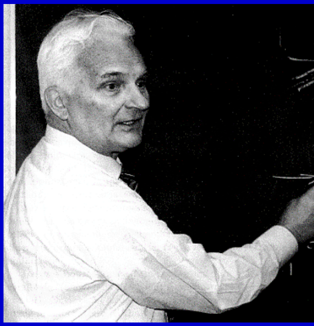
Finite extinction time for the solutions of some parabolic equations with a singular term

Y. Belaud (Tours)

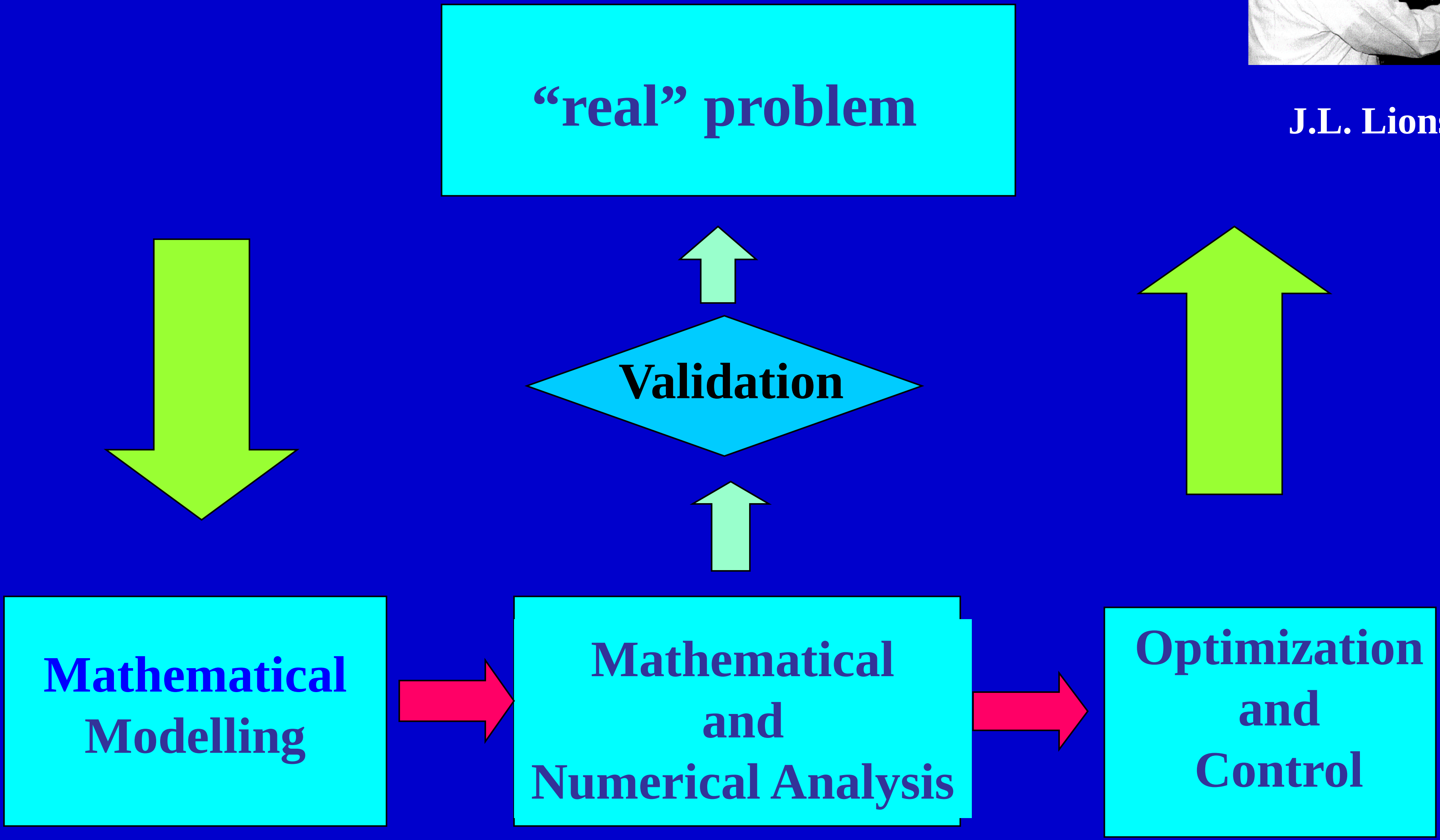
and

J.I. Díaz (UCM)

“The universal trilogy” in Applied Mathematics



J.L. Lions



I - Introduction

II - Estimating the extinction time for abstract Cauchy Problems

III - Singular parabolic equations

I - Introduction

A function $u \in C([0,+\infty) : L^1_{loc}(\Omega))$ satisfies the finite extinction time property if there exists a time $T > 0$ such that $u(x,t) = 0$ a.e. in Ω, $\forall t \geq T$.

