Necessary and sufficient conditions for the very weak solvability of the beam equation

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1. Introduction

- The notion of weak solution of a boundary value problem, on a bounded domain Ω, is associated to functions in some energy space satisfying the equation in a weak form, after multiplying by any test function in such energy space and integrating by parts. Nevertheless, in many relevant cases in the applications the right hand side datum is merely in $L^1_{loc}(\Omega)$ and a different notion of solution is required. For instance, in the case of second order problems the notion of very weak solution is reduced to functions in $L^1(\Omega)$ satisfying the equation passing the second order derivatives to the test functions.

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- Most of the theory on very weak solutions available in the literature deals with second order equations. Recently, sharper results have been obtained, to this case, when the data are merely in $L^1(\Omega, \delta)$, with $\delta = dist\,(x, \partial\Omega))$. That was originally proved by Haim Brezis, at the seventies, in a famous unpublished manuscript *concerning Dirichlet boundary conditions* (see also his paper [**?**], published in 1996 with T. Cazenave, Y. Martel, and A. Ramiandrisoa: for more recent references see J.I. Díaz, J.M. Rakotoson [**?**] [**?**] and our collaboration with J. Hernández [**?**]).

- The main goal of this lecture is to present some new results proving that in the case of **higher order** equations **and Dirichlet boundary conditions** the class of $L^1_{loc}(\Omega)$ data for which the existence and uniqueness of a very weak solution can be obtained is larger than $L^1(\Omega, \delta)$ (the optimal class for the case of second order equations). For instance, for some stationary onedimensional semilinear 4th-order equations we shall prove that the optimal class of data is the space $L^1(\Omega, \delta^2)$. Moreover we shall analyze the optimal solvability also for the case of **other boundary conditions**: something which, as far as we know, was not considered before in the literature.

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- In some sense, the obtained results give an answer to the question about of the greatest weight profile which can support a simple beam such that its two extremes are horizontally supported (for instance to a wall) and do not experience any deflection.

- To fix ideas I will present the results for the relevant model of the Euler-Bernoulli beam model (i. e. a fourth order onedimensional spatial operator) but most of the results remain valid for equations of order $2m$, $m \in \mathbb{N}$. In a first part we shall consider the stationary case:

$$(SP)\begin{cases} \frac{d^4u}{dx^4} = f(x) & x \in \Omega = (0, l), \\ + \text{ boundary conditions } (BC). \end{cases}$$

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$$\left\{\begin{array}{ll} a_0 u(0) = 0, & b_0 u(l) = 0, \\ a_1 u'(0) = 0 & b_1 u'(l) = 0, \\ a_2 u''(0) = 0, & b_2 u''(l) = 0, \\ a_3 u'''(0) = 0, & b_3 u'''(l) = 0. \end{array}\right.$$

- Here the coefficients are taken such that $a_i, b_i \in \{0,1\}$ and $\sum a_i = 2, \sum b_i = 2$ in order to have a simple way to state general results. For instance the usual Dirichlet conditions (supported beam) corresponds to

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- Finally, a very often situation corresponds to a cantilever bar ($x = 0$ clamped and $x = L$ free)

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- In a second part I will consider the Euler-Bernouilli transient hyperbolic problem (with a possible damping term)

$$(HP)\begin{cases} \frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+\frac{\partial^4 u}{\partial x^4}=f(t,x) & t\in(0,T), x\in(0,l), \\ +\text{ boundary conditions}, & t\in(0,T), \\ u(0,x)=u_0(x) & x\in(0,l), \\ u_t(0,x)=v_0(x) & x\in(0,l), \end{cases}$$

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- as well as the so called (Duvaut and Lions 1972) "quasi-static" associated problem (now of parabolic type)

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- We shall see that the optimal weight $w(x)$ in order to solve the above problems is

$$\delta_{\mathbf{ab}}(x) = \max\{a_1 d(x,0)^2, a_2 d(x,0), a_3\} \max\{b_1 d(x,l)^2, b_2 d(x,l), b_3\}.$$

Notice that, for instance for the Dirichlet problem $[\mathbf{a} = (1,1,0,0), \mathbf{b} = (1,1,0,0)]$, we must take $\delta_{\mathbf{ab}}(x) \sim \delta^2(x)$ with $\delta = dist\,(x, \partial\Omega)$.

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- 4. Some numerical experiences.
- Sections 2 and 3 will appear in RACSAM. Section 3 is part of a joint work with I. Arregui and C. Vázquez.

2. Necessary and Sufficient conditions for the existence of solutions for the stationary problem.

To fix ideas I will consider now the case of Dirichlet boundary conditions.
Definition. Given $f \in L^1_{loc}(0,l)$ a function $u \in L^1_{loc}(0,l)$ is a "solution of (SP) in $D\prime(0,l)$" if

$$\left\langle u, \frac{d^4\zeta}{dx^4} \right\rangle_{D'D} = \langle f, \zeta \rangle_{D'D}$$

for any $\zeta \in D(0,l) = C_c^\infty(0,l)$.
We introduce now the space associated to the boundary (BC) as

$$V = \overline{\{\zeta \in C^4([0,l])\colon \zeta \text{ satisfies } (BC)\}}^{W^{4,\infty}(0,l)}.$$

For instance, for the case $W^{4,\infty}(0,l)$ of Dirichlet boundary conditions $V = W^{4,\infty}(0,l) \cap W_0^{2,\infty}(0,l)$.
Definition. Given $f \in L^1(0,l:\delta_{\mathbf{ab}})$ a function $u \in L^1(0,l)$ is a "very weak solution" of (SP) and (BC) if

$$\int_0^l u(x)\frac{d^4\zeta}{dx^4}(x)dx = \int_0^l f(x)\zeta(x)dx$$

for any $\zeta \in V$.

- **Remark 1**. It is not difficult to show that $\zeta \in V$ implies that $|\zeta(x)| \leq c\delta_{\mathbf{ab}}$ for any $x \in (0, l)$ and so the above identity is well justified.

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1. *(Sufficiency) a) Sufficiency. Assume that* $a_2 a_3 = 0$ if $b_2 = b_3 = 1$ (*respectively,*$b_2 b_3 = 0$ *if* $a_2 = a_3 = 1$). *Then, for any* $f \in L^1(0, L : \delta_{\mathbf{ab}})$ *there exists a unique very weak solution of* (SP) *and* (BC). *Moreover we have the estimate (weak maximum principle)*
$$C \left\| u_+ \right\|_{L^1(0,l)} \leq \left\| f_+ \right\|_{L^1(0,l:\delta_{\mathbf{ab}})}, \tag{1}$$
for some $C > 0$ ($C = 24L^4$ *for* (DBC)) *where, in general,* $h_+ = \max(0, h)$. *Moreover* $u \in C^3([0, L])$.

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2. *(Strong maximum principle) Let* $f \in L^1(0, l : \delta_{\mathbf{ab}})$ *with* $f \geq 0$ *a.e.* $x \in (0, l)$*. Then the very weak solution satisfies)*

$$u(x) \geq C \left\| f \right\|_{L^1(0,l:\delta_{\mathbf{ab}})} \delta_{\mathbf{ab}}(x) > 0 \text{ for any } x \in (0, l), \ \textit{for some } C > 0.$$

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3. *(Necessity) Assume that* $f \in L^1_{loc}(0, l)$, *such that* $f \geq 0$ *a.e.* $x \in (0, l)$. *Then if* $\int_0^l f(x)\delta_{\mathbf{ab}}(x)dx = +\infty$ *it can not exists any* $u \in C^3([0, l])$ *satisfying* (BC) *being also solution in* $D'(0, l)$ *of* (SP).

- *Idea of the proof of the Theorem 1.* For the existence part it is enough to use the Green function associated to the boundary conditions (see,e.g., Stakgold 1998). Indeed the expression $u(x) = \int_0^L G(x,y)f(y)dy$ is well justified since we have that $|G(.,y)| \leq C\delta_{\mathbf{ab}}(y)$. For the proof of the L^1-estimate we shall use some "conservation formula". For the case of (DBC) (for other boundary conditions the arguments are similar) we have:

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- We also know (see Chow-Dunninger-Lasota (1973)) that if $f \in L^1_{loc}(0, L)$, *$f \geq 0$ on $(0, L)$ then $u(x) \geq 0$ for any $x \in (0, L)$.*

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- The last ingredient, to prove the L^1-estimate is an abstract result applied usually to hyperbolic equations

- **Lemma 2** (Crandall-Tartar 1980). *Let X, Y two vector lattices and λ_X, λ_Y be nonnegative linear functionals on X and Y respectively. Let $C \subseteq X$ and $f, g \in C$ imply $f \vee g \in C$. Let $T : C \to Y$ satisfy $\lambda_X(f) = \lambda_Y(T(f))$* for $f \in C$. *Then $(a) \Rightarrow (b) \Rightarrow (c)$ where $(a), (b), (c)$ are the properties:*
  (a) $f, g \in C$ and $f \leq g$ imply $T(f) \leq T(g)$,
  (b) $\lambda_Y((T(f) - T(g))_+) \leq \lambda_X((f - g)_+)$ for $f, g \in C$,
  (c) $\lambda_Y(|T(f) - T(g)|) \leq \lambda_X(|f - g|)$.
  Moreover, if $\lambda_Y(F) > 0$ for any $F > 0$, then $(a), (b), (c)$ are equivalent.

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  (c) $\lambda_Y(|T(f) - T(g)|) \leq \lambda_X(|f - g|)$.
  Moreover, if $\lambda_Y(F) > 0$ *for any* $F > 0$, *then* $(a), (b), (c)$ *are equivalent.*
- Now, to prove the L^1-estimate (1) we take $C = X = L^1(0, L : \widehat{\delta}^2)$, $Y = L^1(0, L)$, $\lambda_X(f) = \int_0^L x^2(L-x)^2 f(x)dx$, $\lambda_Y(F) = \int_0^L F(x)dx$ and $T(f) = 24L^4 u$ (with u the very weak solution of (SP) and (DBC)). Then the identity of Lemma 2 coincides with Lemma 1. So we get (b) of Lemma 2 which is the wanted L^1-estimate.

- The proof of the strong maximum principle uses the estimate $|G(.,y)| \leq C\delta_{\mathbf{ab}}(y)$. To prove part c), and more specifically the complete blow up (in the whole interval $(0,L)$) when $f \notin L^1(0,L:\delta_{\mathbf{ab}})$ we truncate f generating $f_n(x) = \min(f(x),n)$. Now, if u_n is the associated solution ($f_n \in L^\infty(0,L) \subset L^1(0,L:\delta_{\mathbf{ab}})$) then $u_n(x) \geq C\,\|f_n\|_{L^1(0,L:\delta_{\mathbf{ab}})}\,\delta_{\mathbf{ab}}(x)$, which implies that $u_n(x) \nearrow +\infty$ for any $x \in (0,L)$.

