is a Hilbert space with the norm

$$\begin{cases} \|\mathbf{v}\|^2_{\mathbf{G}} = \|v_1\|^2_{\widetilde{H}^1_0((0,l_1);L^2(0,l_2))} + \|\partial_2^2 v_3\|^2_{L^2(\Omega)} \\ \simeq \|\partial_1 v_1\|^2_{L^2(\Omega)} + \|\partial_2^3 v_2\|^2_{L^2(\Omega)} \end{cases}$$

Remark *A straightforward comparison with the space* $\mathbf{V}$ *shows that the space* $\mathbf{G}$ *for the limit problem incorporates the two constraints corresponding to the "penalty terms" in* Π_ε *whereas the boundary conditions for* u_3*, which are concerned with the "singular perturbation terms" in* Π_ε *are lost.*

It is worthwhile to state an equivalent definition of the space $\mathbf{G}$ where the functions are defined in terms of a scalar "potential ψ":

Lemma *The space* $\mathbf{G}$ *may equivalently be defined as the space of the triplets* $\mathbf{v} = (v_1, v_2, v_3)$ *such that:*

$$v_1 = \partial_1 \psi, \qquad v_2 = -\partial_2 \psi, \qquad v_3 = -\partial_2^2 \psi.$$

where ψ *is an element of*

$$\tilde{G} = \tilde{H}_0^2((0, l_1); L^2(0, l_2)) \cap L^2((0, l_1); H^4(0, l_2))$$

where

$$\tilde{H}_0^2((0, l_1); L^2(0, l_2)) = \{\psi \in H^2((0, l_1); L^2(0, l_2)); \psi(0, y_2) = \partial_1 \psi(0, y_2) = 0\}.$$

Remark The introduction of the scalar potential φ seems to be new in the shell literature. Some closed, but different, ideas can be associated with the *stress function* introduced by G.B. Airy (1801-1892)

It should prove useful to prove a lemma on density in $\mathbf{G}$.

Lemma *The subspace of* $\mathbf{G}$ *formed by the elements* $\mathbf{v} = (v_1, v_2, v_3)$ *which are smooth, vanish in a neighborhood of* $\{0\} \times [0, l_2]$ *and derive from a "potential"* ψ *is dense in* $\mathbf{G}$.

We are now defining the limit problem. It involves the numerical coefficient $1/C_{1111}$, and B^{2222} where $C_{\alpha\beta\lambda\mu}$ is the matrix inverse of $A^{\alpha\beta\lambda\mu}$, i. e. the matrix of membrane compliances, and $\mathbf{B}$ is the matrix of flection rigidities. They are both strictly positive.

$$\tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{u}}) = C_{\lambda\mu\alpha\beta} \tilde{T}^{\beta\alpha}(\tilde{\mathbf{u}})$$

Problem Π_0. *Find $\mathbf{u} \in \mathbf{G}$ such that*

$$\int_\Omega \frac{1}{C_{1111}} \partial_1 u_1 \partial_1 v_1 d\mathbf{y} + \int_\Omega B^{2222} \partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y} = \int_\Omega F_3 v_3 d\mathbf{y}.$$

$\forall \mathbf{v} \in \mathbf{G}$, *or equivalently, in terms of the potential, find $\varphi \in \widetilde{G}$ such that*

$$\int_\Omega \frac{1}{C_{1111}} \partial_1^2 \varphi \partial_1^2 \psi d\mathbf{y} + \int_\Omega B^{2222} \partial_2^4 \varphi \partial_2^4 \psi d\mathbf{y} = -\int_\Omega F_3 \partial_2^2 \psi d\mathbf{y},$$

$\forall \psi \in \widetilde{G}$.

Obviously, this problem is in the Lax - Milgram framework, as the right hand side is a continuous functional on $\mathbf{G}$. We then have

Theorem *Under the assumption $F_3 \in L^2(\Omega)$, Problem Π_0 has a unique solution.*

Our main convergence result is:

Theorem *Let $\mathbf{u}_\varepsilon$ and $\mathbf{u}$ be the solutions of Π_η and Π_0 respectively. Then, for $\varepsilon \downarrow 0$, we have:*

$$\mathbf{u}^\varepsilon \to \mathbf{u}$$

In other words, the limit $\mathbf{u}^*$ is the solution of the limit problem

The corresponding higher order partial differential equation for φ is obviously

$$(\frac{1}{C_{1111}}\partial_1^4 + B^{2222}\partial_2^8)\varphi = -\partial_2^2 F_3.$$

parabolic according the theory of linear partial differential equations

Remark *if we define the bilinear form*

$$a^0(\mathbf{u},\mathbf{v}) \overset{def}{=} \int_\Omega \frac{1}{C_{1111}}\partial_1 u_1 \partial_1 v_1 d\mathbf{y} + \int_\Omega B^{2222}\partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y},$$

then the symmetry of $a^0(\mathbf{u},\mathbf{v})$ shows that the (unique) solution $\mathbf{u}$ of problem Π_0 can be characterized as the unique element of $\mathbf{G}$ *solving the minimization problem*

$$Min_{\mathbf{G}} J_0(\mathbf{v})$$

where

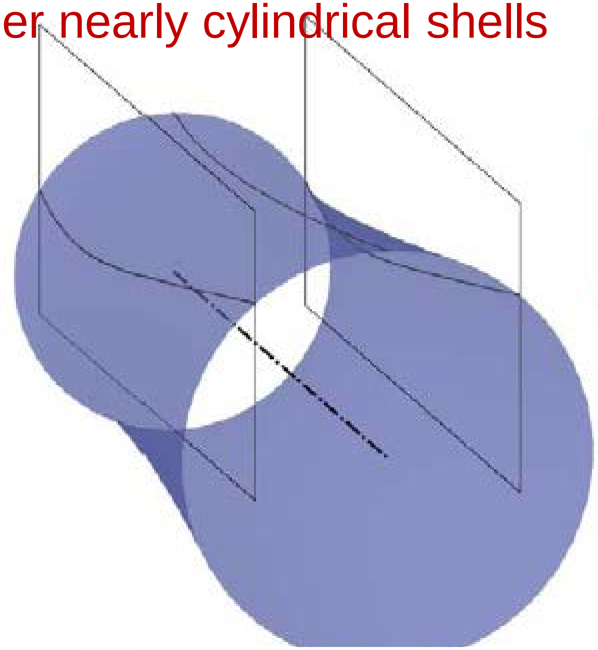
$$J_0(\mathbf{v}) = \frac{1}{2}a^0(\mathbf{v},\mathbf{v}) - \int_\Omega F_3 v_3 d\mathbf{y}.$$