In that case, the extinction time of u is defined as
$\inf\{T,\ \forall t \geq T,\ u(x,t) = 0 \text{ a.e. in } \Omega\}$.

A general exposition on the “finite extinction time” (Antonsev-Díaz-Shmarev 2002)

Abstract results for multivalued operators (Brezis (1974), Díaz (1980))

A new method: semi-classical method: “The asymptotic behaviour in time of solutions of some parabolic equations is strongly related to a family of "first eigenvalue type problems"

V.A. Kondratiev and L. Véron (1997), Y.Belaud, B. Helffer and L. Véron (2001), ...

A very simple ODE example

Let A be a symmetric real matrix of eigenvalues $\lambda_1 \leq ... \leq \lambda_N$.

Let $\lambda_1 = \inf_{||v||^2=1} (Av, v)$ Rayleigh quotient.

$$\frac{du}{dt} + Au = 0. \qquad \frac{1}{2}\frac{d}{dt}||u||^2 + (Au, u) = 0. \quad \frac{1}{2}\frac{d}{dt}||u||^2 + \left(A\frac{u}{||u||}, \frac{u}{||u||}\right)||u||^2 = 0.$$

$$\frac{1}{2}\frac{d}{dt}||u||^2 + \lambda_1 ||u||^2 \leq 0. \qquad ||u(t)|| \leq ||u_0|| e^{-\lambda_1 t}.$$

Definition Given $h > 0$, $\lambda_1(h) = \inf_{||v||^2=h} (Av, v)$.

Consequence : for all $t \geq 0$, $(Au(t), u(t)) \geq \lambda_1(||u(t)||^2)$.

Moreover, since A is linear, $\lambda_1(h) = \lambda_1\, h$.

Let H be an Hilbert space with its inner product $(\ ,\ )$, A a maximal monotone operator (Brezis 1973) of domain $D(A)$ and "principal section" A° given by $A^\circ u = \mathrm{Proj}_{Au} 0$, i.e., $||A^\circ u|| = \min_{v \in Au} ||v||$, $\forall u \in D(A)$.

We assume $u_0 \in D(A)$ with $u_0 \neq 0$ and consider the abstract Cauchy problem

$$\begin{cases} \dfrac{du}{dt} + Au \ni 0, \\ u(0) = u_0. \end{cases}$$

We denote by $T(u_0)$ the extinction time ($T(u_0) = +\infty$ if $||u||$ never vanishes). Since $u_0 \neq 0$, $T(u_0) > 0$.

Definition : $\lambda_1(h) = \inf_{||v||^2 \geq h} (A^\circ v, v)$ for all $0 \leq h \leq ||u_0||^2$.

Remarks :

1) The condition $0 \leq h \leq ||u_0||^2$ implies that the set $\{v, ||v||^2 \geq h\}$ is not empty.

2) $h \mapsto \lambda_1(h)$ is a nondecreasing function (so it is a continuous function except on at most a countable set).

3) If we suppose that $0 \in A0$ then $\lambda_1(h) \geq 0$.

Theorem : Assume that

$$\lambda_1(h) > 0 \text{ for all } 0 < h \leq ||u_0||^2.$$

Then

$$\frac{||u_0||}{||A^\circ u_0||} \leq T(u_0) \leq \frac{1}{2}\int_0^{||u_0||^2} \frac{1}{\lambda_1(h)} dh.$$

Idea of the proof Formally,

$\frac{1}{2}\frac{d}{dt}||u||^2 + (A^\circ u, u) = 0$ a.e. in $[0, +\infty)$.

$\frac{1}{2}\frac{d}{dt}||u||^2 + \lambda_1(||u||^2) \leq 0$ a.e. in $[0, +\infty)$.