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- **Remarks.**

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2. Theorem 1 extends many previous works in the literature: Aftabizadeh (1986), Gupta (1988), Agarwal (1989), O'Regan (1991), Bernis (1996), Pao (1999), Yao (2008)...

3 We also mention that the above versions in the literature on the weak maximum principle (valid under weaker conditions than the above (BC) have a non-quantitative version. Estimate (1) is new in the literature. It seems possible to extend the above result to the case of several dimensions but restricted to balls and under symmetry conditions on f. The maximum principle is false on some ellipsoidal domains (see Boggio 1905 and the conjecture by Hadamard 1908 firstly proved by Duffin (1949) and Garabedian (1951)). For balls see Bachar-Māagli-Masmoudi-Zribi (2003) which also contains ver sharp estimates.

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4 The existence result holds also in the more general class of Radon measures $f \in M(0, L : \delta_{\mathbf{ab}})$: something very useful to justify the engineers study in with the weight on the beam is concentrated in isolated points. Notice that although the usual Radon measure space (without wieight) $M(0, L)$ is a subset of the dual space $H^{-2}(0, L)$ it is not always true that the duality $\langle f, \zeta \rangle_{H^{-2}(0,L), H_0^2(0,L)}$ coincides with the $\langle f, \zeta \rangle_{M(0,L), C^0([0,L])} = \int_0^L \zeta(x) df$ duality.

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- This also holds for the case of Dirichlet conditions: the associated Green function $G(x, y)$ can be explicitly built (for instance by means of the use of Mapple (see Díaz 2010) and it can be shown that if k is large enough then $G(x_0, y_0)$ for some $(x_0, y_0) \in [0, L]^2$.

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and satisfying (DBC). Moreover, if $\widehat{u}$ is the solution for $\widehat{f}$, we have

$$24L^4 \|u - \widehat{u}\|_{L^1(0,L:\delta_{\mathbf{ab}})} \leq \left\| f - \widehat{f} \right\|_{L^1(0,L:\delta_{\mathbf{ab}})} .$$

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- *Idea of the proof of Theorem 2.* The operator $A : D(A) \rightarrow L^1(0, L : \delta_{\mathbf{ab}})$ given $Au = \frac{d^4u}{dx^4}$ if $u \in D(A)$ with $D(A) = \{u \in L^1(0, L : \delta_{\mathbf{ab}}) \cap C^3([0, L]) : \frac{d^4u}{dx^4} \in L^1(0, L : \delta_{\mathbf{ab}})$ and u satisfies the $(DBC)\}$ satisfies that $\exists\ C > 0$ such that $C \left\| u \right\|_{L^1(0,L:\delta_{\mathbf{ab}})} \leq \left\| Au \right\|_{L^1(0,L:\delta_{\mathbf{ab}})}$ for all $u \in D(A)$. So, its inverse operator $J = A^{-1}$, satisfies that $J + C^{-1}I$ is accretive (and also $I - J$ when $C > 1$): see Benilan-Crandall-Pazy 2001.

- Then, in particular, for any accretive operator B on $L^1(\Omega, \delta_{\mathbf{ab}})$ and for any $\lambda > 0$ and $f \in L^1(0, L : \delta_{\mathbf{ab}})$ the problem $\lambda(Jw + Bw + \frac{1}{C}w) + w = f$ has at most one *solution* $w \in L^1(\Omega, \delta_{\mathbf{ab}})$ and we have the continuous dependence estimate $24L^4 \left\| u - \widehat{u} \right\|_{L^1(0,L:\delta_{\mathbf{ab}})} \leq \left\| w - \widehat{w} \right\|_{L^1(0,L:\delta_{\mathbf{ab}})} \leq \left\| f - \widehat{f} \right\|_{L^1(0,L:\delta_{\mathbf{ab}})}$.

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- Nevertheless, it is possible to show the following "positivity result" (which improves Gazzola-Grunau 2009 for the one-dimensional case):
- **Theorem 3** *(eventual positivity). Let $f \in L^1_{loc}(0, +\infty : L^2(0, L))$ with $\frac{\partial f}{\partial t} \in L^1(0, +\infty : L^2(0, L))$ be such that $f(t, x) \to f_\infty(x)$ in $L^2(0, L)$ as $t \to +\infty$, with $f_\infty(x) \geq 0$, $f_\infty \neq 0$.*

Then, for any $u_0 \in H^4(0,L) \cap H_0^2(0,L)$ *and for any* $\varepsilon > 0$ *small enough there exist a time* $T_\varepsilon \geq 0$ *such that the mild solution* $u \in C([0,+\infty) : L^2(0,L))$ *of*

$$(HP) \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = f(t,x) & t \in (0,T), x \in (0,+\infty), \\ u(t,0) = 0, \quad u(t,L) = 0, & \\ \frac{\partial u}{\partial x}(t,0) = 0 \quad \frac{\partial u}{\partial x}(t,L) = 0, & t \in (0,+\infty), \\ u(0,x) = u_0(x) & x \in (0,L), \end{cases}$$

satisfies that $u(t,x) \geq C(\|f_\infty\|_{L^1(0,L:\delta_{\mathbf{ab}})} - \varepsilon)\delta_{\mathbf{ab}}(x) > 0$ *for any* $t \geq T_\varepsilon$ *and for any* $x \in (0,L)$.*Idea of the proof.* It is enough to use that $u(t,x) \to u_\infty(x)$ in $W^{2,\infty}(0,L)$ as $t \to +\infty$ (apply Theorem 3.9 of *Brezis 1973*) with $u_\infty(x)$ given as the unique solution of (SP) and (DBC) with f_∞ as right hand side and to apply the strong maximum principle b) of Theorem1.

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- **4. Some numerical experiences**

Test 1

$$\begin{cases} \Delta^2 u = f\,, & \text{en } \Omega = (0,1)\times(0,1) \\ u = \Delta u = 0\,, & \text{en } \partial\Omega \end{cases}$$

- Segundo miembro:

$$f(x,y) = \frac{1}{|x+\varepsilon|^k}\,\frac{1}{|1+\varepsilon-x|^k}\,\frac{1}{|y+\varepsilon|^k}\,\frac{1}{|1+\varepsilon-y|^k}$$

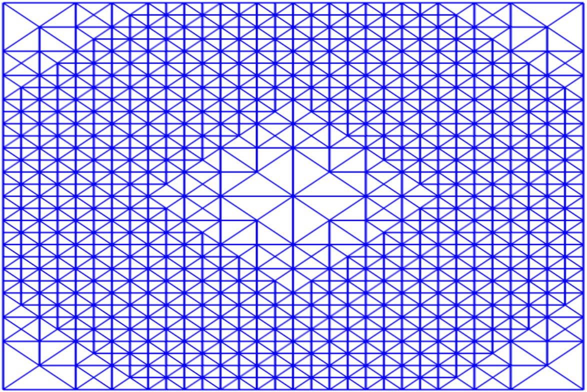
$$k = 1$$

- Resolución, en cada nivel, por un método directo
- Refinamiento según el gradiente del segundo miembro.

Test 1

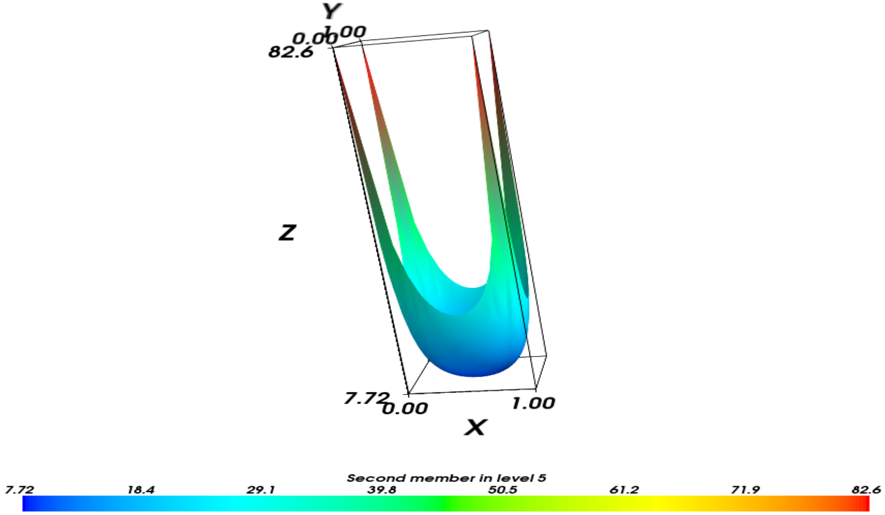
$$\varepsilon = 10^{-1}$$

Mesh 5: 901 nodes, 1712 elements

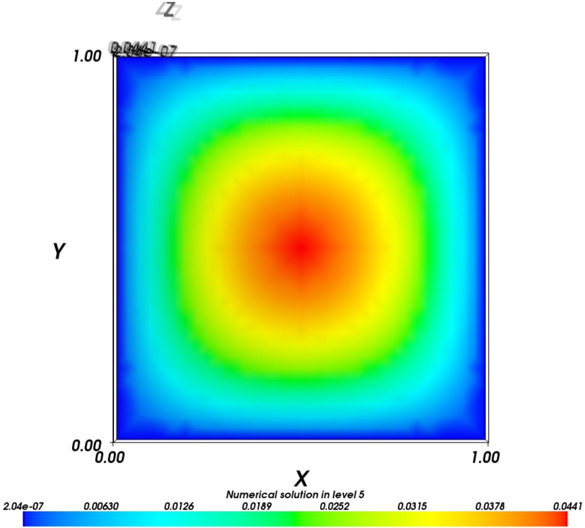


Test 1

Segundo miembro, $\varepsilon = 10^{-1}$



Aproximación numérica de la solución, $\varepsilon = 10^{-1}$



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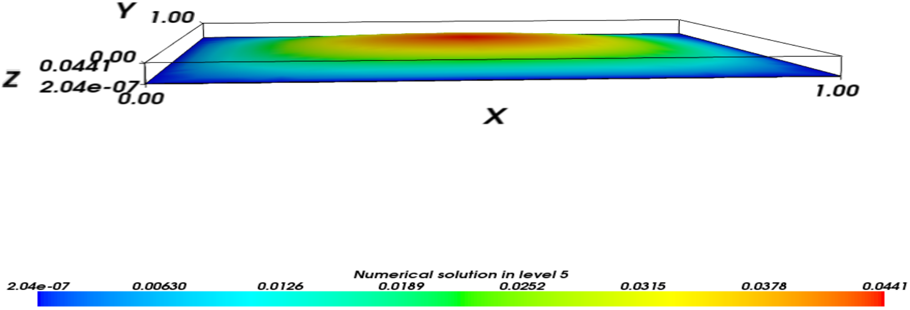
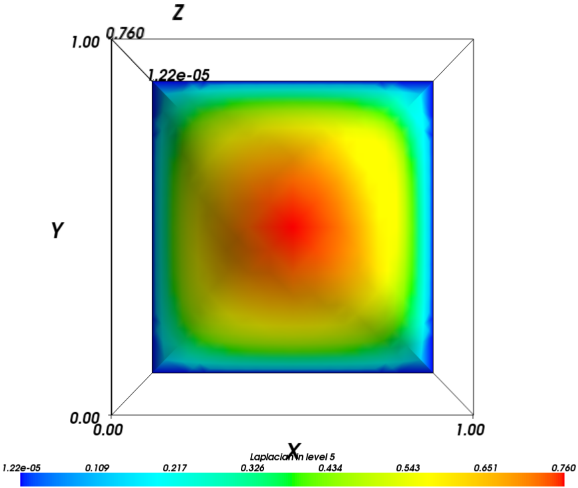


