XXIII Congreso de Ecuaciones Diferenciales y Aplicaciones
XIII Congreso de Matemática Aplicada
Castellón, 9-13 septiembre 2013
(pp. 1–7)

On a nonlinear variation of constants formula and its application to the control of blowing-up trajectories

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Palabras clave: 34A34; 34A60; 49J52

Resumen

We first consider blowing-up solutions $y^0(t)$, $t \in [0, T_{y^0})$, of some ODEs

$$P(T_{y^0}): \quad \frac{dy}{dt}(t) = f(y(t)), y(0) = y_0,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz function and $d \geq 1$. The controllability question we analyze in this work is the following: given $\epsilon > 0$, can we find a continuous deformation of $y^0(t)$, built as solution of the control perturbed problem obtained by replacing $f(y(t))$ by $f(y(t)) + u(t)$, for a suitable control $u \in L^1_{loc}(0, +\infty : \mathbb{R}^d)$ such that $y(t) = y^0(t)$ for any $t \in [0, T_{y^0} - \epsilon]$ and such that $y(t)$ is continued to the whole $[0, +\infty)$? We shall also mention several applications to the case of some nonlinear blowing-up parabolic problems and improve a previous work of the authors [4].

1. Introduction

We consider blowing-up solutions $y^0(t)$, $t \in [0, T_{y_0})$, of some ODEs

$$P(f, y_0) = \begin{cases} \frac{dy}{dt}(t) = f(y(t)) & \text{in } \mathbb{R}^d, \\ y(0) = y_0, \end{cases}$$

where $d \geq 1$, $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz function superlinear near the infinity

$$f(y)y \geq C\,|y|^{p+1} \text{ if } |y| > k, \text{ for some } p > 1 \text{ and } C, k > 0.$$

It is well known that the solutions of $P(f, y_0)$ develop blow-up processes in the sense that the maximal existence interval is of the form $[0, T_{y_0})$, for some finite time T_{y_0} (i.e. there is a complete blow-up after T_{y_0}). From the point of view of Control Theory, it is easy to see (by arguing as in [5]) that we can avoid the blow-up phenomenon by introducing a suitable control function $u(t)$. To be more precise, for any small enough $\epsilon > 0$ we can find a continuous deformation $y(t)$ of the given trajectory, $y^0(t)$, built as solution of the control perturbed problem

$$P(f, y_0, u) = \begin{cases} \frac{dy}{dt}(t) = f(y(t)) + u(t) & \text{in } \mathbb{R}^d, \\ y(0) = y_0, \end{cases}$$

for a suitable control $u \in L^1_{loc}(0, +\infty : \mathbb{R}^d)$ and defined on the whole interval $[0, +\infty)$ such that $y(t) = y^0(t)$ for any $t \in [0, T_{y_0} - \epsilon]$. Indeed, fix any $T_e > T_{y_0} - \epsilon$ and let us consider $w \in C^1[0, +\infty)$ such that $w(t) = y^0(t)$ for any $t \in [0, T_{y_0} - \epsilon]$ and $w(t) = 0$ for any $t \in [T_e, +\infty)$. Then, defining $u(t) = \frac{dw}{dt}(t) - f(w(t))$ if $t \in (T_{y_0} - \epsilon, +\infty)$ we get the required conditions and that, in fact, $y(t) = w(t) = 0$ for any $t \in [T_e, +\infty)$.

In this work our goal is completely different since we do not try to avoid the blow-up phenomenon but to control it in such a way that the solutions let defined in the whole interval $[0, +\infty)$ at least as a $L^1_{loc}(0, +\infty : \mathbb{R}^d)$ function. We shall show that we can çontrol the explosions"by allowing a more singular class of controls.

Definition. *We say that the trajectory $y^0(t)$ of problem $P(f, y_0)$, with blow-up time T_{y^0}, has a controlled explosion if for any small enough $\epsilon > 0$ we can find a continuous deformation, $y(t)$, of the trajectory $y^0(t)$, built as solution of the control perturbed problem $P(f, y_0, u)$, for a suitable control $u \in W^{-1,q'}_{loc}(0, +\infty : \mathbb{R}^d)$* [the dual space of $W^{1,q}_{0,loc}(0, +\infty : \mathbb{R}^d)$], *for some $q > 1$, such that $y(t) = y^0(t)$ for any $t \in [0, T_{y_0} - \epsilon]$, $y(t)$ also blows-up at $t = T_{y_0}$ but $y(t)$ can be extended beyond T_{y_0} as a function $y \in L^1_{loc}(0, +\infty : \mathbb{R}^d)$.*