$1 \leq -\frac{\frac{d}{dt}||u||^2}{2\lambda_1(||u||^2)}$ a.e. in $[0, T(u_0))$. Integrating between 0 and $T(u_0)$ gives the result.

$$||u_0|| = \left|\left|\int_0^{T(u_0)} \frac{du}{dt}(t)\ dt\right|\right| = \left|\left|\int_0^{T(u_0)} A^\circ u(t)\ dt\right|\right| \leq \int_0^{T(u_0)} ||A^\circ u(t)||\ dt \leq T(u_0)\ ||A^\circ u_0||.$$

Corollary 1 : Let $A = \partial\varphi$ where φ is a lower semi-continuous convex proper function. Assume

$$u_0 \in \overline{D(A)} \text{ and } \lambda_1(h) > 0 \text{ for } 0 < h \leq ||u_0||^2.$$

Then,

$$T(u_0) \leq \frac{1}{2}\int_0^{||u_0||^2} \frac{1}{\lambda_1(h)} dh.$$

Moreover, if u_0 belongs to $D(\varphi)$ then

$$\frac{||u_0||^2}{\varphi(u_0)} \leq T(u_0).$$

Corollary 2

A k-homogeneous ($A(au)=a^k A(u)$). Assume that $\lambda(1)=\inf\{(A°v,v)$ with $||v||^2 \geq 1\}$ is finite. Then if $(-1)<k<1$ there is extinction in finite time

Remark. If $k>1$ there is not extinction in finite time.

N. Alikakos and R. Rostamian (1982)

Parabolic equations with a singular term.

Consider the abstract perturbated problem

$$\begin{cases} \dfrac{du}{dt} + Au + Bu \ni 0, \\ u(0) = u_0 \neq 0. \end{cases}$$

We define $\bar{\lambda}_1(h) = \inf\limits_{||v||^2 \geq h} (A^\circ v, v) + (Bv, v)$ for all $0 \leq h \leq ||u_0||^2$.

Theorem 2 : Assume that

$$\bar{\lambda}_1(h) > 0 \text{ for all } 0 < h \leq ||u_0||^2.$$

Then

$$T(u_0) \leq \frac{1}{2} \int_0^{||u_0||^2} \frac{1}{\bar{\lambda}_1(h)} dh.$$

III - Application to some singular parabolic equations

Given $-1 < q < 1$ and $a(x)$ a nonnegative, measurable and bounded function on Ω (bounded regular domain), we consider

$$\begin{cases} u_t - \Delta u + a(x) u^q \chi_{u>0} &= 0 \quad \text{in} \quad \Omega \times (0, +\infty), \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0, +\infty), \\ u(x,0) &= u_0(x) \quad \text{on} \quad \Omega, \end{cases}$$

$H = L^2(\Omega)$, $Au = -\Delta u$ and $Bu = a(x) u^q \chi_{u>0}$

$$\lambda_1(h) = \inf \left\{ \int_\Omega |\nabla v|^2 + a(x)|v|^{1+q}\, dx, v \in W_0^{1,2}(\Omega), ||v||^2_{L^2(\Omega)} = h \right\}.$$

It is easy to see that there exist a real number $\nu(h)$ and a nonnegative function $v_h \in W_0^{1,2}(\Omega)$ with $||v_h||^2_{L^2(\Omega)} = h$ such that

$$\lambda_1(h) = \int_\Omega |\nabla v_h|^2 + a(x)|v_h|^{1+q}\, dx,$$

and

$$\begin{cases} -\Delta v_h + \frac{1+q}{2} a(x) v_h^q \chi_{v_h>0} &= \nu(h) v_h \quad \text{in} \quad \Omega, \\ v_h &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Previous remarks for two special cases

1) If $a(x) \geq \gamma > 0$ on Ω then any solution of the parabolic equation vanishes in a finite time.
2) If, by the contrary, $a(x) = 0$ on an open domain $\omega \subset \Omega$, then defining,

$$-\Delta\psi = \lambda_\omega \psi \text{ in } \omega, \psi = 0 \text{ on } \partial\omega$$

we get that if $u_0 \geq \psi$ then $u(x,t) \geq \psi e^{-\lambda_\omega t}$ and so $T(u_0) = +\infty$.

Our next result consider, for instance, the case in which $a(x_0) = 0$ in an isolated point $x_0 \in \Omega$.

