Perimeter Symmetrization of Some Dynamic and Stationary Equations Involving the Monge-Ampère Operator

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Abstract We apply the perimeter symmetrization to a two-dimensional pseudo-parabolic dynamic problem associated to the Monge-Ampère operator as well as to the second order elliptic problem which arises after an implicit time discretization of the dynamical equation. Curiously, the dynamical problem corresponds to a third order operator but becomes a singular second order parabolic equation (involving the 3-Laplacian operator) in the class of radially symmetric convex functions. Using symmetrization techniques some quantitative comparison estimates and several qualitative properties of solutions are given.

Keywords Perimeter symmetrization • Pseudoparabolic dynamic Monge-Ampère equation • Two-dimensional domain

AMS (MOS) Subject Classification 35K55, 35J65, 35B05

1 Introduction

Starting with the pioneering paper by Giorgio Talenti [37] in 1981, many results were obtained concerning the comparison of solutions to some stationary equations, which can be written in terms of suitable perturbations of the Monge-Ampère operator in a general domain, with the radially symmetric solutions to some auxiliary stationary boundary value problems on an associated ball. In contrast with

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the case of many stationary problems given by operators in divergence form, the main tool is not the Schwarz (neither the Steiner) radially symmetric rearrangement of the solution but now the *perimeter rearrangement* of that function (see, e.g., [9, 12, 13, 19, 41, 42]).

The main difficulty to extend the previous papers concerning several stationary problems to the case of parabolic problems comes from the fact that it seems very complicated to relate the terms

$$\frac{d}{dt}\int_{u<\theta} u(x,t)\,dx \quad \text{and} \quad \frac{d}{dt}\int_{u^\star<\theta} u^\star(x,t)\,dx$$

when $u^\star(\cdot,t)$ is the rearrangement of $u(\cdot,t)$ with respect to the perimeter of its level sets. This contrasts with what happens in the case of the Schwarz radially symmetric rearrangement (since there, by construction, both level sets $\{u<\theta\}$ and $\{u^\star<\theta\}$ keep the same measure): see, e.g. the results relating both time differential terms by Bandle [4, 5], Mossino-Rakotoson [32] and Nagai [33], among many other authors.

Due to that, and following a previous work by Brandolini [10], we shall consider the following dynamic problem associated to the Monge-Ampère operator:

$$\begin{cases} (k_u(x,t)u)_t - \det D^2u = f(x,t) & \text{in } \Omega\times(0,+\infty) \\ u=0 & \text{on } \partial\Omega\times(0,+\infty) \\ u(x,0)=u_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here the subscript t means the derivative with respect to the time variable t, Du means the gradient of u with respect to the space variables $x=(x_1,x_2)\in\mathbb{R}^2$, D^2u denotes the Hessian matrix of u with respect to x and $k_u(\cdot,t) = \operatorname{div}\left(\frac{Du(\cdot,t)}{|Du(\cdot,t)|}\right)$ is the curvature of the level line of $u(\cdot,t)$ passing through the point $(\cdot,t)$. As we shall justify later, our main interest will focus on negative convex solutions to problem (1.1).

Notice that problem (1.1) is a pseudoparabolic dynamic problem and that, as we shall see, curiously enough, this third order operator becomes a singular second order parabolic equation (involving the 3-Laplacian operator) in the class of radially symmetric convex functions. For some recent results for other pseudoparabolic problems see e.g. [34]. The main reason for the consideration of the penalization factor $k_u(\cdot,t)$ in the inertia term comes from the fact that, in the class of radially symmetric functions, $\det D^2u$ behaves, formally, in a similar manner to the expression

$$\frac{1}{|x|}\operatorname{div}(|Du|\,Du)$$

and, as we shall show, this is exactly the behavior that brings, in the class of convex radially symmetric functions, the product $k_u\,u$.

The first goal of this work (a preliminary version of an extended paper Brandolini-Díaz [11]) is to obtain some quantitative comparison estimates for the solution u to (1.1) and the solution z to the symmetrized problem, sharpening in this way the results in [10]. Moreover, we shall extend the mentioned comparison result to the case of negative convex solutions to the stationary Dirichlet problem

$$\begin{cases} -\det D^2 u + k_u u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We shall give also many indications on the existence and uniqueness of solutions to problem (1.2); nevertheless, for the limited extension of this work, we shall delay to [11] the presentation of the corresponding indications for the dynamic problem (1.1). Finally, we shall apply the rearrangement comparison results in order to get some qualitative properties of solutions to (1.2) and (1.1).