Test 1

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-1}$



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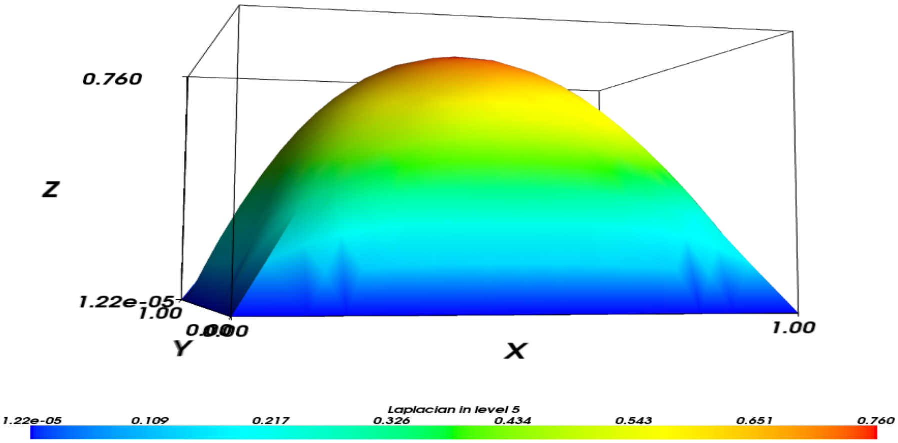

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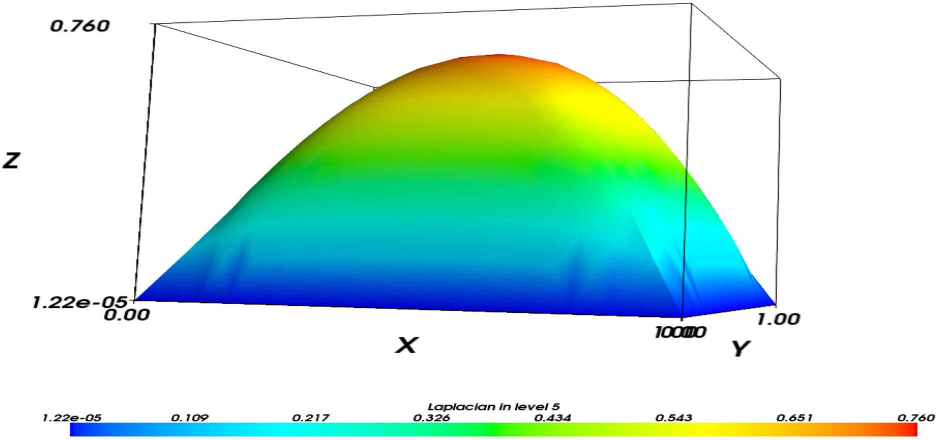

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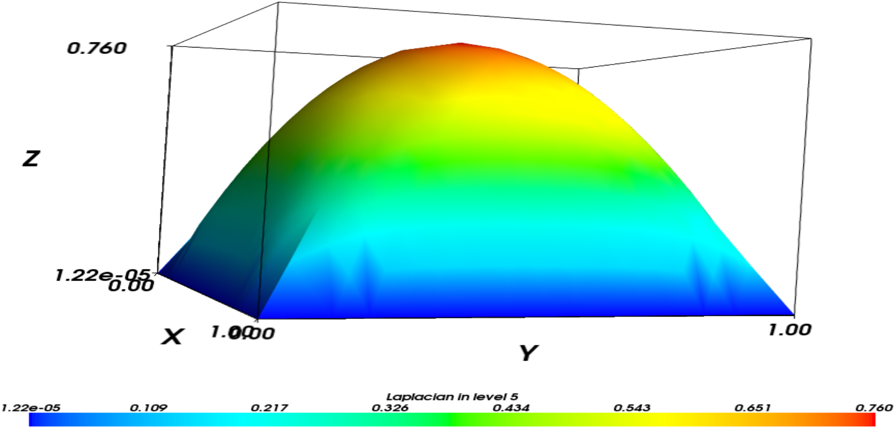

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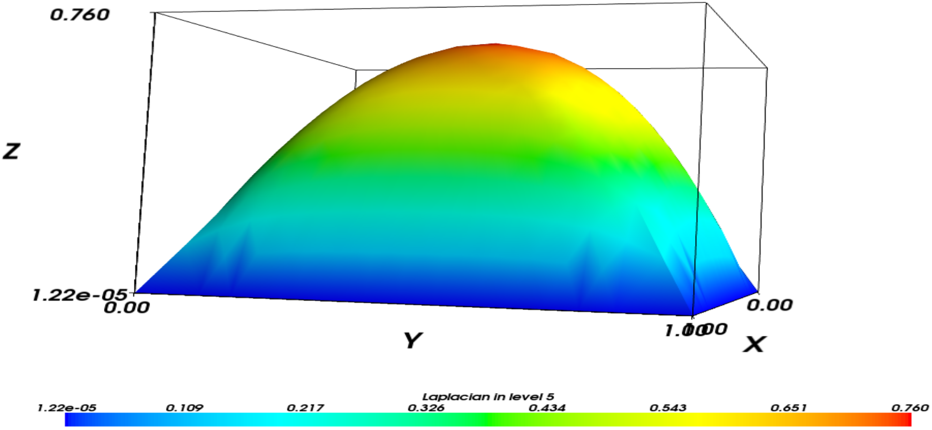
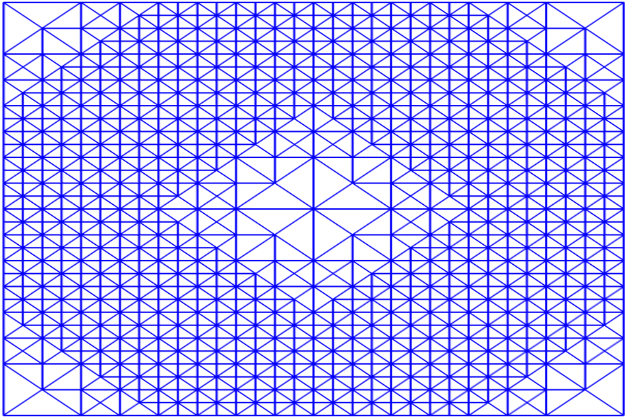
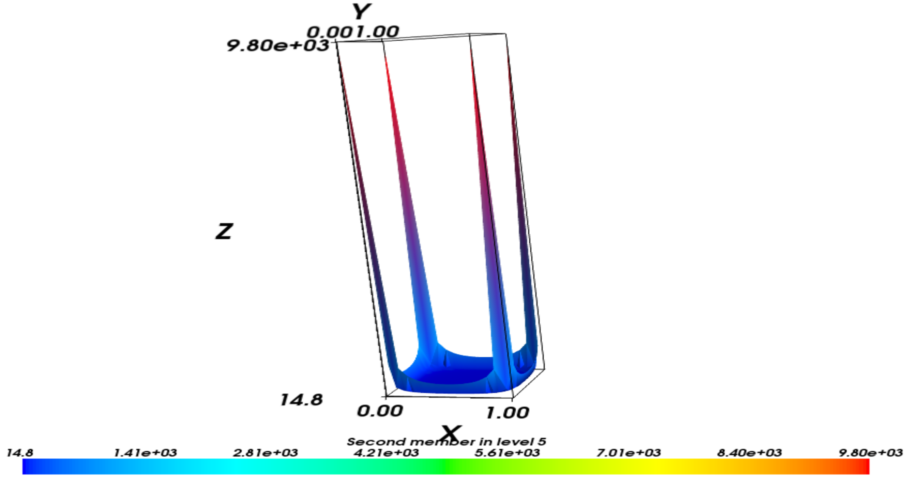


$$\varepsilon = 10^{-2}$$

Mesh 5: 901 nodes, 1712 elements

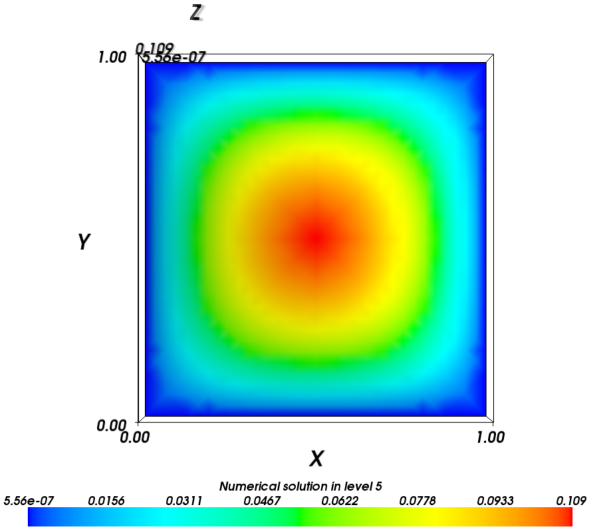


Segundo miembro, $\varepsilon = 10^{-2}$



Test 2

Aproximación numérica de la solución, $\varepsilon = 10^{-2}$



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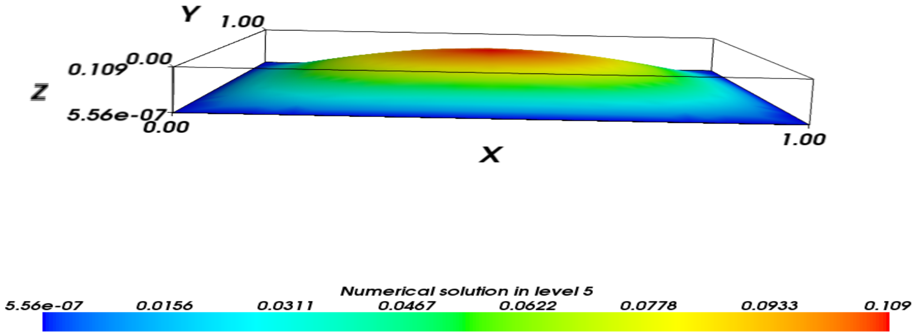