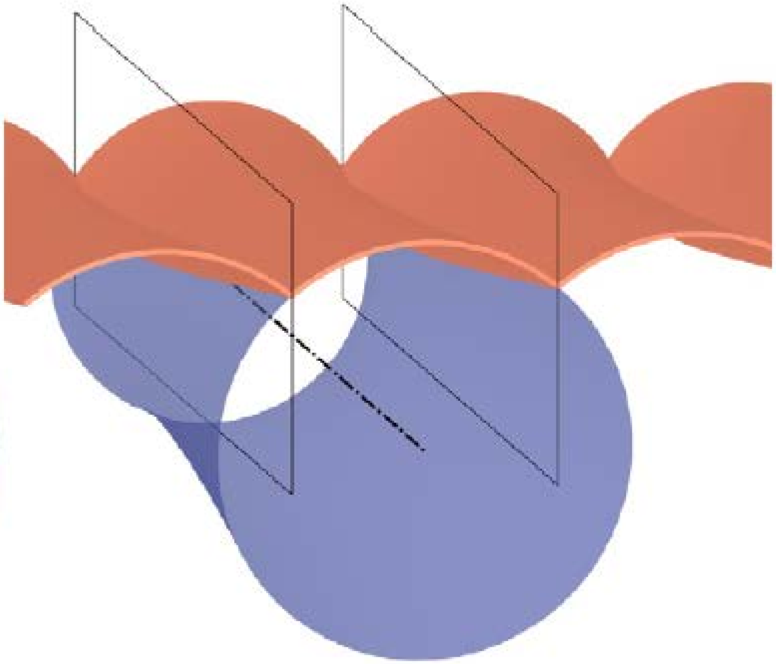
We can formulate, equivalently, this property in terms of the potential φ
So, the (unique) solution $\varphi \in \widetilde{G}$ of problem Π_0 can be characterized as the unique element of $\widetilde{G}$ solving the minimization problem

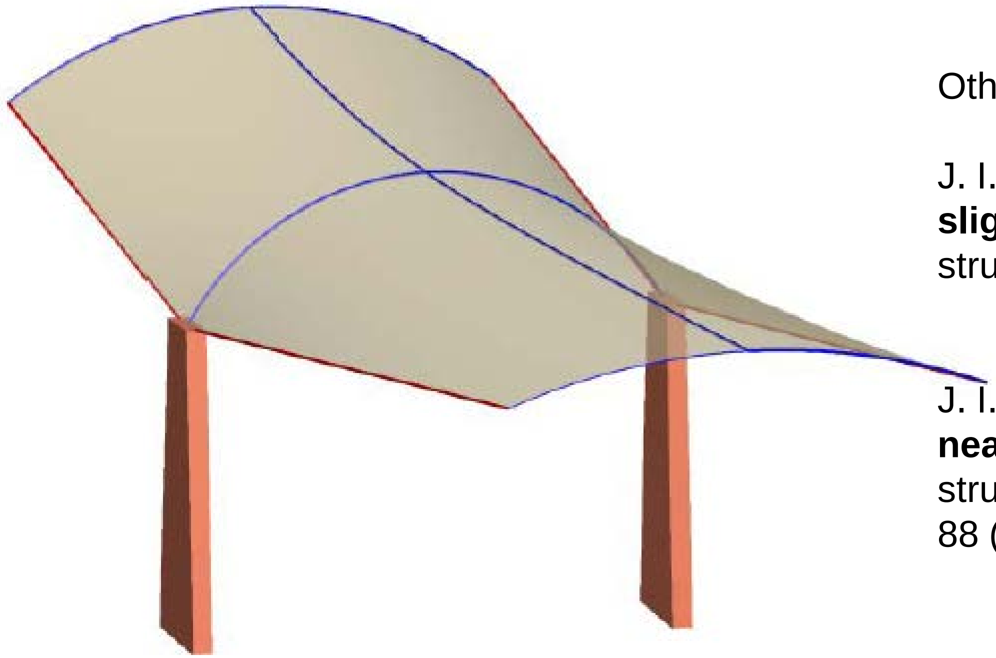
$$Min_{\widetilde{G}} \widetilde{J}_0(\psi)$$

$$\widetilde{J}_0(\psi) = \frac{1}{2C_{1111}}\int_\Omega \left|\partial_1^2\psi\right|^2 d\mathbf{y} + \frac{B^{2222}}{2}\int_\Omega \left|\partial_2^4\psi\right|^2 d\mathbf{y} + \int_\Omega F_3 \partial_2^2 \psi d\mathbf{y}$$