Theorem 3. Let $N \geq 3$ and let Ω be a C^1 bounded domain of $\mathbb{R}^N$. Assume that $a(x)$ is measurable, positive a.e., bounded on Ω and such that

$$\left|\ln \frac{1}{a}\right|^s \in L^1(\Omega),$$

for some $s > N/2$. Then any solution vanishes in a finite time.

Moreover, if $M \geq ||a||_{L^\infty(\Omega)}$ is such that

$$\forall h \in \left(0, ||u_0||^2_{L^2(\Omega)}\right], \; \frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}} \frac{K(N)^{1-q}}{(C(N,\Omega))^{\frac{(1-q)N}{4}}} < M,$$

then, the extinction time $T(u_0)$ can be estimated by

$$T(u_0) \leq \frac{1}{2} \int_0^{||u_0||^2_{L^2(\Omega)}} \frac{dh}{h\left((\gamma-\beta)\ln h - \ln K - C\frac{2\beta}{\alpha}\ln\left((\gamma-\beta)\ln\left(\frac{h}{||u_0||^2_{L^2(\Omega)}}\right)+y_0\right)\right)^{\frac{2}{\alpha}}},$$

where

$$\alpha = \frac{N}{s}, \quad \beta = \frac{4+(1-q)N}{4}, \quad \gamma = \frac{2(1+q)+(1-q)N}{4},$$

$$C = C(N,\Omega)^{\frac{N}{2s}}\left(\int_\Omega \left(\ln\frac{M}{a(x)}\right)^s dx\right)^{\frac{1}{s}}, \quad K = \frac{K(N)^{1-q}}{M\,(C(N,\Omega))^{\frac{(1-q)N}{4}}},$$

and

$$y_0 + C\frac{2\beta}{\alpha}\ln y_0 = (\gamma-\beta)\ln\left(||u_0||^2_{L^2(\Omega)}\right) - \ln K.$$

Remarks:

1 $a(x) = \exp\left(\frac{-1}{|x|^\alpha}\right)$. $\alpha < 2$ gives $T(u_0) < +\infty$.

2 we prove that

$$\lambda_1(h) \geq C_2\, h\,(-\ln h)^{\frac{2}{\alpha}}, \quad C_2 > 0, \qquad \text{which implies that } \int_0^1 \frac{dh}{\lambda_1(h)} \text{ converges.}$$