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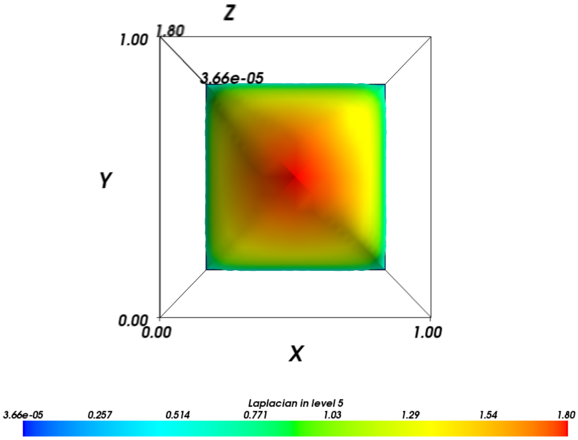

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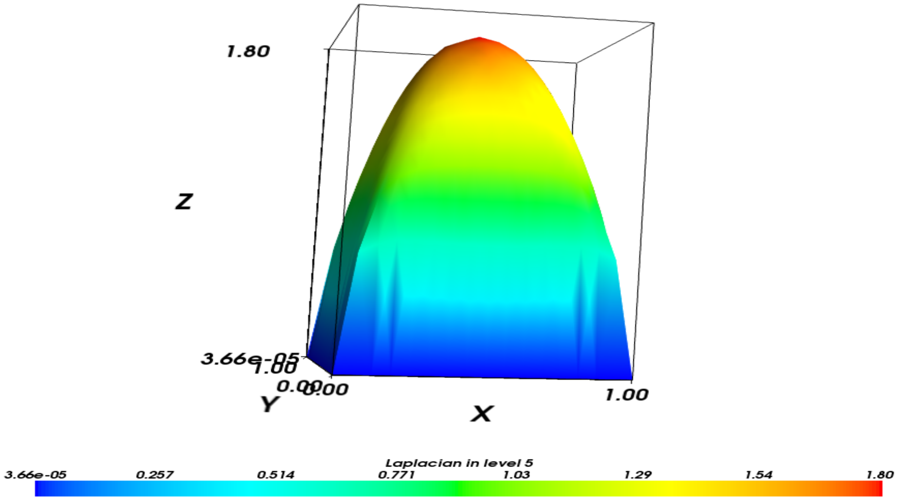

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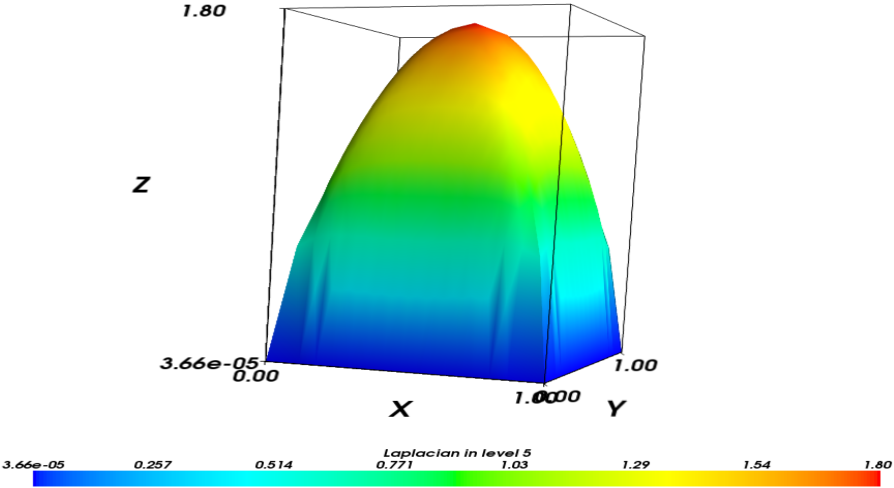

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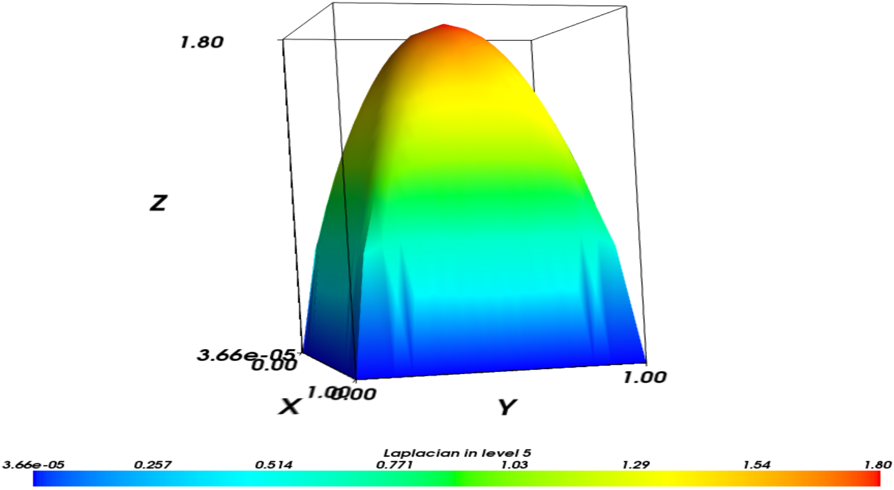

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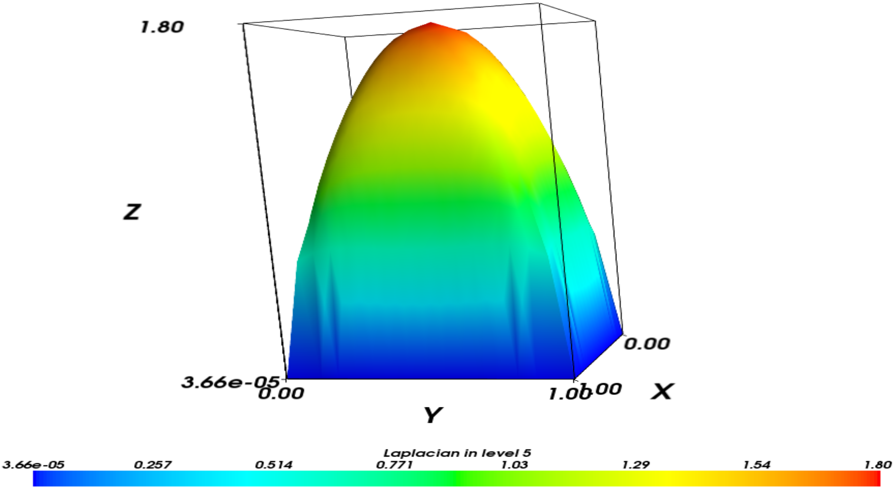
Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-2}$



Test 3

$$\varepsilon = 10^{-4}$$

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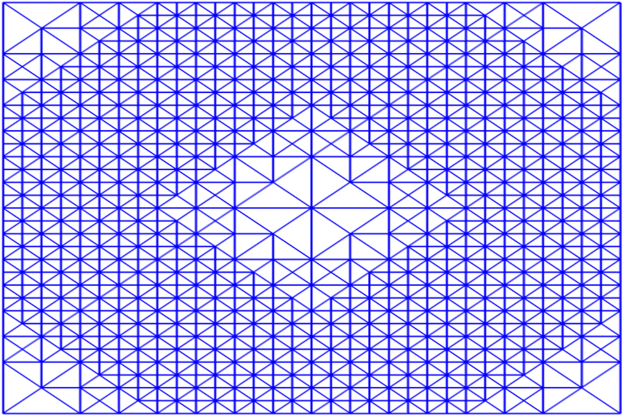
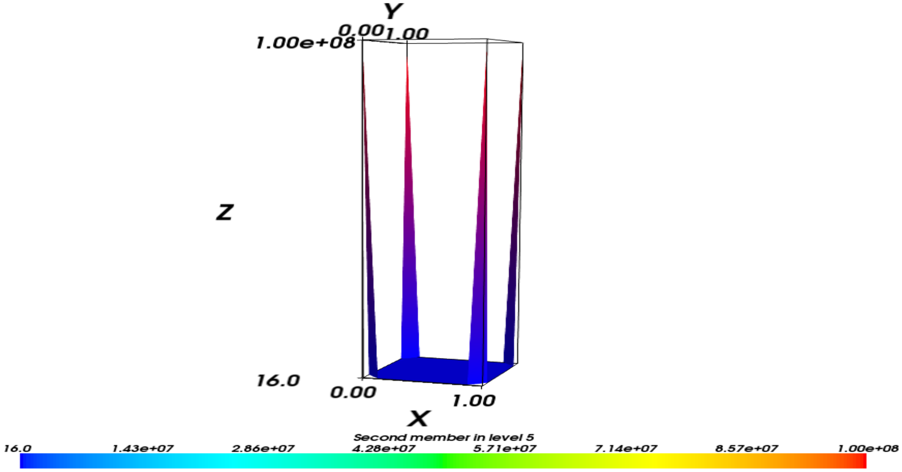