Slender nearly cylindrical shells







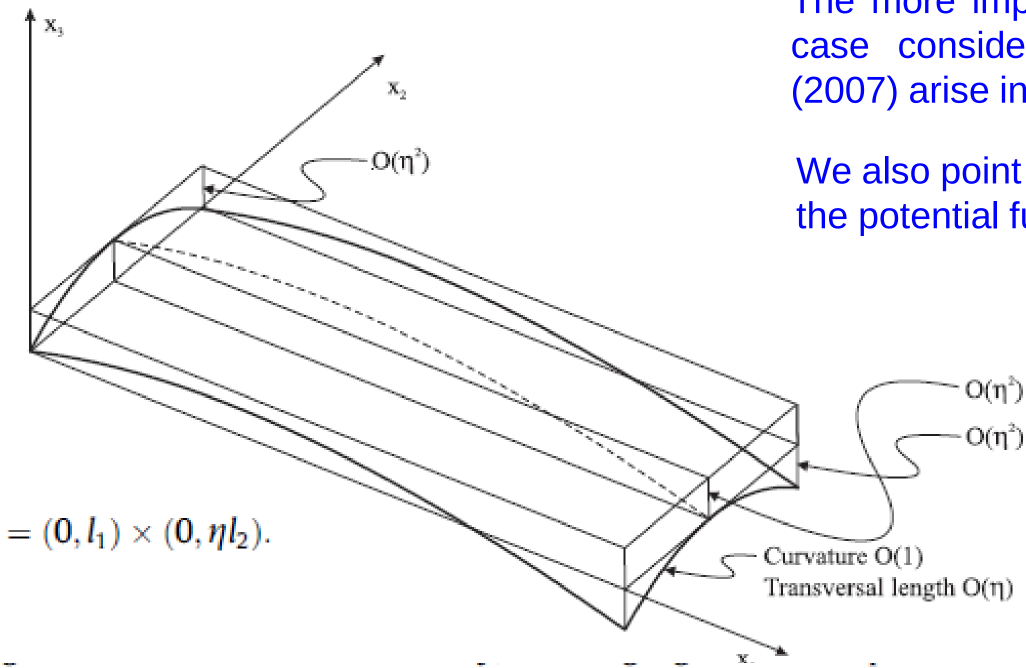
Other cases:

J. I. Díaz, E. Sánchez-Palencia, On a problem of slender **slightly hyperbolic** shells suggested by Torroja's structures. CRAS Mechanique, 337 (2009) 1-7.

J. I. Díaz, E. Sanchez-Palencia, A problem on slender **nearly cylindrical** shells suggested by Torroja's structures. International Journal of Engineering Science. 88 (2015) 83–98

The more important differences with respect to the case considered in Díaz and Sánchez-Palencia (2007) arise in the description of the space G

We also point out that the higher order equation for the potential function φ add some new terms



$\Omega_\varepsilon = (0, l_1) \times (0, \eta l_2).$

As a matter of fact, the above described perturbation of the cylinder corresponds (in the framework of shallow theory) to a middle surface of the form:

$$x_3 = bx_2^2 + x_1^2 + \eta^2 a \quad \text{with} \quad a, b \text{ constants}, \tag{6}$$

and this is a very restricted perturbation of the exact cylinder $x_3 = bx_2^2$. Indeed, the curvature lines (within shallow approximation) are x_2 = constant or x_1 = constant, according to $\partial_1 \partial_2 x_3 = 0$ in (6). This does not allow a kind of perturbation which was handled by Torroja (the shell roofs of the Madrid Racecourse) corresponding to a "kind of cylinder" with curvature depending on the longitudinal variable, *i.e*.

$$x_3 = b(x_1)x_2^2 \quad \text{with} \quad b(x_1) \geqslant c > 0. \tag{7}$$

$$\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle,$$

Concerning the applied forces, we shall give a normal loading depending on ε by the factor ε^3, specifically

$$\langle \mathbf{f}, \mathbf{v} \rangle = \varepsilon^3 \int_{\Omega_\varepsilon} F_3(x_1, x_2/\eta)\tilde{v}_3(x_1, x_2)d\mathbf{x}.$$

Thus, the shape of the profile of the applied loading in x_2 is independent of ε but applied to the points x_2/η. Defining $y_2 = x_2/\eta$ (see also the scaling (25) hereafter), the function $F_3(x_1, y_2)$ is independent of ε. We shall always assume that

$$F_3 \in L^2(\Omega),$$

where

$$\Omega = (0, l_1) \times (0, l_2).$$

We introduce the change of variables

$$\begin{cases} \mathbf{x} = (x_1, x_2) \Rightarrow \mathbf{y} = (y_1, y_2), \\ y_1 = x_1, \quad y_2 = \eta^{-1} x_2, \end{cases} \qquad \partial_1 = \tilde{\partial}_1, \quad \partial_2 = \eta \tilde{\partial}_2; \quad \partial_\alpha = \frac{\partial}{\partial y_\alpha}.$$

Moreover, we shall perform the change of unknowns

$$\begin{cases} \tilde{u}_1(\mathbf{x}) = \eta^6 u_1(\mathbf{y}), \\ \tilde{u}_2(\mathbf{x}) = \eta^5 u_2(\mathbf{y}), \\ \tilde{u}_3(\mathbf{x}) = \eta^4 b_{22}^{-1} u_3(\mathbf{y}), \end{cases}$$

$$\int_\Omega A^{\alpha\beta\lambda\mu} \gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \gamma^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu} \rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \rho^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} = \int_\Omega F_3(y_1, y_2) v_3(y_1, y_2) d\mathbf{y}.$$

Lemma 2.1. *The estimates:*

$$\|\partial_1 v_1 + \gamma v_3\|^2_{L^2(\Omega)} \leqslant c a^\varepsilon(\mathbf{v}, \mathbf{v}),$$

$$\|\eta^{-1} \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2)\|^2_{L^2(\Omega)} \leqslant c a^\varepsilon(\mathbf{v}, \mathbf{v}),$$

$$\|\eta^{-2}(\partial_2 v_2 + v_3)\|^2_{L^2(\Omega)} \leqslant c a^\varepsilon(\mathbf{v}, \mathbf{v}),$$

$$\|\partial_2^2 v_3\|^2_{L^2(\Omega)} \leqslant c a^\varepsilon(\mathbf{v}, \mathbf{v}),$$

$$\|\eta \partial_1 \partial_2 v_3\|^2_{L^2(\Omega)} \leqslant c a^\varepsilon(\mathbf{v}, \mathbf{v}),$$

$$\|\eta^2 \partial_1^2 v_3\|^2_{L^2(\Omega)} \leqslant c a^\varepsilon(\mathbf{v}, \mathbf{v}),$$

hold true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

Lemma 2.2. *The estimate:*

$$\|v_3\|^2_{L^2((0,l_1);H^2(0,l_2))} \leqslant ca^{\varepsilon}(\mathbf{v},\mathbf{v})$$

holds true for a certain $c > 0$ *independent of* ε *and* $\mathbf{v} \in \mathbf{V}$.