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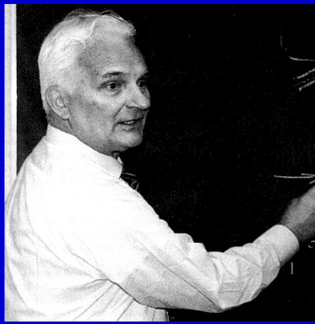
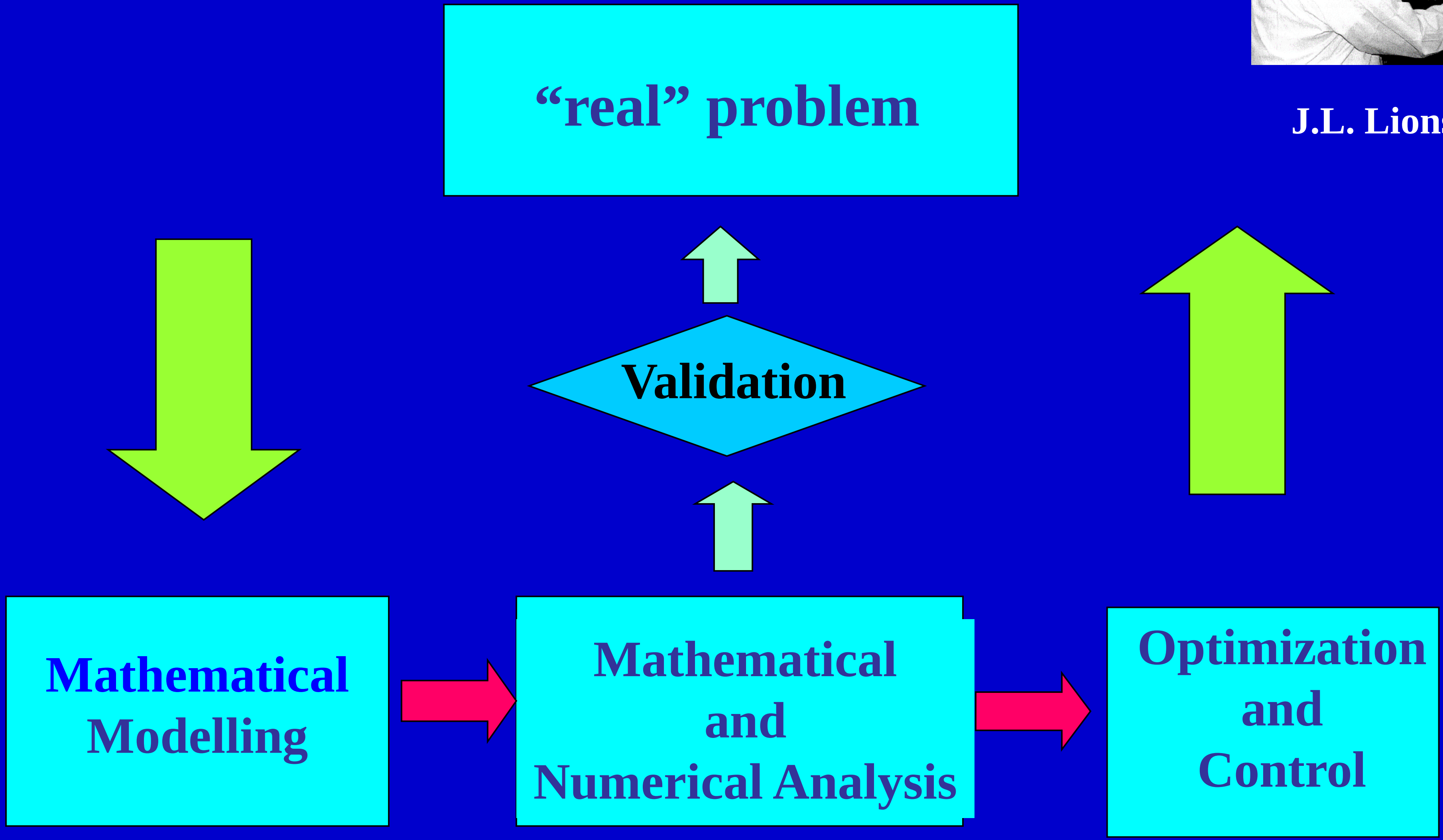


J.L. Lions



Remark. Some related numerical experiences: **control for EBM**

J.I. Díaz and A. M. Ramos, In CD-Rom Proceedings European Congress Computational Methods in Applied Sciences and Engineering (ECCOMAS 2000).

$$y_t - y_{xx} + y^3 = arctg y + u(t)\delta_{1/2} \text{ in } (0,1)\times(0,T),$$

$$y(0,t) = y(1,t) = 0 \qquad t \in (0,T),$$

$$y(x,0) = y^0(x) \qquad x \in (0,1),$$

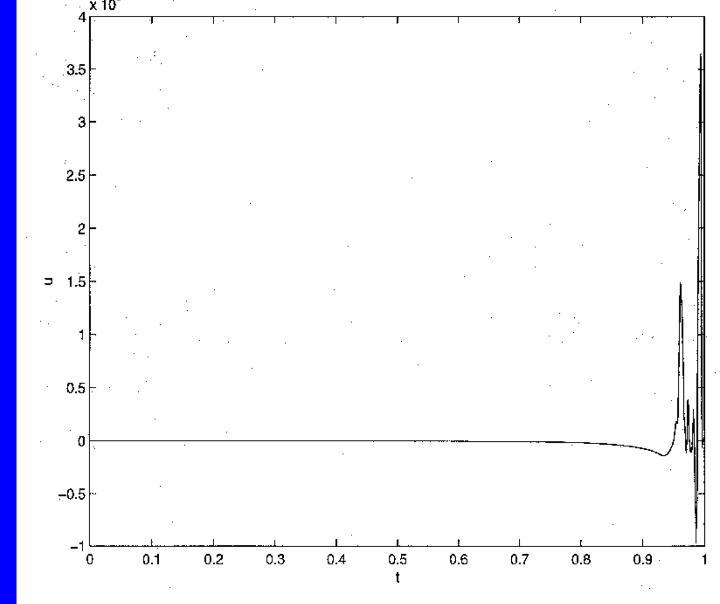
$$J_k(u) = \frac{1}{2}\|u\|_{L^2(0,T)} + \frac{k}{2}\|y(T,.:u) - y_d\|_{L^2(0,1)}$$