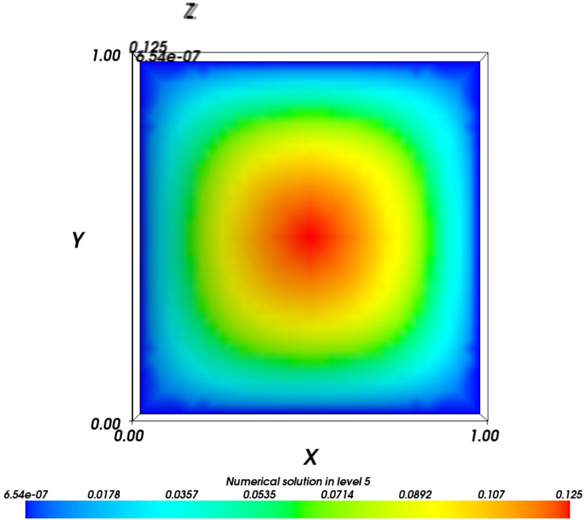
Test 3

Segundo miembro, $\varepsilon = 10^{-4}$



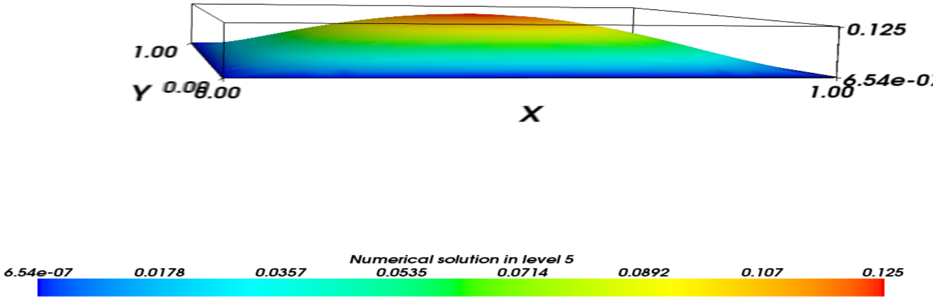
Test 3

Aproximación numérica de la solución, $\varepsilon = 10^{-4}$



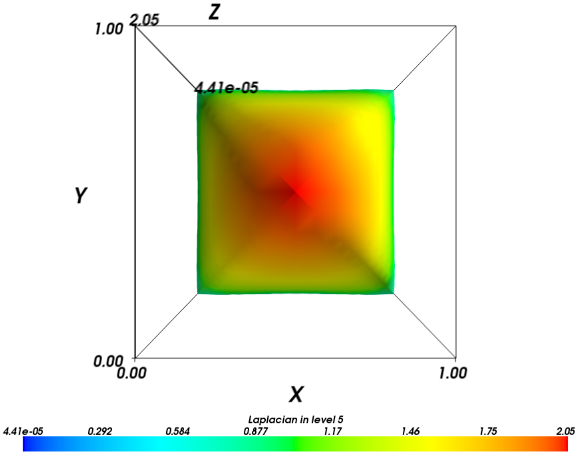
Test 3

Aproximación numérica de la solución, $\varepsilon = 10^{-4}$



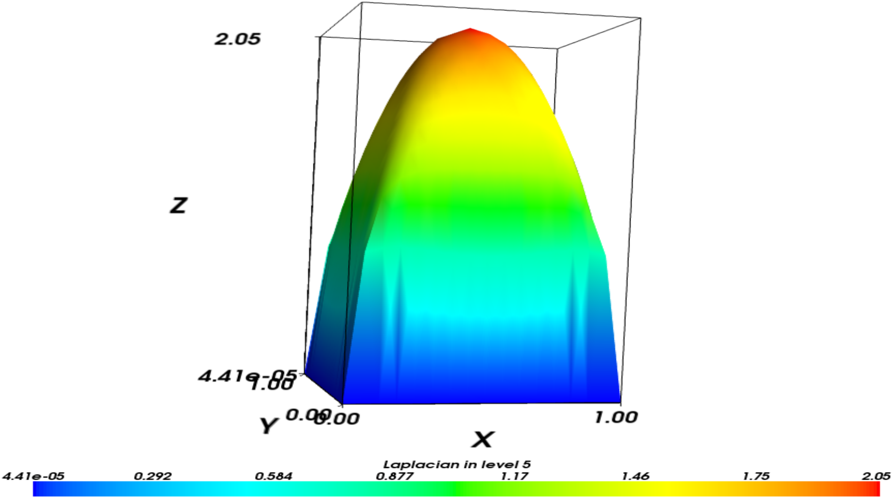
Test 3

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-4}$



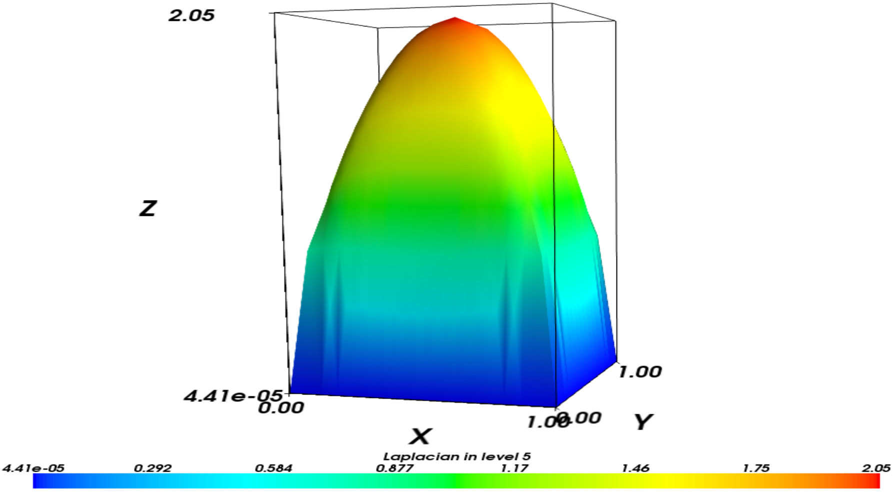
Test 3

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-4}$



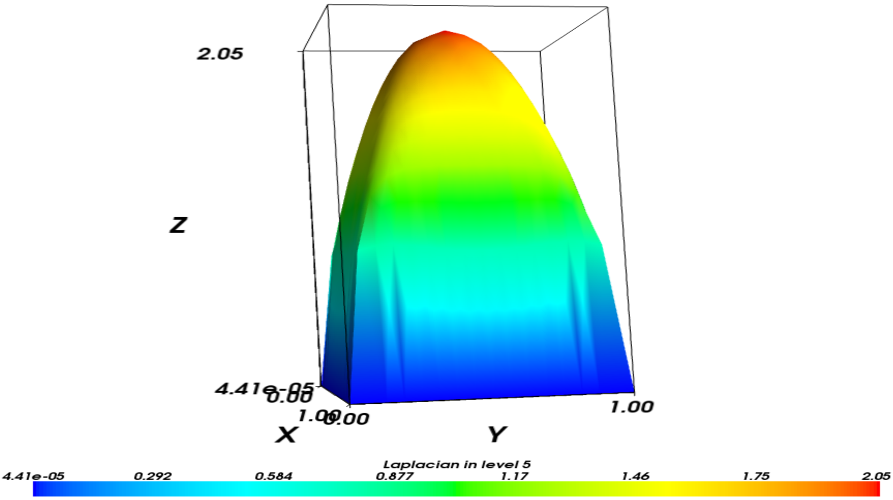
Test 3

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-4}$



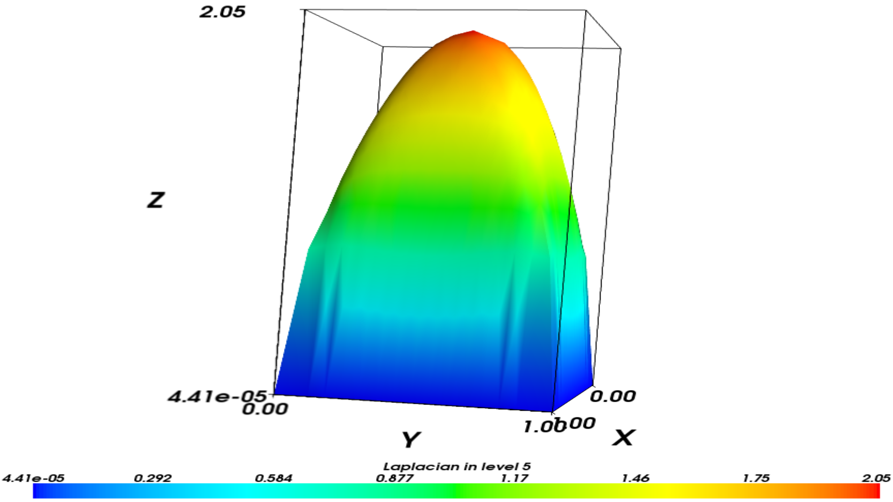
Test 3

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-4}$



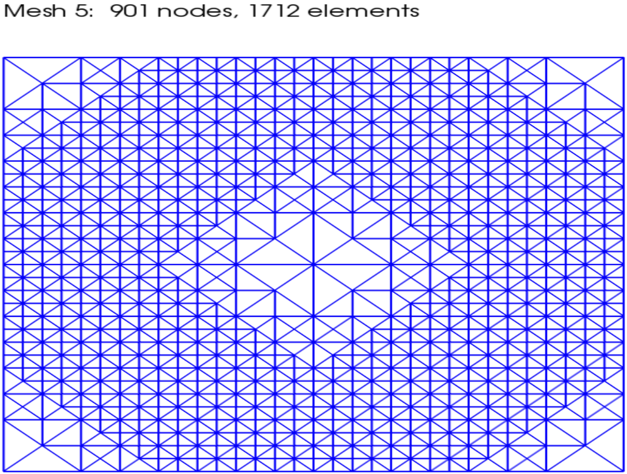
Test 3

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-4}$



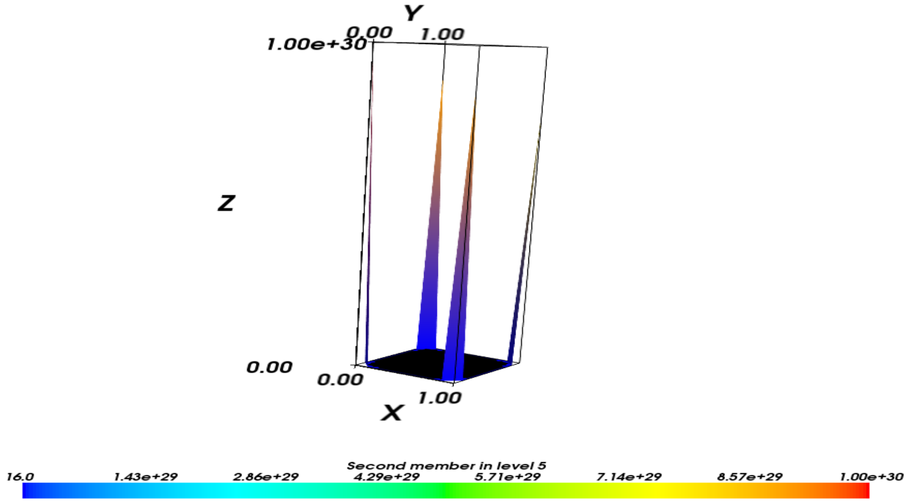
Test 4

$$\varepsilon = 10^{-15}$$



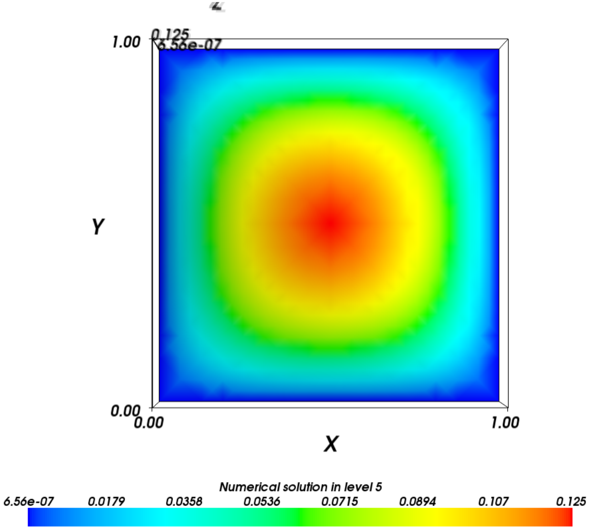
Test 4

Segundo miembro, $\varepsilon = 10^{-15}$



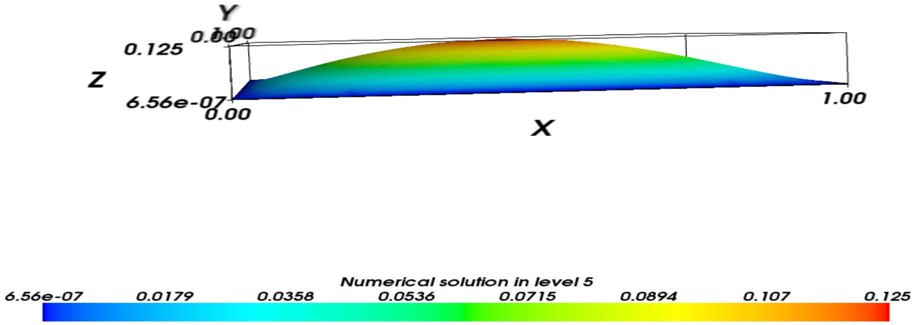