Lemma 2.7. *Let* $\mathbf{u}^{\varepsilon}$ *be the solution of* Π_{ε}. *The following convergences (as* $\varepsilon \to 0$*) hold true (in the sense of subsequences, the limits being not necessarily unique):*

$$\begin{aligned} u_1^{\varepsilon} &\to u_1^* \quad \text{weakly in } \widetilde{H}_0^1((0,l_1);L^2(0,l_2)) \\ u_2^{\varepsilon} &\to u_2^* \quad \text{weakly in } \widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2)) \\ u_3^{\varepsilon} &\to u_3^* \quad \text{weakly in } L^2((0,l_1);H^2(0,l_2)) \end{aligned}$$

In order to formulate the limit problem we introduce the following space

$$\mathbf{G} = \{\mathbf{v} = (v_1, v_2, v_3) \in \widetilde{H}_0^1((0,l_1);L^2(0,l_2)) \times \widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2)) \times L^2((0,l_1);H^2(0,l_2)), \partial_2 v_1 + \partial_1 v_2 = 0, \quad \partial_2 v_2 + v_3 = 0\}.$$

It is easily checked that v_1 defines completely v_2 and then v_3 and that $\mathbf{G}$ is a Hilbert space with the norm

$$\begin{cases} \|\mathbf{v}\|^2_{\mathbf{G}} = \|v_1\|^2_{\widetilde{H}_0^1((0,l_1);L^2(0,l_2))} + \|\partial_2^2 v_3\|^2_{L^2(\Omega)} \\ \simeq \|\partial_1 v_1\|^2_{L^2(\Omega)} + \|\partial_2^3 v_2\|^2_{L^2(\Omega)}. \end{cases}$$

The space **G** involves the two constraints corresponding to the "penalty terms" in Π_{ε}

whereas the boundary conditions for u_3, which are concerned with the "singular perturbation terms"

As in Díaz and Sánchez-Palencia (2007), it is useful to give an equivalent definition of the space **G** in terms of a scalar "potential ψ".

Lemma 2.8. *The space* **G** *may equivalently be defined as the space of the triplets* $\mathbf{v} = (v_1, v_2, v_3)$ *such that:*

$$v_1 = \partial_1\psi, \quad v_2 = -\partial_2\psi, \quad v_3 = \partial_2^2\psi,$$

where ψ *is an element of*

$$\widetilde{G} = \widetilde{H}_0^2((0,l_1);\ L^2(0,l_2)) \cap L^2((0,l_1);\ H^4(0,l_2)),$$

where

$$\widetilde{H}_0^2((0,l_1);L^2(0,l_2)) = \{\psi \in H^2((0,l_1);\ L^2(0,l_2));\ \psi(0,y_2) = \partial_1\psi(0,y_2) = 0\}.$$

Obviously, the norm of $\widetilde{G}$ is

$$\|\varphi\|_{\widetilde{G}}^2 = \int_\Omega \left(|\partial_1\varphi|^2 + |\partial_2^4\varphi|^2\right)dy.$$

Lemma 2.10. *The expression*

$$e(\varphi) = \int_\Omega \left(|\partial_1\varphi + \gamma\partial_2^2\varphi|^2 + |\partial_2^4\varphi|^2\right)dy$$

is the square of a norm on $\widetilde{G}$ *, which is equivalent to the natural norm*

Lemma 2.11. *There exists a constant c such that for any* $\varphi \in \widetilde{G}$*;*

$$\|\partial_2^2\varphi\|_{L^2(\Omega)}^2 \leqslant ce(\varphi).$$

We are now in position of formulating the limit problem (the convergence will be proved later). It involves the numerical coefficients $1/C_{1111}$, and B^{2222} where $C_{\alpha\beta\lambda\mu}$ is the matrix inverse of $A^{\alpha\beta\lambda\mu}$, *i.e.* the matrix of membrane compliances, whereas **B** is the matrix of flexion rigidities. They are both strictly positive.

Problem. Π_0. Find $\mathbf{u} \in \mathbf{G}$ such that

$$\int_\Omega \frac{1}{C_{1111}}(\partial_1 u_1 + \gamma u_3)(\partial_1 v_1 + \gamma v_3)d\mathbf{y} + \int_\Omega B^{2222}\partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y} = \int_\Omega F_3 v_3 d\mathbf{y}.$$

$\forall \mathbf{v} \in \mathbf{G}$, or equivalently, in terms of the potential, find $\varphi \in \widetilde{G}$ such that

$$\int_\Omega \frac{1}{C_{1111}}(\partial_1^2 + \gamma\partial_2^2)\varphi(\partial_1^2 + \gamma\partial_2^2)\psi d\mathbf{y} + \int_\Omega B^{2222}\partial_2^4\varphi\partial_2^4\psi d\mathbf{y} = -\int_\Omega F_3\partial_2^2\psi d\mathbf{y},$$

$\forall \psi \in \widetilde{G}$.

Theorem 2.1. *Under the assumption $F_3 \in L^2(\Omega)$, Problem Π_0 has a unique solution.*

Concerning the convergence of the solutions is given by.

Theorem 2.2. *Let $\mathbf{u}^\varepsilon$ and $\mathbf{u}$ be the solutions of Π_ε and Π_0 respectively. Then, for $\varepsilon \downarrow 0$, we have:*

$$\mathbf{u}^\varepsilon \to \mathbf{u}$$

Our next result improves the convergence under additional hypotheses on the rigidity coefficients.