Theorem 1. *Assume f locally Lipschitz continuous and superlinear. Then, for any $y_0 \in \mathbb{R}^d$ the blowing up trajectory $y^0(t)$ of the associated problem $P(f : y_0)$ has a controlled explosion by means of the control problem $P(f, y_0, u)$.*

Our main tools are the study of a suitable delayed feedback problems (in the spirit of a previous work by the authors [Casal, Díaz and Vegas [4]] and the application of a powerful *nonlinear variation of constants formula.* This type of formula was first established in the literature for nonlinear terms of class C^2 [Alekseev [2], Laksmikantham and Leela [6], ...]. In this work we shall show that, as a matter of fact, the formula holds also for Lipschitz functions f (which at this stage can be assumed to be in fact globally Lipschitz) and with a very general perturbation term (which in fact can be even multivalued). For instance, given such a f and a family of maximal monotone operators $\beta(t, y)$, on the space $H = \mathbb{R}^d$, with $\beta(t, \cdot) \in L^1_{loc}(0, +\infty : \mathbb{R}^d)$, we consider the perturbed problem

$$P^*(f, \beta, \xi) = \begin{cases} \frac{dy}{dt}(t) \in f(y(t)) + \beta(t, y(t)), \text{ in } \mathbb{R}^d, \\ y(t_0) = \xi. \end{cases} \tag{1}$$

We know that once that f is globally Lipschitz function, the solutions of $P(f, \beta, \xi)$ are well defined, as absolutely continuous functions on $[0, T]$, for any given $T > 0$ (this is an easy consequence of the general theory: see [3]). Now, we reformulate the trajectory $y^0(t)$ in more general terms (by modifying the initial time and the initial condition) as

$y^0(t) = \phi(t, t_0, \xi)$ with $\phi(t, t_0, \xi)$ the unique solution of the ODE

$$P^*(f, 0, \xi) = \begin{cases} y'(t) = f(y(t)) & \text{in } \mathbb{R}^d, \\ y(t_0) = \xi. \end{cases} \tag{2}$$

We introduce the formal notation $\Phi(t, t_0, \xi) = \partial_\xi \phi(t, t_0, \xi)$, where ∂_ξ denotes partial differentiation. Then we shall prove:

Theorem 2. *The flow map ϕ is Lipschitz continuous, Φ is absolutely continuous and the solution $y(t)$ of the "perturbed problem" $P^*(f, \beta, \xi)$ has the integral representation*

$$y(t) = y^0(t) + \int_{t_0}^{t} \Phi(t, s, y(s))\beta(s, y(s))ds, \text{ for any } t \in [0, T], \tag{3}$$

where $y^0(t) = \phi(t, t_0, \xi)$ is the solution of the "unperturbed" problem $P^*(f, 0, \xi)$.

In the above formula we assumed, for simplicity, that $\beta(t,\cdot)$ is single-valued but a suitable similar expression can be stated if $\beta(t,\cdot)$ is multivalued. Applications of this arguments to parabolic partial differential equations (see Remark 3) will be presented elsewhere.

2. Case 1. $f \in C^2$ and superlinear (e.g. $f(y) = |y|^{p-1} y$ with $p > 1$).

For the sake of presentation we shall start with the study of regular superlinear functions f.

Theorem 3. *Assume $f \in C^2$ and superlinear. Then, for any $y_0 \in \mathbb{R}^d$ the blowing up trajectory $y^0(t)$ of the associated problem $P(f : y_0)$ has a controlled explosion.*

Proof. Step 1 (the strategy). Define $\tau = T_{y_0} - \epsilon$. We make the change of variable

$$\widetilde{t} = t - \tau$$

and consider the delayed problem

$$\widetilde{P}(f, y^0, B) = \begin{cases} y'(t) = f(y(t)) + B'(t)g(y(t - \tau)), & 0 < t < \tau \\ y(\theta) = \chi(\theta), & -\tau \leq \theta \leq 0 \end{cases} \tag{4}$$

(where, for simplicity we denote again $\widetilde{t}$ by t, so that, for any $-\tau \leq \theta \leq 0$ we are identifying $\chi(\theta)$ with $\chi(\theta + T_{y_0} - \epsilon)$, for some suitable functions $B(t)$,and where $g(r)$ is any C^2 function (for instance $g(r) = r$). Our goal is to show that we can chose the control term

$$u(t) := B'(t)g(y(t - \tau))$$

such that the solution of $\widetilde{P}(f, y^0, B)$ is defined on the whole interval $[0, \tau)$ and that $u \in W^{-1,q'}(0, \tau : \mathbb{R}^d)$. Since $y(t - \tau) = \chi(t - \tau)$ for any $t \in [0, T_{y_0} - \epsilon]$, this will prove the result by iteration on the intervals $\tau < t < 2\tau, ..., n\tau < t < (n + 1)\tau$, $n \in \mathbb{N}$.