Test 4

Aproximación numérica de la solución, $\varepsilon = 10^{-15}$



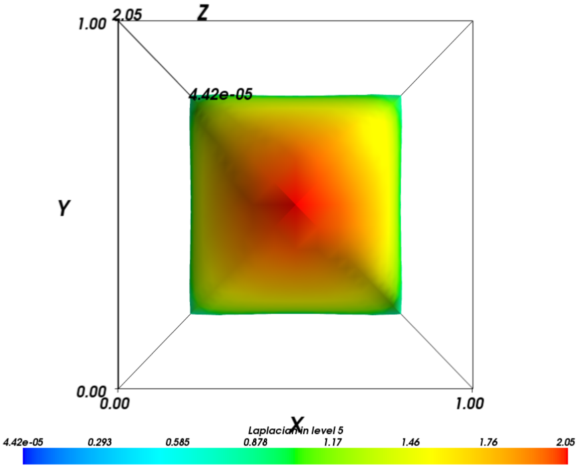
Test 4

Aproximación numérica de la solución, $\varepsilon = 10^{-15}$



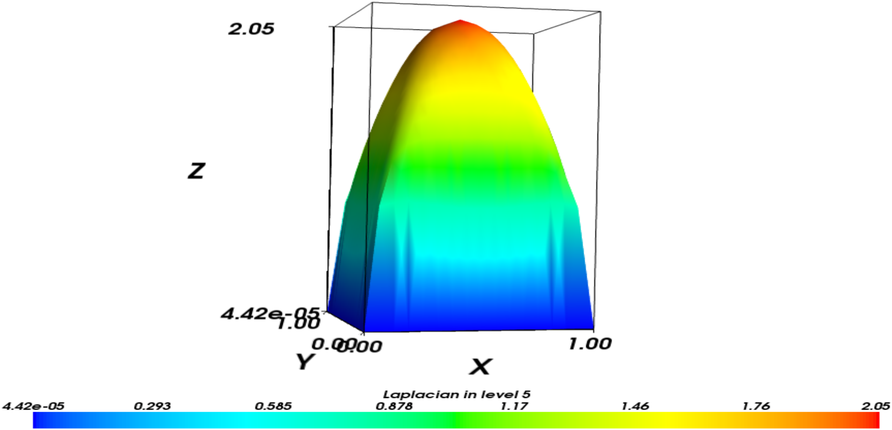
Test 4

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$



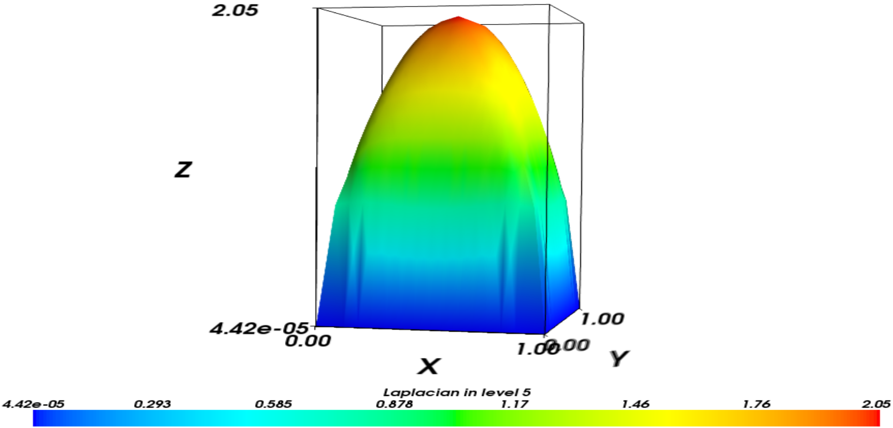
Test 4

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$



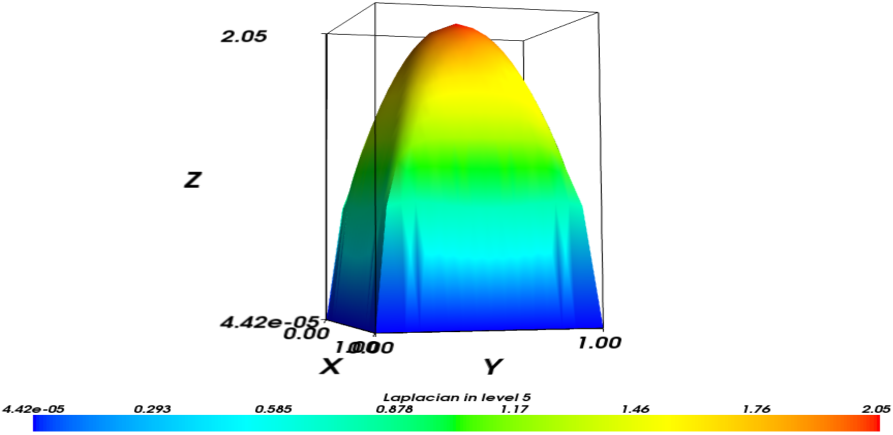
Test 4

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$



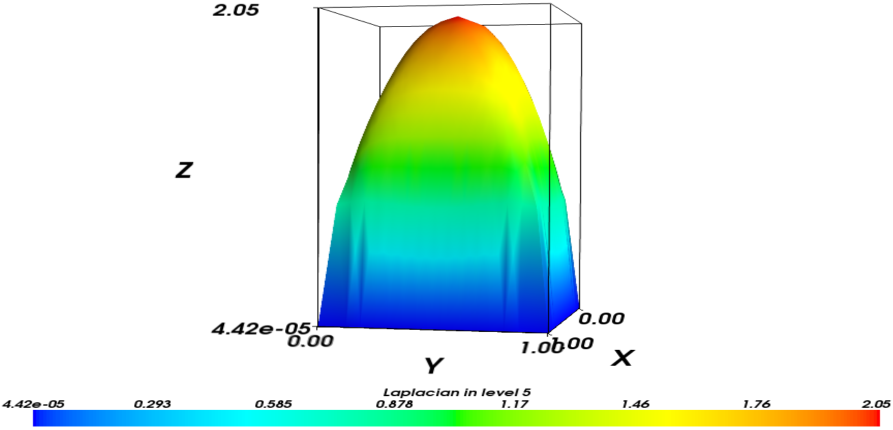
Test 4

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$



Test 4

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$



Test 5

$$\begin{cases} \Delta^2 u = f\,, & \text{en } \Omega = (0,1)\times(0,1) \\ u = \Delta u = 0\,, & \text{en } \partial\Omega \end{cases}$$

- Segundo miembro:

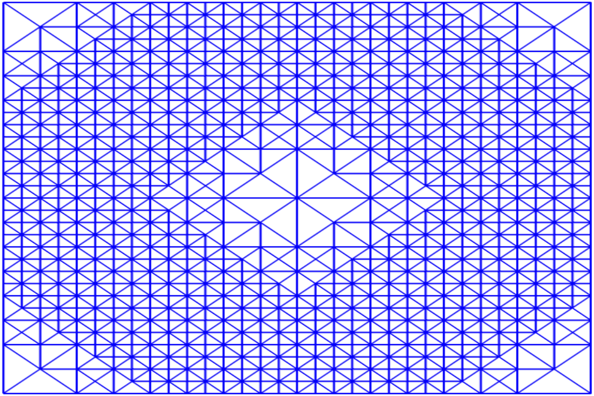
$$f(x,y) = \frac{1}{|x+\varepsilon|^k}\,\frac{1}{|1+\varepsilon-x|^k}$$
$$k = 1$$

- Resolución, en cada nivel, por un método directo
- Refinamiento según el gradiente del segundo miembro.