Theorem 2.3. *Assume that*

$$A^{11\lambda\mu} = 0 \text{ if } \lambda > 1 \quad \text{or} \quad \mu > 1.$$

Let $\mathbf{u}^\varepsilon$ and $\mathbf{u}$ be the solutions of Π_ε and Π_0 respectively. Then

$$u_1^\varepsilon \to u_1^* \quad \text{strongly in } \widetilde{H}_0^1((0,l_1);L^2(0,l_2)),$$
$$u_2^\varepsilon \to u_2^* \quad \text{strongly in } \widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2)),$$
$$u_3^\varepsilon \to u_3^* \quad \text{strongly in } L^2((0,l_1);H^2(0,l_2))$$

for $\varepsilon \downarrow 0$.

We point out that the limit problem is given by the variational formulation (93). The corresponding partial differential equation (of higher order) for φ is

$$\left(\frac{1}{C_{1111}}\partial_1^4 + 2\gamma\partial_1^2\partial_2^2 + \gamma^2\partial_2^4\right)\varphi + B^{2222}\partial_2^8\varphi = -\partial_2^2 F_3,$$

Generalizations and remarks

It is worth while noticing that all the previous results hold true when the fixation conditions

$$u_1(0,y_2) = u_2(0,y_2)$$

are prescribed *only on a part* (with positive measure) of the boundary $y_1 = 0$.

It is not hard to prove such kind of estimates in more general situations. For instance in cases when the shape of the shell is not exactly rectangular. Let us consider for example the case described (after scaling to the variables y_1, y_2) in the Fig. 2.

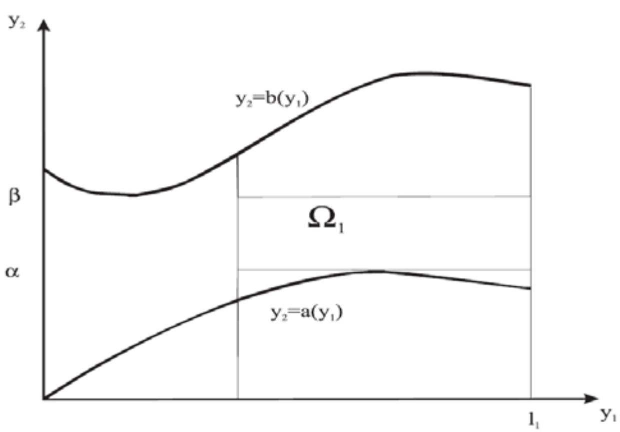


Fig. 2.

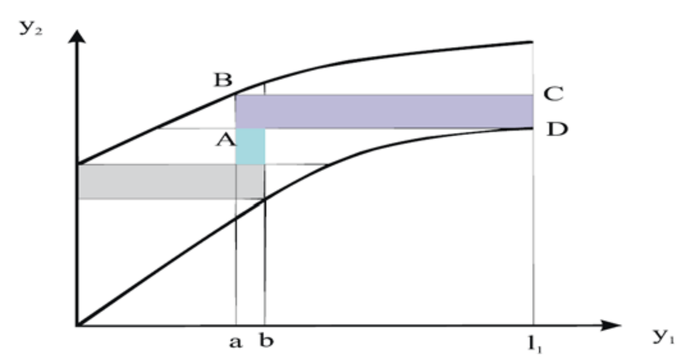


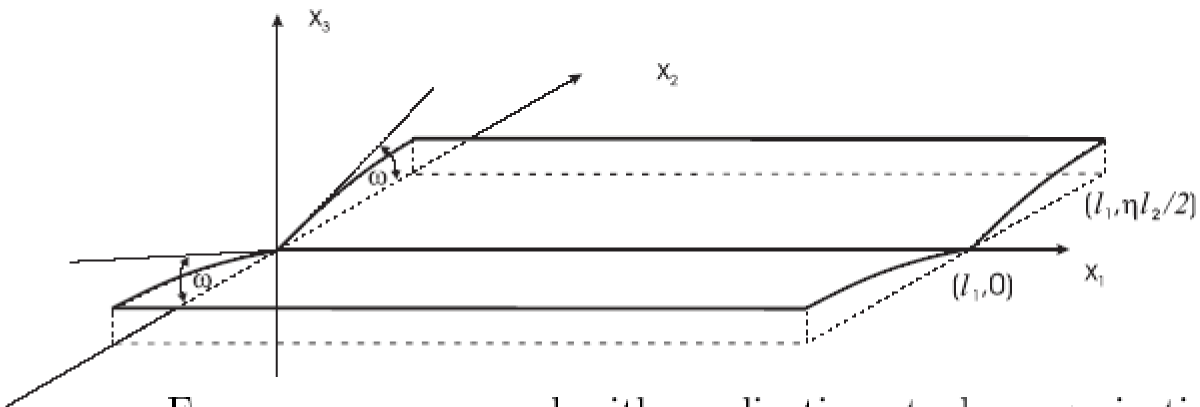
Fig. 3.

Remark 3.1. Let us comment a little on the results of this paper and their interpretation in terms of global rigidity of the shell and its comparison with the flexion rigidity of a plate (i.e. the rigidity power of the curvature). The starting formulation (13) is the standard one for flexion of plates or shells. There is a factor ε^3 in front of the flexion bilinear form b as the flexion rigidity of a plate or shell is proportional to ε^3 (we shall explain this point later). Accordingly, the applied flexion loading (23) is also of order ε^3, and this leads in classical plate or flexion shell theory (i.e. for "non-inhibited shells which correspond to shells with middle surface not geometrically rigid) to the existence and uniqueness of a (finite and non-vanishing) limit. But, in the present problem, the (finite and non-vanishing) limit is obtained after the scaling in u_3 defined by the third relation (28). This amounts to saying that the flexion rigidity of the (curved) plate or shell in the present case is $\eta^{-4} = \varepsilon^{-1}$ times that of an analogous plane plate.