2 Preliminary Results

2.1 Rearrangements and Main Properties

First of all we recall the definition of decreasing rearrangement of a measurable function $\varphi : \Omega \to \mathbb{R}$, where Ω is a bounded open subset of $\mathbb{R}^2$ with measure A. The distribution function of φ is defined by

$$\mu_\varphi(\theta) = |\{x \in \Omega : |\varphi(x)| > \theta\}|, \qquad \theta \geq 0,$$

while the decreasing rearrangement of φ is defined as the generalized left-continuous inverse of μ_φ, i. e.

$$\varphi^*(s) = \inf\{\theta \geq 0 : \mu_\varphi(\theta) < s\}, \qquad s \in [0, +\infty[.$$

Note that $\varphi^*(s) = 0$ if $s \geq A$. By definition, φ and φ^* are equidistributed functions, that is they share the same distribution function. In particular, φ^* is the unique decreasing left-continuous function in $[0, +\infty[$ equidistributed with φ.

By using the previous notions we can also introduce the decreasing spherically symmetric rearrangement of φ, also known as Schwarz symmetrand of φ, as follows

$$\varphi^\sharp(x) = \varphi^*(\pi|x|^2), \qquad x \in \Omega^\sharp,$$

where $\Omega^\sharp$ denotes the disc, centred at the origin, having the same measure A as Ω. By definition, $\varphi^\sharp$ is the unique spherically symmetric function, which is decreasing along the radii and equidistributed with φ.

Being φ, φ^* and $\varphi^\sharp$ equidistributed, if $\varphi \in L^p(\Omega)$ for some $p \in [1, +\infty[$, clearly it holds true that

$$||\varphi||_{L^p(\Omega)} = ||\varphi^*||_{L^p(0,A)} = ||\varphi^\sharp||_{L^p(\Omega^\sharp)}.$$

The theory of rearrangements is well-known and exhaustive treatments can be found, for example, in [31] or [38]. Here we just recall the following celebrated inequality that will be useful in the sequel.

Proposition 1 (Hardy-Littlewood Inequality) *Let φ, ψ be measurable functions in Ω. Then*

$$\int_\Omega |\varphi(x)\psi(x)|\, dx \le \int_0^{+\infty} \varphi^*(s)\psi^*(s)\, ds = \int_{\Omega^\sharp} \varphi^\sharp(x)\psi^\sharp(x)\, dx.$$

The above definitions will be useful in the following sections, but the crucial notion we are dealing with concerns the perimeter $\lambda_\varphi(\theta)$ of the level sets of φ. From now on we consider a bounded, convex, open set Ω in $\mathbb{R}^2$ and we denote by L its perimeter. Let φ be a smooth convex function in Ω, vanishing on the boundary; the sublevel sets of such a function φ are convex subsets of Ω and their perimeter $\lambda_\varphi(\theta)$ coincides with

$$\text{length}\,\{x \in \Omega :\ \varphi(x) = \theta\}, \qquad \theta \le 0.$$

We define

$$\tilde{\varphi}(s) = \sup\{\theta \le 0 : \lambda_\varphi(\theta) < s\}, \qquad s \in [0, L] \tag{2.3}$$

and the rearrangement of φ with respect to the perimeter of its level sets as

$$\varphi^\star(x) = \tilde{\varphi}(2\pi|x|), \qquad x \in \Omega^\star,$$

where $\Omega^\star$ is the disc, centred at the origin, with the same perimeter L as Ω (we explicitly observe that $\Omega^\sharp \subseteq \Omega^\star$). Differently from φ^*, $\tilde{\varphi}$ is in general not equidistributed with φ. But, the classical isoperimetric inequality states that

$$\mu_\varphi(-\theta) \le \frac{1}{4\pi}\lambda_\varphi(\theta)^2, \qquad \theta \le 0, \tag{2.4}$$

and then $\min\limits_\Omega \varphi = \min\limits_{[0,L]} \tilde{\varphi} = \min\limits_{\Omega^\star} \varphi^\star$, while for every $1 \le p < +\infty$

$$||\varphi||_{L^p(\Omega)} \le \frac{1}{2\pi}||s\tilde{\varphi}(s)||_{L^p(0,L)} = ||\varphi^\star||_{L^p(\Omega^\star)}.$$