Step 2 (choice of function B and reformulation as neutral equation). Given $q > 1$, $a > 0$ and $\alpha \in (0, \frac{1}{q})$ and a continuous function m (taken, for instance, in order to have $B(0) = 0$) we define

$$B(t) = \frac{a}{|t-t^*|^\alpha} + m(t), \quad t \in [0,\tau], \tag{5}$$

with $t^* = \epsilon$ (i.e. $t = T_{y^0}$ in the original time scale). We assume that $t^* \in (0,\tau)$, i.e. $2\epsilon < T_{y^0}$. As in [4] we can reformulate $\widetilde{P}(f, y^0, B)$ as the neutral problem

$$\begin{cases} \frac{d}{dt}\left[y(t) - B(t)g(y(t-\tau))\right] \\ \qquad\qquad = f(y(t)) - B(t)\frac{d}{dt}\left[g(y(t-\tau))\right], t > 0, \\ y(\theta) = \chi(\theta), \quad -\tau \le \theta \le 0 \end{cases} \tag{6}$$

Instead, we will change our strategy and make use of a very useful, but little-known mathematical device: *Alekseev's nonlinear variation of constants formula* [2]. We now briefly recall this result in a very simple setting (a more general statements will be obtained in the next section).

Proposition. [Alekseev's formula, [2]] *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^2 and $G : \mathbb{R} \to \mathbb{R}$ be L^1_{loc}. Let $y = \phi(t, t_0, \xi)$ represent the unique solution of the ODE*

$$\begin{cases} y' = f(y(t)), \\ y(t_0) = \xi, \end{cases} \tag{7}$$

and let $\Phi(t, t_0, \xi) = \partial_\xi \phi(t, t_0, \xi)$, where ∂_ξ denotes partial differentiation. Then ϕ is C^2, Φ is C^1 and the solution $z(t)$ of the so-called "perturbed problem"

$$\begin{cases} z' = f(z(t)) + G(t), \\ z(t_0) = \xi, \end{cases} \tag{8}$$

has the integral representation

$$z(t) = y(t) + \int_{t_0}^t \Phi(t, s, z(s))G(s)ds, \tag{9}$$

where $y(t) = \phi(t, t_0, \xi)$ is the "unperturbed" or "reference" solution.

Remark. Notice that $\Phi(t, t_0, \xi)$ satisfies $\Phi(t, t, \xi) = 1$. Notice also that Alekseev's formula is usually stated under stronger regularity conditions on G. However, it is very simple to check by direct differentiation that the function $z(t)$ defined by (9) is an absolutely continuous solution of the (Carathéodory) equation. Alekseev's formula is usually applied to the more ambitious setting of having G depending on t *and* z, which is typical of control theory. (9) then becomes an integral equation and a more delicate analysis is required.

Fortunately, we can consider the retarded term as an external "forcing"

$$G(t) = B'(t)g(\xi(t-\tau)), \tag{10}$$

and by setting $t_0 = 0$, $\xi = z(0) = \chi(0)$, $y(t) = \phi(t, 0, \xi)$, write (formally):

$$z(t) = y(t) + \int_0^t \Phi(t, s, z(s))B'(s)g(y^0(s-\tau))ds, \tag{11}$$

and integrate by parts:

$$\begin{aligned} z(t) &= y(t) + [\Phi(t,s,z(s))B(s)g(\chi(s-\tau))]_{s=0}^{s=t} \\ &\quad - \textstyle\int_0^t B(s)\frac{d}{ds}\left[\Phi(t,s,z(s))g(\chi(s-\tau))\right] ds \\ &= y(t) + \Phi(t,t,z(t))B(t)g(\chi(t-\tau)) \\ &\quad - \textstyle\int_0^t B(s)\frac{d}{ds}\left[\Phi(t,s,z(s))g(\chi(s-\tau))\right] ds. \end{aligned} \tag{12}$$