This result easily understood using a heuristic reasoning in terms of kinematics of flexion of plates and beams. According to Kirchhoff theory, the rigidity of a plate or beam is proportional to the inertia moment of the surface of a section with respect to the "neutral fiber" of the flexion. In a plate of thickness $O(\varepsilon)$ the surface of the section (by unit length) is $O(\varepsilon)$ and the distances to the neutral fiber are also $O(\varepsilon)$, so that the inertia moment is $O(\varepsilon^3)$, as we saw. Oppositely, in the present problem, the surfaces (by unit length) are again $O(\varepsilon)$, whereas the distances to the neutral fiber are $O(\eta^2)$ because of the curvature, so that the inertia moment is $O(\varepsilon\eta^4) = O(\varepsilon^2)$, *i.e.* $O(\varepsilon^{-1})$ times greater. In fact, the plate behaves as some kind of a beam with sections of a very large inertia moment because of the curvature. It should be noticed that these remarkable rigidity properties are a consequence of the transversal curvature, and, as a consequence, they are common to the shells addressed in Díaz and Sánchez-Palencia (2007, 2009) and the present paper, and even in analogous situations of slightly folded plates (Sanchez Palencia, 2006). The slight longitudinal curvature considered in Díaz and Sánchez-Palencia (2009) and in this work does not modify the order of magnitude of the rigidity, it only has an influence on the corresponding coefficients, which are only accessible by numerical computation.

4. The shell with an edge with slight folding

In this section we consider a case slightly more complicated than the basic problem, when the section by $y_1 = const.$ is as sketched in Fig 2.



For reasons concerned with applications to homogenization problems

tangent plane on $y_2 = -l_2/2$ and $y_2 = l_2/2$ is horizontal. This amounts to saying that the angle of the folding is 2ω, with $\omega = b\eta l_2/2$ (see Fig 2) where b always denote the (constant) curvature. Denoting by $\tilde{u}_i^-$ and $\tilde{u}_i^+$ the traces on $x_2 = 0$, the continuity of the displacement $\tilde{\mathbf{u}}$ at the folding gives in the projections along x_1, its normal in the "base plane" and the axis Z (see Fig. 2) respectively:

In order to avoid irrelevant and cumbersome expressions, as ω is small, we shall take $cos\omega = 1$, $sin\omega = \omega$. Moreover, we shall see in the sequel that the components u_3 are asymptotically larger than u_2, and we shall neglect $\omega^2 u_2$ with respect to u_3. Then we shall consider

$$\begin{cases} \tilde{u}_1^+ = \tilde{u}_1^- \\ -\omega\tilde{u}_3^+ + \tilde{u}_2^+ = \omega\tilde{u}_3^- + \tilde{u}_2^- \\ \tilde{u}_3^+ = \tilde{u}_3^- \end{cases}$$

so that we merely may keep in mind that $\tilde{u}_1$ and $\tilde{u}_3$ are continuous across $x_2 = 0$ and

$$\tilde{u}_2^+ - \tilde{u}_2^- = 2\omega\tilde{u}_3.$$

Let us denote

$$\Omega_\varepsilon^+ = (0, l_1) \times (0, \eta l_2/2) \ \text{ and } \ \Omega_\varepsilon^- = (0, l_1) \times (-\eta l_2/2, 0)$$

and we shall also denote by Ω_ε the union of Ω_ε^+ and Ω_ε^-. The space of configuration will be denoted by $\mathbf{V}_\varepsilon$. It is the subspace of

$$H^1(\Omega_\varepsilon^+) \times H^1(\Omega_\varepsilon^+) \times H^2(\Omega_\varepsilon^+) \times H^1(\Omega_\varepsilon^-) \times H^1(\Omega_\varepsilon^-) \times H^2(\Omega_\varepsilon^-)$$

formed by the functions satisfying the kinematic boundary conditions

$$\begin{aligned} 0 &= \tilde{u}_1^+ = \tilde{u}_2^+ = \tilde{u}_3^+ \ \text{ on } \{0\} \times [0, \eta l_2/2], \\ 0 &= \tilde{u}_1^- = \tilde{u}_2^- = \tilde{u}_3^- \ \text{ on } \{0\} \times [-\eta l_2/2, 0], \end{aligned}$$

and the transmission conditions

The "variational or weak formulation" of the elasticity problem for this structure takes again the form

$$\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle$$

with

$$\begin{aligned} a(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) &= \int_{\Omega_\varepsilon^+} A^{\alpha\beta\lambda\mu} \tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}}^+) \tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}}^+) d\mathbf{x} + \int_{\Omega_\varepsilon^-} A^{\alpha\beta\lambda\mu} \tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}}^-) \tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}}^-) d\mathbf{x} \\ b(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) &= \int_{\Omega_\varepsilon^+} B^{\alpha\beta\lambda\mu} \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}}^+) \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}^+) d\mathbf{x} + \int_{\Omega_\varepsilon^-} B^{\alpha\beta\lambda\mu} \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}}^-) \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}^-) d\mathbf{x}, \end{aligned}$$

where we are using the obvious decomposition $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}^+, \tilde{\mathbf{v}}^-)$ for any element of the energy space $\mathbf{V}_\varepsilon$.

The scaling and other developments are then analogous to those of the "basic problem". The space of configuration after scaling will be denoted by $\mathbf{V}$. It is the subspace of

$$H^1(\Omega^+) \times H^1(\Omega^+) \times H^2(\Omega^+) \times H^1(\Omega^-) \times H^1(\Omega^-) \times H^2(\Omega^-)$$

formed by the functions satisfying the transmission and kinematic boundary conditions

$$u_2^+ - u_2^- = l_2 u_3$$

and

$$0 = u_1 = u_2 = u_3 \text{ on } \{0\} \times [-l_2/2, l_2/2],$$

Theorem *Let $\mathbf{u}_\varepsilon$ and $\mathbf{u}$ be the solutions of the above coupled problems respectively. Then, for $\varepsilon \downarrow 0$, we have:*

$$\mathbf{u}^\varepsilon \to \mathbf{u}$$

with convergence of each component $\mathbf{u}^\varepsilon = (\mathbf{u}^{\varepsilon+}, \mathbf{u}^{\varepsilon-})$