The perimeter function $\lambda_\varphi(\theta)$ and the rearrangement $\tilde{\varphi}(s)$ defined by (2.3) satisfy some properties analogous to those ones of the distribution function $\mu_\varphi(\theta)$ and the decreasing rearrangement $\varphi^*(s)$. For the seek of simplicity and completeness we list some of these properties below (see [37, 41, 42]).

Proposition 2 (Regularity Properties) *Let Ω be a bounded, convex, open set in $\mathbb{R}^2$ and let $\varphi, \psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ be convex functions, vanishing on the boundary of Ω.*

i) $\lambda_\varphi(\theta) \in C([\min_\Omega \varphi, 0]) \cap C^2([\min_\Omega \varphi, 0))$*; moreover it is an increasing, concave function on the interval* $[\min_\Omega \varphi, 0]$ *and* $\lambda_\varphi(\min_\Omega \varphi) = 0$, $\lambda_\varphi(0) = L$;

ii) *for every* $\theta \in (\min_\Omega \varphi, 0)$

$$\lambda'_\varphi(\theta) = \int_{\varphi=\theta} \frac{k_\varphi}{|D\varphi|} \tag{2.5}$$

where, denoted $x = (x_1, x_2) \in \mathbb{R}^2$,

$$k_\varphi = \operatorname{div}\left(\frac{D\varphi}{|D\varphi|}\right) = |D\varphi|^{-3}\left(\begin{pmatrix} \varphi_{x_2x_2} & -\varphi_{x_1x_2} \\ -\varphi_{x_1x_2} & \varphi_{x_1x_1} \end{pmatrix} D\varphi, D\varphi\right) \geq 0$$

is the curvature of the level line $\{\varphi = \theta\}$;

iii) $\tilde{\varphi}(\lambda_\varphi(\theta)) = \theta$ *for every* $\theta \in [\min_\Omega \varphi, 0]$;

iv) $\tilde{\varphi} \in C([0, L]) \cap C^2([0, L))$*; it is an increasing, convex function on the interval* $[0, L]$ *and* $\tilde{\varphi}(0) = \min_\Omega \varphi$, $\tilde{\varphi}(L) = 0$*; moreover*

$$0 \leq \tilde{\varphi}'(s) \leq \frac{1}{2\pi} \sup_\Omega |D\varphi|, \qquad s \in [\min_\Omega \varphi, 0);$$

v) $\varphi^\star \in C(\overline{\Omega^\star}) \cap C^2(\Omega^\star)$*; moreover it is a convex function on* $\Omega^\star$ *and it vanishes on the boundary of* $\Omega^\star$.

Proposition 3 (General Properties of Rearrangements) *Under the same assumptions of Proposition 2, it holds that:*

vi) *if* $\varphi \leq \psi$ *in* Ω, *then* $\tilde{\varphi} \leq \tilde{\psi}$ *in* $[0, L]$;

vii) *for every* $c > 0$, $\widetilde{(c\varphi)} = c\tilde{\varphi}$;

viii) *for every* $c \in \mathbb{R}$, $\widetilde{(\varphi + c)} = \tilde{\varphi} + c$;

ix) *if* $\Psi : (-\infty, 0] \to (-\infty, 0]$ *is a strictly increasing, continuous, convex function, then*

$$\widetilde{(\Psi \circ \varphi)} = \Psi(\tilde{\varphi});$$

x) *for every* $s \in [0, A]$, $\tilde{\varphi}(2\pi\sqrt{s}) \leq (-\varphi^*(s))$;

xi) *the rearrangement operator is continuous from* $L^\infty(\Omega)$ *to* $L^p(0,L)$ *and for every* $s \in [0,L]$

$$|\tilde{\varphi}(s) - \tilde{\psi}(s)| \le ||\varphi - \psi||_{L^\infty(\Omega)}.$$