By the remark above, $\Phi(t,t,z(t)) = 1$. On the other hand, as we saw before, for $\chi \in W^{1,q}(-\tau,0)$ and $g \in C^1$ the composite function $s \mapsto g(\chi(s-\tau))$ is also $W^{1,q}(-\tau,0)$ and so is its product by the C^1 function $\Phi(t,s,z(s))$. Therefore, its derivative belongs to $L^q(-\tau,0)$ and the indefinite integral, as in all the previous cases, is an absolutely continuous function. This means that the integration by parts is legitimate and we may state the following result, which is an extension of the previous ones. We may summarize the previous comments in the following way:

The initial value problem

$$\widetilde{P}(f,\chi,B) = \begin{cases} y'(t) = f(y(t)) + B'(t)g(y(t-\tau)), & 0 < t < \tau \\ y(\theta) = \chi(\theta), & -\tau \le \theta \le 0 \end{cases} \tag{13}$$

with $f \in C^2(\mathbb{R}^2)$, $g \in C^1(\mathbb{R}^2)$ and initial function χ in $W^{1,q}(-\tau,0)$ has a precise integral sense in $[0,\tau]$ by means of the neutral equivalent equation and its unique solution z admits the integral representation

$$z(t) = y(t) + B(t)g(\chi(t-\tau)) - \int_0^t B(s)\frac{d}{ds}\left[\Phi(t,s,z(s))g(\chi(s-\tau))\right] ds, \tag{14}$$

(where $y(t) = \phi(t,0,\chi(0))))$. Then, for every $\xi \in W^{1,r}(0,\tau)$ (where $1/q + 1/r = 1$) the neutral Cauchy problem has a unique solution given by the identity (14). Therefore $z \in L^q(0,\tau)$ and $z(t)-\ B(t)g(\chi(t-\tau))$ is an absolutely continuous function and we may write symbolically

$$z(t) = B(t)g(\chi(t-\tau)) + AC \tag{15}$$

where “AC” means “an absolutely continuous function”. As a consequence, the singularities of the solution on $[0,\tau]$ are also singularities of B. Thus, in particular, let $t^* = \epsilon$ (notice that $t^* = T_{y^0}$ in the original scale of time), $0 < \alpha < 1$, let m be continuous on $[0,\tau]$ and let

$$B(t) = \frac{a}{|t-t^*|^\alpha} + m(t), \tag{16}$$

Since the initial function χ satisfies $\chi(t^*-\tau) = \chi(\epsilon) \neq 0$, then t^* is also a singularity of z and

$$z(t) \simeq \frac{a}{|t-t^*|^\alpha} g(\chi(\epsilon)), \quad \text{as } t \to t^*, \tag{17}$$

is an asymptotic expansion of z near $t^* = T_{y_0}$, which gives the qualitative picture of the behavior of the solution near singularities of B. Obviously, from the choice of α we get that the control $u(t) := B'(t)g(y(t-\tau))$ is in $W^{-1,q'}(0,\tau : \mathbb{R}^d))$.

3. Case 2. *Controllable explosions for* f locally Lipschitz and superlinear: a generalization of the *nonlinear variation of constants formula.*

The proof of Theorem 1 is exactly the same than the one of Theorem 3 once we let able to show Theorem 2. Notice that since what we need is merely to have a control of the way in which the solution growths near the blow-ups time T_{y_0} the proof of Theorem 2 is only needed for globally Lipschitz functions f.

Proof of Theorem 2. Let $f_n \in C^1(\mathbb{R}^d : \mathbb{R}^d)$ be a sequence approximating f in $W^{1,s}(\mathbb{R}^d : \mathbb{R}^d)$, for any $s \in [1, +\infty)$, and such that

$$\|\partial_x f_n(\cdot)\|_{L^\infty(\mathbb{R}^d:\mathcal{M}_{d\times d})} \leq \|\partial_x f(\cdot)\|_{L^\infty(\mathbb{R}^d:\mathcal{M}_{d\times d})} := M \text{ for any } n \in \mathbb{N} \text{ and} \tag{18}$$

(see, for instance, Adams [1]). Let $y_n^0 = \phi_n(t, t_0, \xi)$ be the unique solution of the unperturbed ODE

$$P^*(f_n, 0, \xi) = \begin{cases} y'(t) = f_n(y(t)) \ \text{ in } \mathbb{R}^d, \\ y(t_0) = \xi, \end{cases} \tag{19}$$

and let $\Phi_n(t, t_0, \xi) = \partial_\xi \phi_n(t, t_0, \xi)$,. Let us consider the sequence of perturbed problems

$$P^*(f_n, \beta, \xi) = \begin{cases} \frac{dy_n}{dt}(t) \in f_n(y_n(t)) + \beta(t, y_n(t)), \text{ in } \mathbb{R}^d, \\ y(t_0) = \xi. \end{cases} \tag{20}$$