Equivalently, the limit $\mathbf{u}$ can be obtained through its potential $\varphi = (\varphi^+, \varphi^-) \in \widetilde{G}$, solution of

$$\int_{\Omega+} \frac{1}{C_{1111}} \partial_1^2 \varphi^+ \partial_1^2 \psi^+ d\mathbf{y} + \int_{\Omega-} \frac{1}{C_{1111}} \partial_1^2 \varphi^+ \partial_1^2 \psi^+ d\mathbf{y}$$
$$+ \int_{\Omega+} B^{2222} \partial_2^4 \varphi^+ \partial_2^4 \psi^+ d\mathbf{y} + \int_{\Omega-} B^{2222} \partial_2^4 \varphi^- \partial_2^4 \psi^- d\mathbf{y}$$
$$= -\int_{\Omega+} F_3 \partial_2^2 \psi^+ d\mathbf{y} - \int_{\Omega-} F_3 \partial_2^2 \psi^- d\mathbf{y},$$

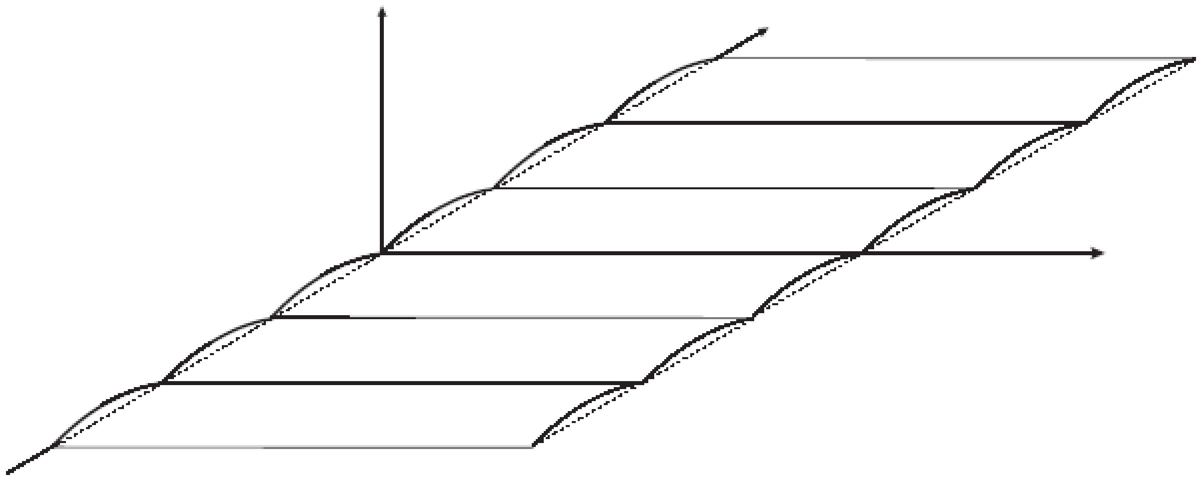
$\forall \psi = (\psi^+, \psi^-) \in \widetilde{G}$.

Remark. Similar treatment for the case of Slender nearly cylindrical shells

5. On periodic or pseudo-periodic slender nearly cylindrical shells

Case of cylindrical shells:

We consider now the case in which the shell is $2\eta l_2-$periodic with respect the section by $x_1 = const.$projected on the band $(0, l_1) \times (-\infty, +\infty)$ and having a slight folding at any section of the form $(0, l_1) \times \{k\eta l_2\}$ with $k \in \mathbb{Z}$, as sketched in Fig 3.



along the "small sides" at $\{0\} \times [-\eta l_2, \eta l_2]$, which implies again kinematic boundary conditions similar to those indicated

We then consider periodic loadings and search for periodic solutions.

The convergence arguments follows as in previous Subsections with easy modifications.

Case of slender nearly cylindrical shells:

We take an ondulated surface with wave length η and curvature $O(1)$; namely:

$$x_3 = \eta^2 \Phi(x_1/\eta)$$

where $\Phi(z)$ is $l_2 - periodic$. As a rule, Φ is smooth and changing sign, so involving points with zero curvature, which constitutes a non-trivial generalization of previous theory. For the time being, we shall consider

$$\Phi''(y) \neq 0 \; a.e.$$

where a. e. means, in this work, for "unless at the inflexion points of Φ".