Proposition 4 *Let* Ω *be a bounded, convex, open set in* $\mathbb{R}^2$ *and let* $\varphi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be a convex function, vanishing on the boundary of* Ω*. For every convex subset* E *of* Ω *with perimeter* $P(E)$ *it holds*

$$\int_E \varphi(x)\, dx \ge \frac{1}{2\pi}\int_0^{P(E)} s\,\tilde{\varphi}(s)\, ds.$$

Proposition 5 (Hardy-Littlewood Type Inequality) *Let* Ω *be a bounded, convex, open set in* $\mathbb{R}^2$ *and let* $\varphi, \psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be convex functions, vanishing on the boundary of* Ω*. Then*

$$\int_\Omega \varphi(x)\psi(x)\, dx \le \frac{1}{2\pi}\int_0^L s\,\tilde{\varphi}(s)\tilde{\psi}(s)\, ds = \int_{\Omega^\star} \varphi^\star(x)\psi^\star(x)\, dx. \tag{2.6}$$

Remark 1 Actually, inequality (2.6) can be improved as follows

$$\int_\Omega \varphi(x)\psi(x)dx \le \int_{\Omega^\star} \varphi^\star(x)\left(-\psi^\sharp(x)\right) dx.$$

Proposition 6 (Pólya-Szegő Type Inequality) *Let* Ω *be a bounded, convex, open set in* $\mathbb{R}^2$ *and let* $\varphi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be a convex function, vanishing on the boundary of* Ω*. Then*

$$\int_\Omega (-\varphi)\det D^2\varphi\, dx \ge 2\pi \int_0^L (\tilde{\varphi}(s)')^3 ds = \int_{\Omega^\star} (-\varphi^\star)\det D^2\varphi^\star dx,$$

equality holding if Ω *is a disc.*

Proof By a direct computation it is easy to verify that the Hessian determinant of φ can be written in divergence form as follows

$$\det D^2\varphi = \frac{1}{2}\mathrm{div}\left\{\begin{pmatrix} \varphi_{x_2x_2} & -\varphi_{x_1x_2} \\ -\varphi_{x_1x_2} & \varphi_{x_1x_1}\end{pmatrix} D\varphi\right\}. \tag{2.7}$$

Then, by using divergence theorem and co-area formula, we obtain

$$\int_\Omega (-\varphi)\det D^2\varphi\, dx = \frac{1}{2}\int_\Omega k_\varphi |D\varphi|^3\, dx = \frac{1}{2}\int_{-\infty}^0 d\theta \int_{\varphi=\theta} k_\varphi |D\varphi|^2.$$

By Hölder inequality and (2.5) we get

$$\int_{\varphi=\theta} k_\varphi |D\varphi|^2 \geq \frac{\left(\int_{\varphi=\theta} k\right)^3}{\left(\lambda'_\varphi(\theta)\right)^2}.$$

Gauss-Bonnet theorem ensures that

$$\int_{\varphi=\theta} k = 2\pi.$$

Thus

$$\int_{\varphi=\theta} k_\varphi |D\varphi|^2 \geq \frac{8\pi^3}{\left(\lambda'_\varphi(\theta)\right)^2} = 8\pi^3 (\tilde{\varphi}(s)'|_{s=\lambda_\varphi(\theta)})^2 = \int_{\varphi^\star=\theta} k_{\varphi^\star} |D\varphi^\star|^2,$$

and the thesis immediately follows. □

Proposition 7 *Let Ω be a bounded, convex, open set in $\mathbb{R}^2$ and let $\varphi, \psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ be convex functions, vanishing on the boundary of Ω. Then, the following statements are equivalent:*

1) $\int_0^s r\tilde{\varphi}(r)\,dr \leq \int_0^s r\tilde{\psi}(r)\,dr$, *for* $s \in [0, L]$;
2) for every increasing, negative function $\phi \in C^1([0, L])$ such that $\phi(L) = 0$,

$$\int_0^L s\tilde{\varphi}(s)\phi(s)\,ds \geq \int_0^L s\tilde{\psi}(s)\phi(s)\,ds.$$

Proof 1) $\Rightarrow$ 2) is a consequence of the following identity

$$\int_0^L s\tilde{\varphi}(s)\phi(s)\,ds = -\int_0^L \left(\int_0^s r\tilde{\varphi}(r)\,dr\right) d\phi(s) + \phi(L)\int_0^L s\tilde{\varphi}(s)\,ds.$$

2) $\Rightarrow$ 1) is deduced from Proposition 4 above, after observing that if $\psi = \chi_E$ and $P(E) = s$, then $\tilde{\psi} = -\chi_{[0,s[}$. □

2.1.1 Accretive Operators in Banach Spaces

We start this subsection recalling some definitions contained in [18].