Then, by the classical version of the Alekseev formula we know that

$$y_n(t) = y_n^0(t) + \int_{t_0}^{t} \Phi_n(t, s, y_n(s))\beta(s, y_n(s))ds, \text{ for any } t \in [0, T], \tag{21}$$

(as before, in the above formula we assumed, for simplicity, that $\beta(t,\cdot)$ is single-valued but a suitable similar expression can be obtained if $\beta(t,\cdot)$ is multivalued). But since $f_n \to f$ and f is locally Lipschitz we know that $y_n^0(\cdot) \to y_n^0(\cdot)$ and $y_n(\cdot) \to y_n(\cdot)$ strongly in $AC([0, T] : \mathbb{R}^d)$ for any fixed $T > 0$ (this is an easy application of Theorem 4.2 of Brezis [3]). Moreover since any maximal monotone operator is strongly-weakly closed we know that, at least, $\beta(\cdot, y_n(\cdot)) \rightharpoonup \beta(\cdot, y_n(\cdot))$ in $L^2(0, T : \mathbb{R}^d)$. Moreover, from the classical Peano theorem we know that there exists a $\Phi(t, s, y)$ such that

$$\Phi_n(t, \cdot, y_n(\cdot)) \to \Phi(t, \cdot, y(\cdot)), \text{ for a.e. } t \in (0, T),$$

strongly in $L^2(0, T : \mathcal{M}_{d\times d})$. Indeed, $\Phi_n(t, t_0, \xi)$ is the solution of the problem

$$\begin{cases} \Phi'(t) = H_n(t, t_0, \xi)\Phi(t) \ \text{ in } \mathcal{M}_{d\times d}, \\ \Phi(t_0) = I, \end{cases}$$

where

$$H_n(t, t_0, \xi) = \partial_x f_n(\phi_n(t, t_0, \xi)).$$

But, we know that, if M is given by (18) then

$$\|H_n(t, t_0, \xi)\|_{L^\infty(t_0,T:\mathcal{M}_{d\times d})} \ \leq M \ \text{ for any } t_0 \in (0, T) \text{ and for any } \xi \in \mathbb{R}^d.$$

Thus, by Gronwall inequality, there exists a positive constant $\widetilde{M} = \widetilde{M}(t_0, \xi)$ such that

$$\|\Phi_n(\cdot, t_0, \xi)\|_{W^{1,\infty}(0,T)} \leq \widetilde{M}$$

which implies that there exists a Lipschitz function $\Phi(t, s, \xi)$ such that $\Phi_n(t,\cdot, y_n(\cdot)) \rightharpoonup \Phi(t,\cdot, y(\cdot))$ in $W^{1,q}(0, T : \mathcal{M}_{d\times d})$ for any $q \in (1, \infty)$. This leads to the strong convergence in $L^2(0, T : \mathcal{M}_{d\times d})$. Then we can pass to the limit in formula (21) and get that

$$y(t) = y^0(t) + \int_{t_0}^{t} \Phi(t, s, y(s))\beta(s, y(s))ds, \text{ for any } t \in [0, T].$$

Remark 2. Notice that since our main interest is to study the asymptotic, near T_{y_0}, we do not need to identify the limit matricial function $\Phi(t, s, y)$. This is a complicated task over the set of points $y \in \mathbb{R}^d$ where f is not Frechet differentiable in y (see a nonlinear characterization in Mirica [7]).

Remark 3. Several applications to the case of the some nonlinear blowing-up parabolic problems of the type

$$(P_N) \quad \begin{cases} \frac{\partial y}{\partial t} - \Delta y = |y|^{p-1} y + u(t, x) & \text{for } (t, x) \in (0, +\infty) \times \Omega, \\ \frac{\partial y}{\partial n}(t, x) = 0, & \text{for } (t, x) \in (0, +\infty) \times \partial\Omega, \\ y(0, x) = y_0(x), & \text{for } x \in \Omega, \end{cases} \tag{22}$$

once we assume $p > 1$, for suitable conditions on $y_0 \in L^2(\Omega)$ and for an appropriate choice of the control function (taken as a suitable delayed feedback control) can be given in a similar way to the results presented in [4]. By limitations in the length of this work, those results will be given elsewhere.

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