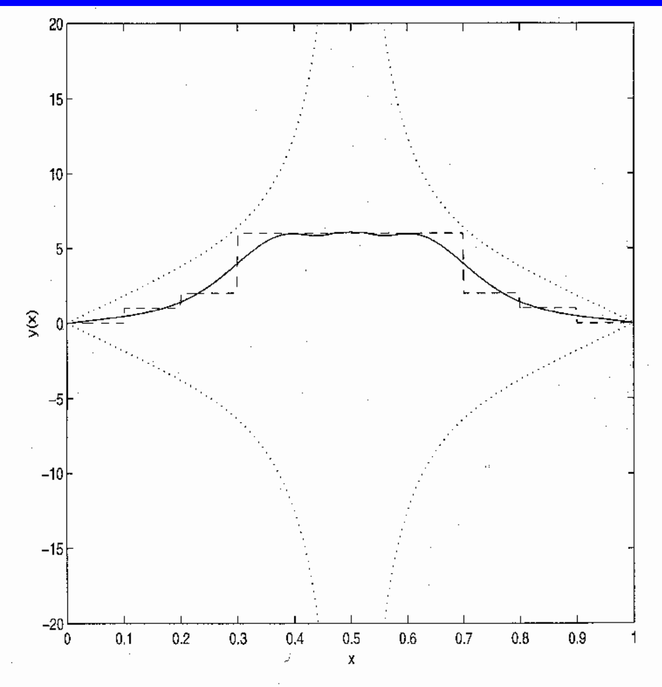
$$k = 10^{12}$$

The cost of control decreases with complexity:

J.I. Díaz and J.L. Lions (Semilinear heat equation with blow-up)

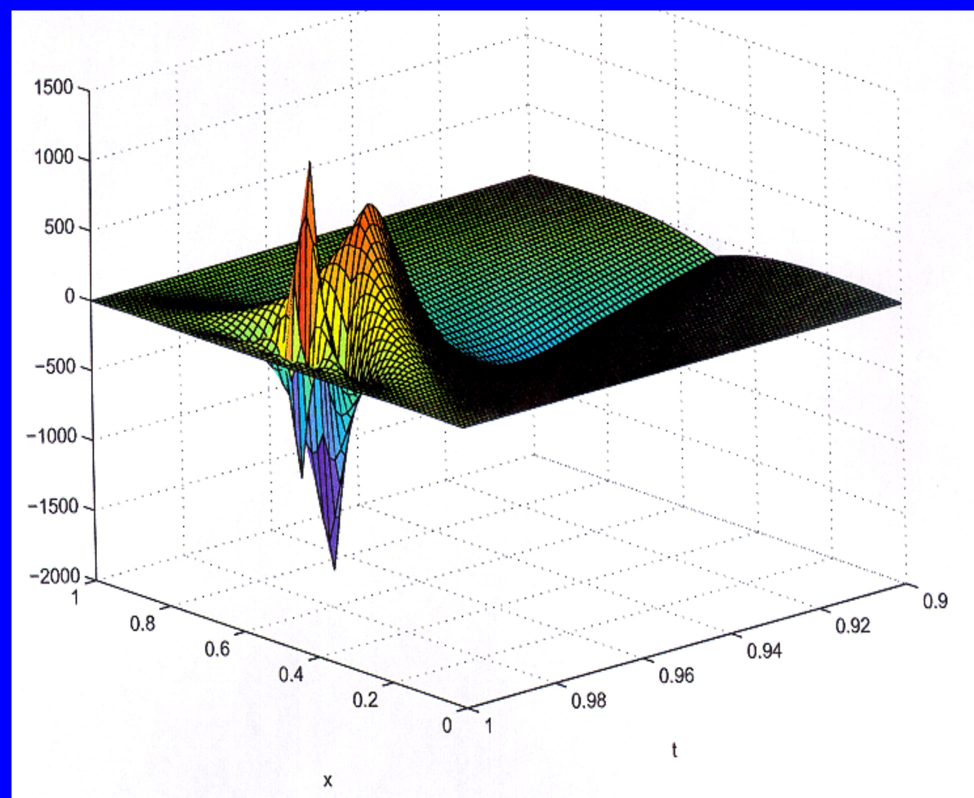
CRAS, 2000, IFIP (Chemonix, 2000: Birkhauser) control





Target state

State



Thanks for your attention