Let $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}(N) \to \mathbb{R}$, where $\mathscr{S}(N)$ is the set of symmetric $N \times N$ matrices. We recall that F is said to be proper if

$$F(x, r, p, X) \leq F(x, s, p, X) \quad \text{whenever} \quad r \leq s,$$

and F is said degenerate elliptic if

$$F(x,r,p,X) \leq F(x,r,p,Y) \quad \text{whenever} \quad Y \leq X.$$

Lemma 1 *The formal operator*

$$F(u,Du,D^2u) = -\det D^2u + k_u u$$

is degenerate elliptic and proper in the class of C^2, convex and negative functions u.

Proof This property was already shown for the Monge-Ampère part $F_1(u,Du,D^2u) := -\det D^2u$ in [18]. So it remains to prove it for the part

$$F_2(u,Du,D^2u) := k_u\, u = u \operatorname{div}\left(\frac{Du}{|Du|}\right),$$

that can be written in the class of negative functions as follows:

$$F_2(u,Du,D^2u) = -\,|u|\ \operatorname{div}\left(\frac{Du}{|Du|}\right).$$

It is well-known that the Laplacian acts as an ordinary differential operator along the lines of steepest descent; more precisely, the value of Δu at a point only involves derivatives of u along the line of steepest descent passing through that point and the mean curvature of the level line through the point:

$$\Delta u = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) + \frac{D^2u\, Du \cdot Du}{|Du|^2},$$

that is

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = \frac{1}{|Du|}\operatorname{trace}\left[\left(I - \frac{Du \otimes Du}{|Du|^2}\right) D^2u\right].$$

Then, if $r \leq 0$, we get that the operator

$$F_2(r,p,X) = -\frac{|r|}{|p|}\operatorname{trace}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) X\right]$$

is decreasing in X (for r and p prescribed), that is F_2 is degenerate elliptic. Analogously, in the class of convex functions we can assume that $\operatorname{div}\left(\frac{Du}{|Du|}\right) \geq 0$ and then $F_2(r,p,X)$ is increasing in r (for p and X prescribed) so it is proper. □

Now we recall some definitions and properties about accretive and T-accretive operators first abstractly, then in C^0 and L^∞. For all the proofs and applications of the theory of accretive operators to both elliptic and parabolic equations we remind the interested reader for instance to [6, 7, 16, 17, 29].

Let $\mathbb{X}$ be a real Banach space with norm $||\cdot||$ and let $A:\ D(A)\subset\mathbb{X}\to\mathbb{X}$. A is said to be accretive in $\mathbb{X}$ if

$$||x-\hat{x}||\leq||x-\hat{x}+\lambda(A(x)-A(\hat{x}))||,\qquad \text{for all }\ x,\hat{x}\in D(A),\ \lambda>0.$$

If, in addition, $R(I+\lambda A)=\mathbb{X}$ for some $\lambda>0$, A is m-accretive, in which case $R(I+\lambda A)=\mathbb{X}$ for all $\lambda>0$.

For $x,y\in\mathbb{X}$ we define the pairing

$$[y,x]_+=\inf_{\lambda>0}\frac{||x+\lambda y||-||x||}{\lambda}.$$

Clearly, $[\cdot,\cdot]_+:\ \mathbb{X}\times\mathbb{X}\to\mathbb{R}$ is upper semicontinuous and A is accretive if and only if

$$[A(x)-A(\hat{x}),x-\hat{x}]_+\geq 0,\qquad x,\hat{x}\in\mathbb{X}.$$

Moreover, the accretiveness of A in $\mathbb{X}$ can be determined by the normalized duality map. Indeed, if $\mathbb{X}'$ is the dual space of $\mathbb{X}$, then it can be proved that