Test 5

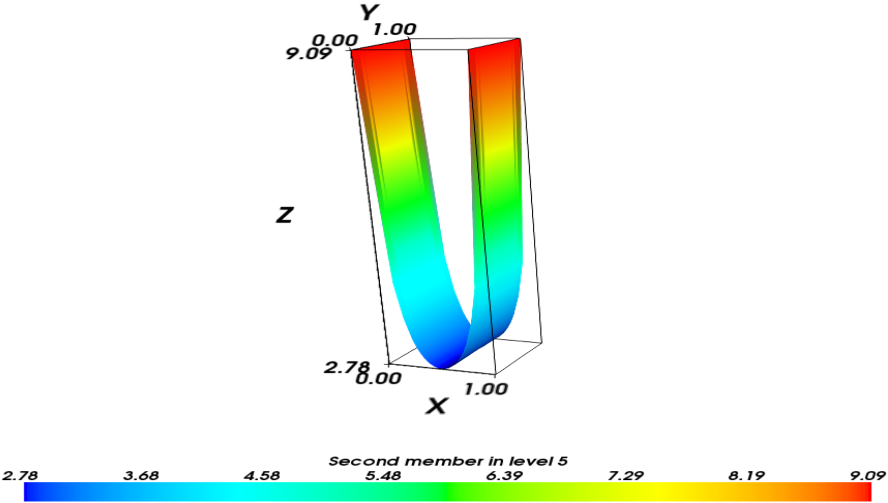
$$\varepsilon = 10^{-1}$$

Mesh 5: 901 nodes, 1712 elements



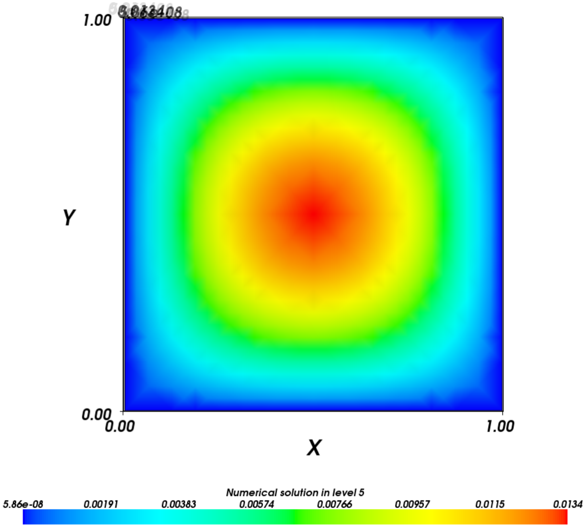
Test 5

Segundo miembro, $\varepsilon = 10^{-1}$



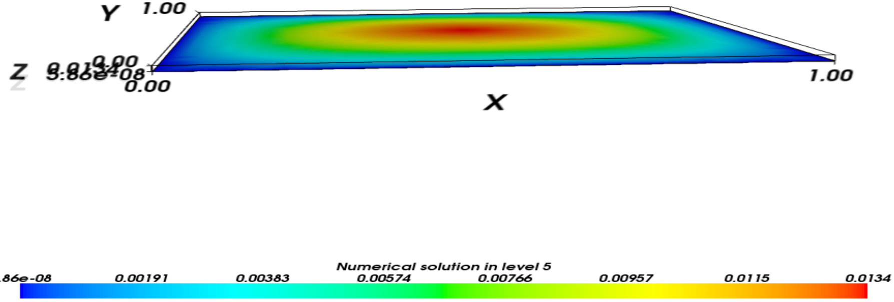
Test 5

Aproximación numérica de la solución, $\varepsilon = 10^{-1}$



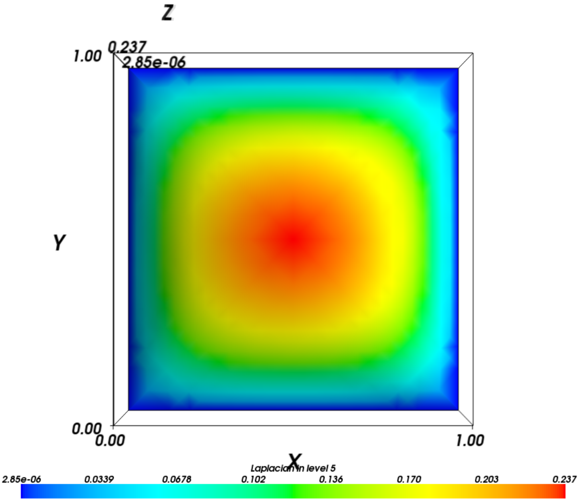
Test 5

Aproximación numérica de la solución, $\varepsilon = 10^{-1}$



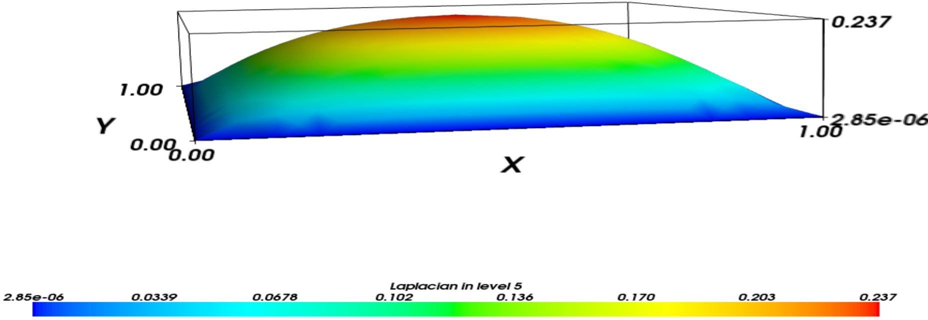
Test 5

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-1}$



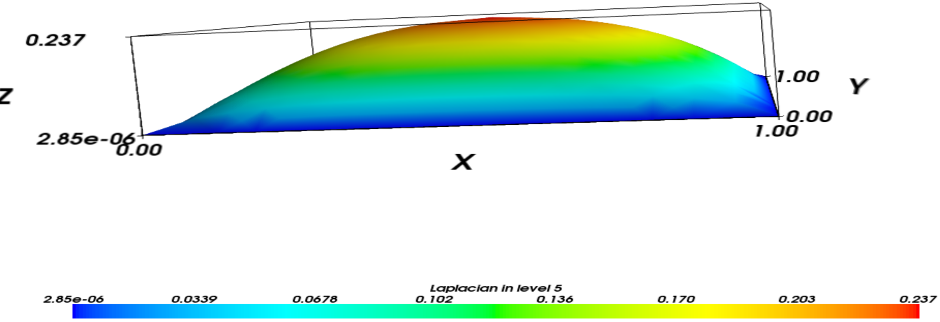
Test 5

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-1}$



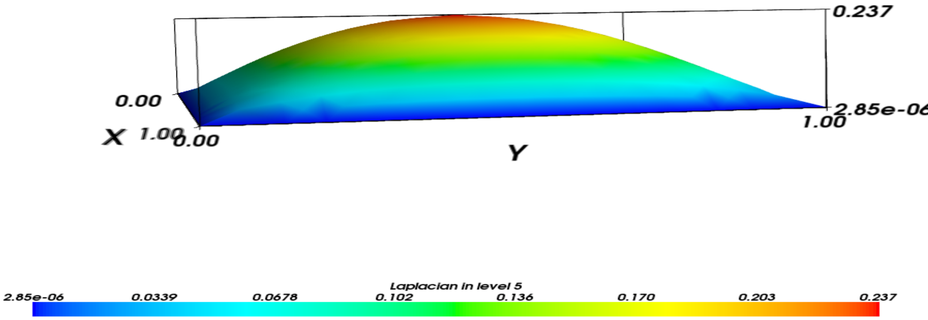
Test 5

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-1}$



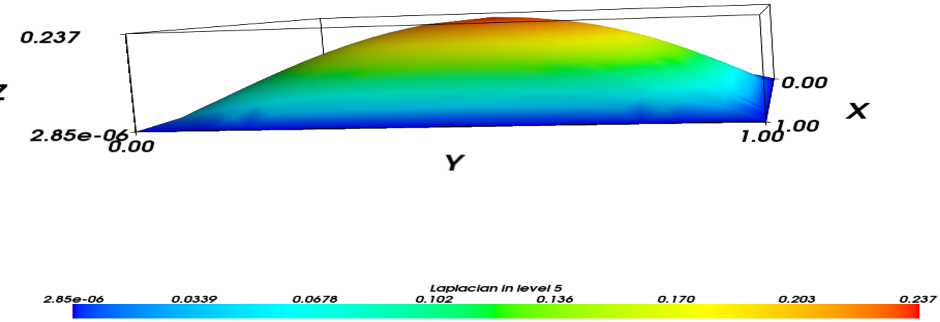
Test 5

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-1}$



Test 5

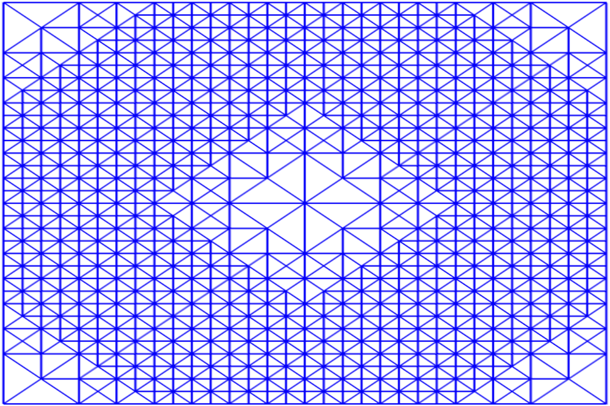
Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-1}$