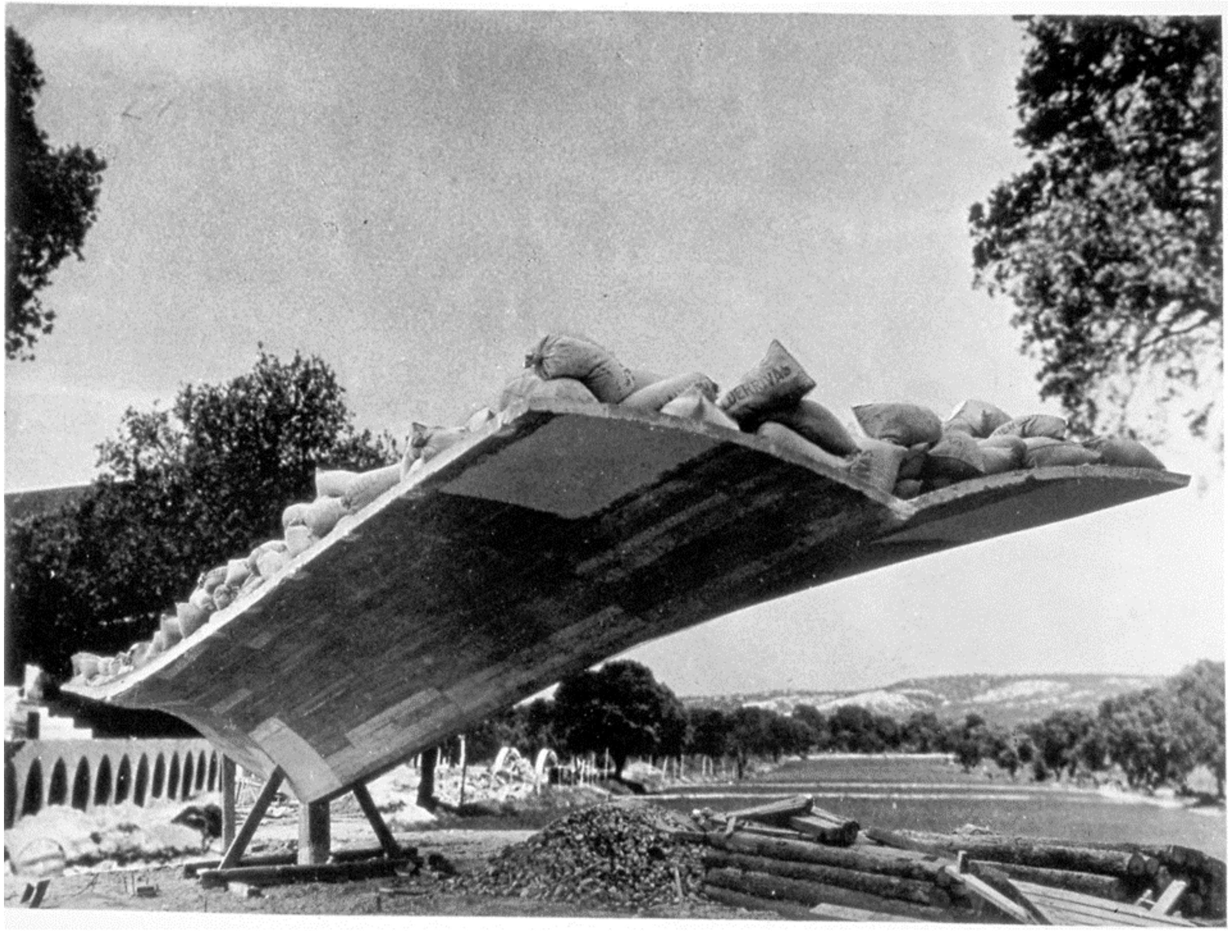
The implementation of the formal asymptotics uses as an essential element the approximation:

$$u_i^\varepsilon = u_i^0(x_1, y) + \eta u_i^1(x_1, y) + \eta^2 u_i^2(x_1, y) + ...$$

where the terms are Y-periodic in y or equivalently, 1-periodic in y_2.

$$C^{-1}_{1111} \partial_1^4 \varphi + B^{2222} \partial_2^2 [c^{-1} \partial_2^4 (c^{-1} \partial_2^2 \varphi)] = \partial_2^2 F_3$$

Our results concerning the potential función shows that the tests of Torroja with an isolated band constituted a sure test for the periodic case, since the space where it was minimized was larger



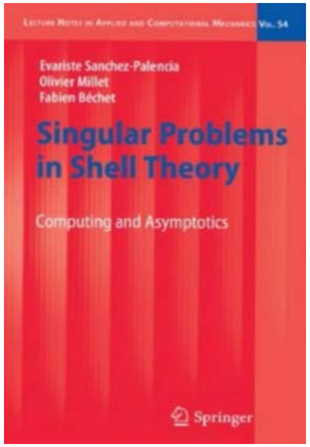
6. Final remarks

- Extension to the obstacle problem

J. I. Díaz, E. Sánchez-Palencia, On slender shells and related problems suggested by Torroja's structures, Asymptotic Analysis, 52, 2007, 259-297

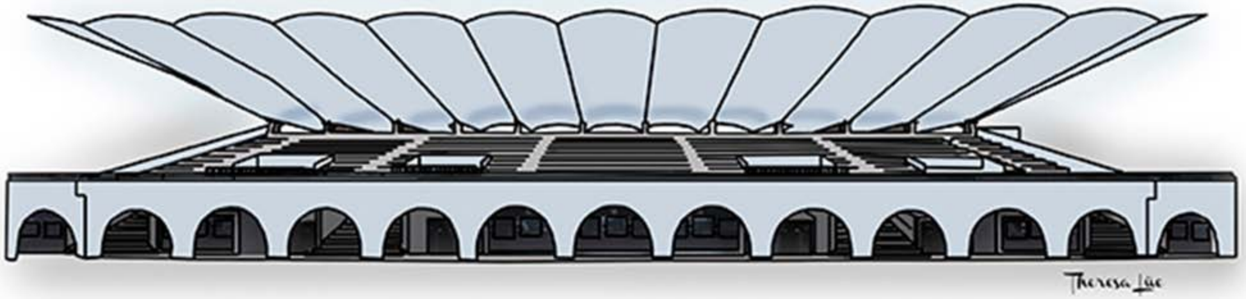
- Many alternatives, waiting some answer, for the case of nonlinear elastic shells (Ciarlet school,…

- Very careful numerical analysis ??



E. Sanchez-Palencia, O. Millet, and F. Béchet, *Singular Problems in Shell Theory,*
Vol. 54:Lecture Notes in Applied and Computational Mechanics, Springer, New York, 2010.

Thanks for your attention



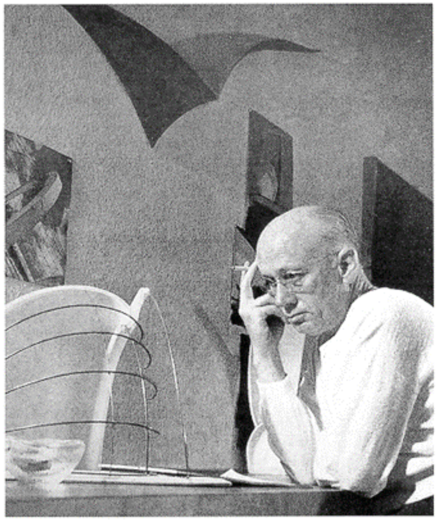


Foto: M. García Moya.