$$[y,x]_+=\max_{f\in H(x)}<f,y>_{\mathbb{X}',\mathbb{X}},\tag{2.8}$$

where $H(x)=\{f\in\mathbb{X}':\ ||f||_{\mathbb{X}'}=1,\ <f,x>_{\mathbb{X}',\mathbb{X}}=||x||\}$. So, A is accretive if and only if there exists $f\in\mathbb{X}'$, $||f||_{\mathbb{X}'}=1$, and

$$<f,x-\hat{x}>_{\mathbb{X}',\mathbb{X}}=||x-\hat{x}||,\quad <f,A(x)-A(\hat{x})>_{\mathbb{X}',\mathbb{X}}\geq 0,\qquad x,\hat{x}\in\mathbb{X}.$$

Finally, A is said to be T-accretive (T stands for truncation) in $\mathbb{X}$ if

$$||(x-\hat{x})_+||\leq||(x-\hat{x}+\lambda(A(x)-A(\hat{x})))_+||,\qquad \text{for all }\ x,\hat{x}\in D(A),\ \lambda>0.$$

Here $a_+=\max\{a,0\}$. Equivalently, A is T-accretive in $\mathbb{X}$ if there exists $f\in\mathbb{X}'$, $f\geq 0$, $||f||_{\mathbb{X}'}=1$, and

$$<f,x-\hat{x}>_{\mathbb{X}',\mathbb{X}}=||(x-\hat{x})_+||,\quad <f,A(x)-A(\hat{x})>_{\mathbb{X}',\mathbb{X}}\geq 0,\qquad x,\hat{x}\in\mathbb{X}.$$

If $\Omega\subset\mathbb{R}^N$ is a bounded domain and $\mathbb{X}=C(\overline{\Omega})$, equipped with the supremum norm, the following representation holds

$$[v,u]_+=\max\{v(x_0)\mathrm{sign}(u(x_0)):\ x_0\in\overline{\Omega},\ |u(x_0)|=||u||\},$$

while, when $\mathbb{X} = L^\infty(\Omega)$, we have

$$[v, u]_+ = \lim_{\epsilon \to 0} \text{ess sup}_{\Omega(u,\epsilon)} v(x)\text{sign}(u(x)), \quad u \not\equiv 0,$$

where $\Omega(u, \epsilon)$ is defined (up to a set of measure zero) by

$$\Omega(u, \epsilon) = \{x \in \Omega : |u(x)| > ||u||_{L^\infty(\Omega)} - \epsilon\}$$

(see [29]). Thus, A is accretive in $L^\infty(\Omega)$ if and only if

$$\lim_{\epsilon \to 0} \text{ess sup}_{x \in \Omega(u - \hat{u},\epsilon)} (A(u(x)) - A(\hat{u}(x)))\text{sign}(u(x) - \hat{u}(x)) \geq 0$$

where

$$\Omega(u - \hat{u}, \epsilon) = \{x \in \Omega : |u(x) - \hat{u}(x)| \geq ||u - \hat{u}||_{L^\infty(\Omega)} - \epsilon\}.$$

Finally, thanks to (2.8), A is T-accretive in $L^\infty(\Omega)$ if and only if there is a finitely additive, absolutely continuous positive set function Φ with total variation 1, such that, for any $u, \hat{u} \in L^\infty(\Omega)$,

$$\int_{u > \hat{u}} (u - \hat{u})(x)\Phi(dx) = ||(u - \hat{u})_+||_{L^\infty(\Omega)}, \quad \int_\Omega (A(u) - A(\hat{u}))(x)\Phi(dx) \geq 0.$$