Test 6

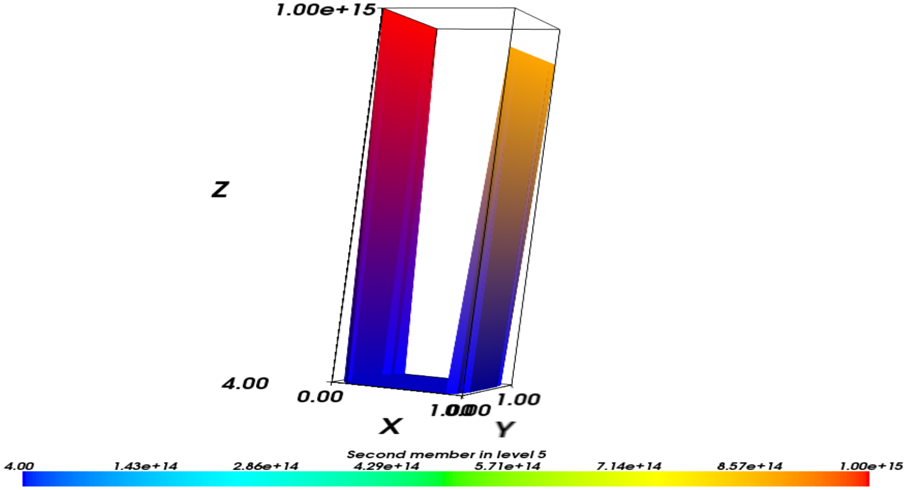
$$\varepsilon = 10^{-15}$$

Mesh 5: 901 nodes, 1712 elements



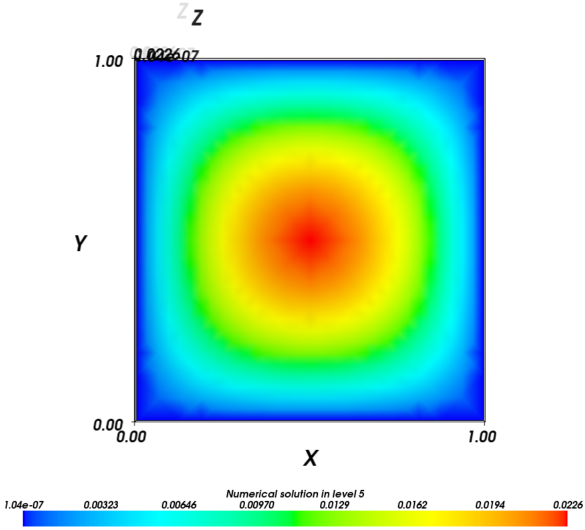
Test 6

Segundo miembro, $\varepsilon = 10^{-15}$



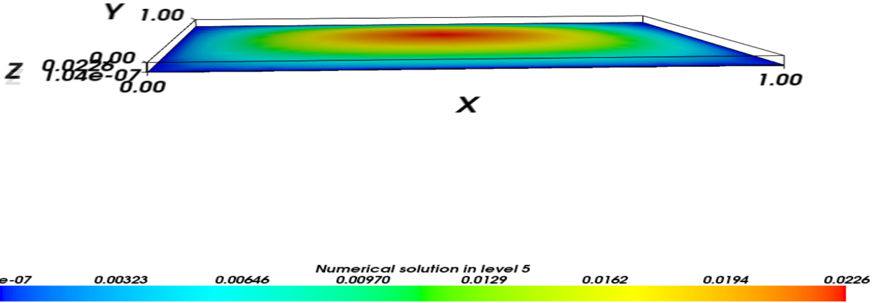
Test 6

Aproximación numérica de la solución, $\varepsilon = 10^{-15}$



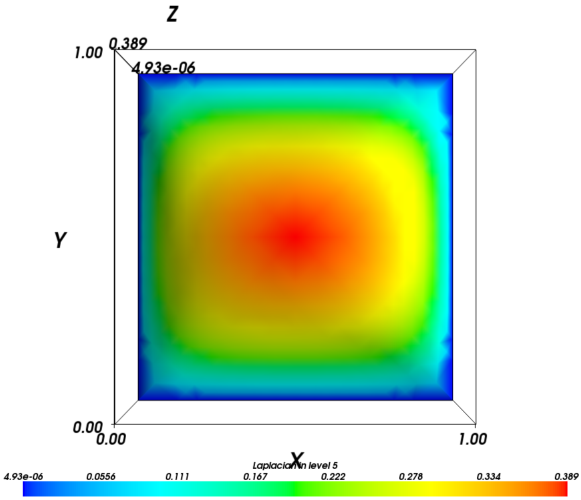
Test 6

Aproximación numérica de la solución, $\varepsilon = 10^{-15}$



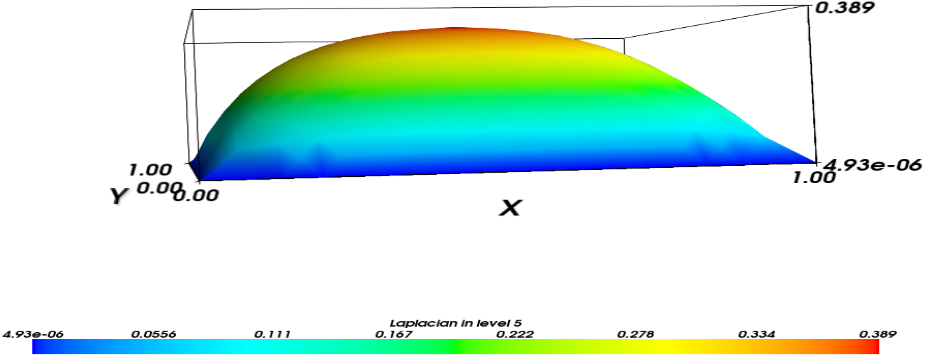
Test 6

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$



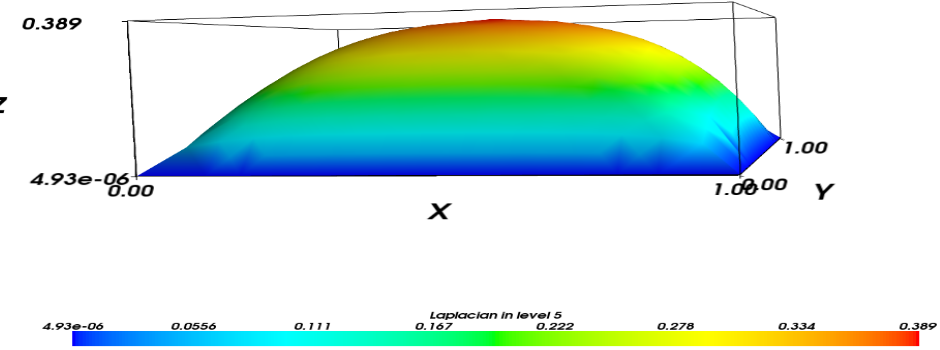
Test 6

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$



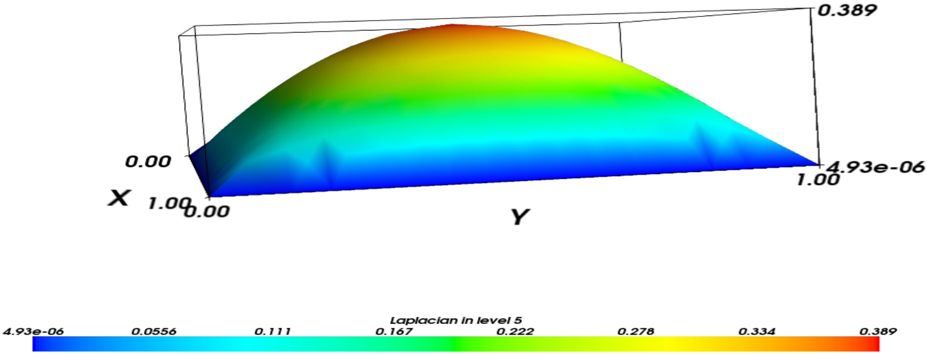
Test 6

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$



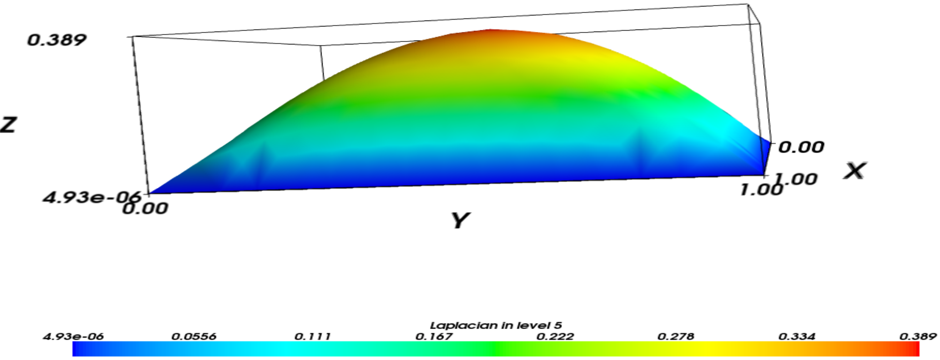
Test 6

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$

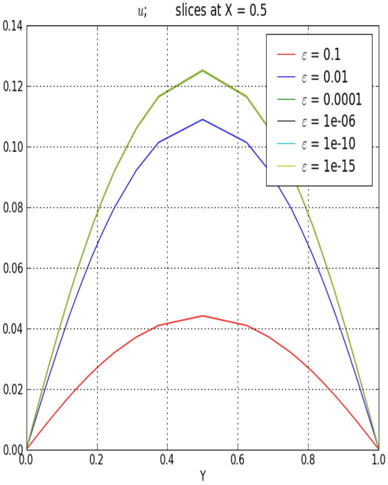


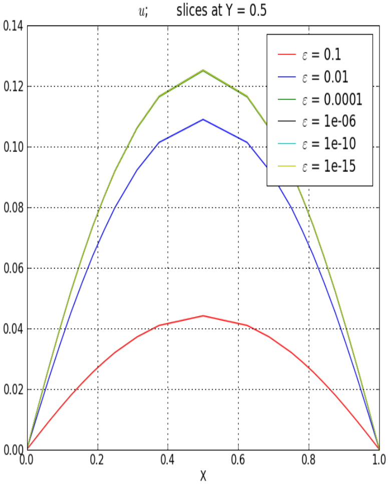
Test 6

Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-15}$

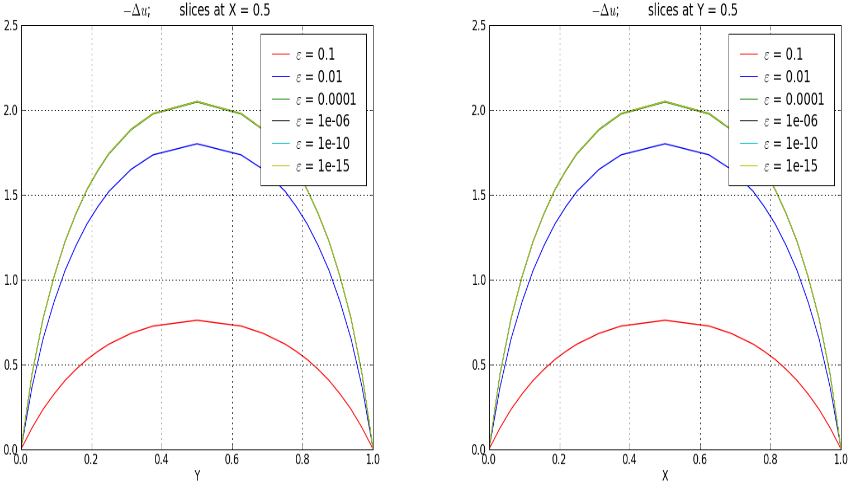


Tests with rhs exploiting on the whole boundary

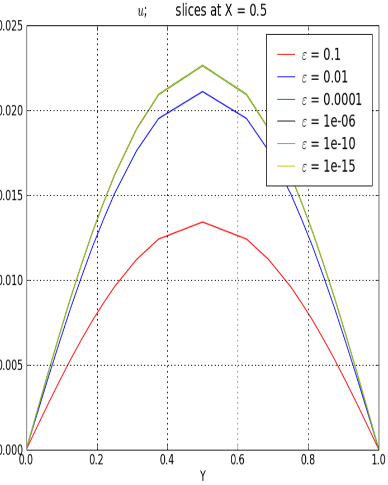


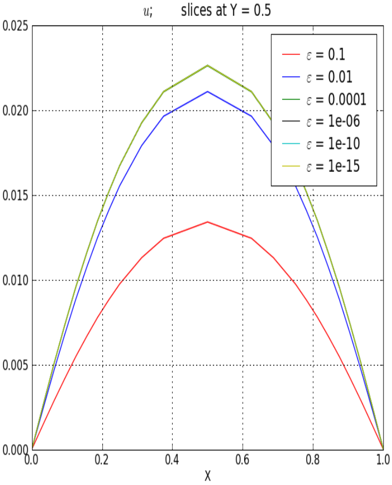


Tests with rhs exploiting on the whole boundary



Tests with rhs exploiting on $x = 0$ and $x = 1$





Tests with rhs exploiting on $x = 0$ and $x = 1$