3 The Stationary Case

In this section we concentrate on the following Dirichlet problem

$$P(\Omega) : \quad \begin{cases} -\det D^2 u + k_u u = f \text{ in } \Omega \\ u = 0 \qquad\qquad\qquad \text{on } \partial\Omega \\ u \text{ convex in } \Omega, \end{cases}$$

where Ω is a planar, bounded, convex, open set. We look for convex, and then negative, solutions; then we need $f \leq 0$ in Ω as compatibility condition. As in Sect. 2, we denote by $\Omega^\star$ the disc, centered at the origin, with the same perimeter L as Ω. If $g(x)$ is a smooth, radially symmetric, negative function defined in $\Omega^\star$, which is increasing with respect to the radii, our main goal will be proving a suitable comparison result between u and the solution z to the following symmetrized problem

$$P(\Omega^\star) : \quad \begin{cases} -\det D^2 z + |x|^{-1} z = g \text{ in } \Omega^\star \\ z = 0 \qquad\qquad\qquad \text{on } \partial\Omega^\star \\ z \text{ convex in } \Omega^\star. \end{cases}$$

First of all let us discuss the notion of solutions we shall use in this paper. We immediately note that, as we shall see in the case of the radially symmetric problem, the presence of the term $u\,\mathrm{div}\left(\frac{Du}{|D_u|}\right)$ make quite difficult to get classical solutions (for instance in the radially symmetric case the term $\frac{z(x)}{|x|}$ will never be a bounded function since $z(x)$ will be a bounded function). Then it is natural to start our study by considering the truncated problems

$$P_N(\Omega):\quad \begin{cases} -\det D^2 u_N + T_N(k_{u_N})u_N = f \text{ in } \Omega \\ u_N = 0 \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega \\ u_N \text{ convex in } \Omega, \end{cases}$$

and

$$P_N(\Omega^\star):\quad \begin{cases} -\det D^2 z_N + T_N(|x|^{-1})z_N = g \text{ in } \Omega^\star \\ z_N = 0 \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega^\star \\ z_N \text{ convex in } \Omega^\star, \end{cases}$$

where

$$T_N(s) = \min\{s, N\} \ \text{ for } \ s \geq 0.$$

Proposition 8 *Given $f \in C(\overline{\Omega})$, $f \leq 0$ in Ω, there exists a unique C-viscosity solution u_N to $P_N(\Omega)$. Moreover*

$$\underline{u} \leq u_N \leq u_{N'} \leq 0 \quad \textit{in } \ \Omega,$$

where $\underline{u}$ is the unique C-viscosity solution to the unperturbed problem

$$P_{M-A}(\Omega):\quad \begin{cases} -\det D^2 \underline{u} = f \ \textit{in } \Omega \\ \underline{u} = 0 \qquad\qquad \textit{on } \partial\Omega \\ \underline{u} \ \textit{convex in } \Omega \end{cases}$$

and $u_{N'}$ is the C-viscosity solution to $P_{N'}(\Omega)$ for $N' > N$.

Proof It is not difficult to verify that the comparison principle holds for problem $P_N(\Omega)$. Thus we can apply the Perron method (see Theorem 4.1 in [18]) starting with the supersolution $\overline{u} = 0$ and the subsolution $\underline{u}$. Moreover since $T_N(s) \leq T_{N'}(s)$ if $N' > N$, we immediately get that $u_N \leq u_{N'}$ in Ω. □

Now we introduce the notion of limit solution.

Definition 1 A function $u \in C(\overline{\Omega}) \cap W^{2,1}_{\mathrm{loc}}(\Omega)$ such that u is convex and $-\det D^2 u + k_u u \in L^\infty(\Omega)$ is called a *limit solution* to $P(\Omega)$ if

$$u(x) = \lim_{N \to +\infty} u_N(x),$$

with u_N solution to $P_N(\Omega)$.

Proposition 9 *Given* $f \in C(\overline{\Omega}), f \leq 0$ *in* Ω*, there exists a unique limit solution* u_N *to* $P(\Omega)$.

Proof It suffices to use the Beppo-Levi monotone convergence theorem and the comparison principle. □

Now we can prove that the following operator (jointly with the Dirichlet boundary condition)

$$\mathscr{A}_N u = -\det \mathrm{D}^2 u - T_N \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) \right) |u|$$

is T-accretive in $C(\overline{\Omega})$ once we define suitably its domain $\mathrm{D}(\mathscr{A}_N)$. Since the *formal* operator

$$F(u, Du, D^2u) = -\det \mathrm{D}^2 u - T_N \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) \right) |u|$$

is not uniformly elliptic but merely degenerate elliptic we must use the notion of C-viscosity solution for the associated problem (see details and references for instance in [26]).

Definition 2 We say that $u \in \mathrm{D}(\mathscr{A}_N)$ if $u \in C(\overline{\Omega})$ is a convex function, with $u = 0$ on $\partial\Omega$, and there exists a nonpositive continuous function v in Ω such that u is a C-viscosity solution to

$$\begin{cases} -\det \mathrm{D}^2 u - T_N \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) \right) |u| = v & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

We denote by $\mathscr{A}_N u$ the set of all such $v \in C(\overline{\Omega})$.

Corollary 1 *The operator* $\mathscr{A}_N$ *is* T*-accretive in the Banach space* $\mathbb{X} = C(\overline{\Omega})$ *equipped with the supremum norm.*

Proof It is essentially a consequence of the maximum principle (see Theorem 3.3 and Section 5B in [18]). □

Remark 2 The extension to the accretiveness in $L^\infty(\Omega)$ is standard since the norm is given in a similar way. Notice that without the truncation function $T_N \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) \right)$ the corresponding operator is not well defined as an operator from $\mathbb{X}$ to $\mathbb{X}$ since, as we already pointed out, the expression $\operatorname{div} \left(\frac{Du}{|Du|} \right)$ is in general not an element of $\mathbb{X}$.

We continue this section with some considerations on the existence of solutions to the radially symmetric problem $P(\Omega^\star)$. The convexity condition is not always satisfied. So we shall need some extra conditions on the right hand side. Without any interest in getting the more general result at all, we shall proceed under some additional conditions. We denote by $R^\star$ the radius of $\Omega^\star$ and we assume

$$g(x) = g(|x|), \quad g \in W^{2,1}(\Omega^\star), \quad 0 \geq g(|x|) \geq -\frac{M}{|x|} \text{ for some } M > 0 \tag{3.9}$$

and, with $r = |x|$,

$$g''(r)r + 2g'(r) \geq 0 \quad \text{for a.e. } r \in (0, R^\star). \tag{3.10}$$

We get the following result.

Lemma 2 *Under the assumptions (3.9) and (3.10), there exists a unique convex solution $z \in W_0^{1,3}(\Omega^\star)$ to $P(\Omega^\star)$ with* $\det D^2 z$, $\frac{z}{|x|} \in L^3(\Omega^\star)$.

Proof If we set $w = z + M$, we can equivalently prove the existence of a unique convex solution w to

$$\begin{cases} -\det D^2 w + \dfrac{w}{|x|} = g(|x|) + \dfrac{M}{|x|} & \text{in } \Omega^\star \\ w = M & \text{on } \partial\Omega^\star. \end{cases}$$

From the assumption (3.9) and the maximum principle we know that necessarily $0 \leq w \leq M$ in $\Omega^\star$. Now we define

$$\Gamma(|x|) := g(|x|) + \frac{M}{|x|}$$

and we construct w as the unique solution to the following radially symmetric problem

$$\begin{cases} -\dfrac{1}{2}\Delta_3 w + w = |x|\Gamma(|x|) & \text{in } \Omega^\star \\ w = M & \text{on } \partial\Omega^\star. \end{cases} \tag{3.11}$$

Since $|x|\Gamma(|x|) \in L^\infty(\Omega^\star)$, by well-known results there is a unique (radially symmetric) solution $w \in W^{1,3}(\Omega^\star)$ to problem (3.11). Then $z \in W_0^{1,3}(\Omega^\star)$ and by the Hardy inequality $\frac{z}{|x|} \in L^3(\Omega^\star)$. In order to show that z is convex, since it is radially symmetric, it is enough to show that $z''(r) \geq 0$, being $r = |x|$. But the nonnegative function $\gamma(x) = |x|\,\Gamma(|x|) = |x|\,g(|x|) + M$ is a subsolution to problem (3.11), i.e.

$$\begin{cases} -\dfrac{1}{2}\Delta_3 \gamma + \gamma \leq |x|\,\Gamma(|x|) & \text{in } \Omega^\star \\ \gamma \leq M & \text{on } \partial\Omega^\star. \end{cases}$$

Indeed, by (3.10) we have for a.e. $r \in (0, R^\star)$

$$\begin{aligned}\Delta_3 \gamma = \operatorname{div}(|D\gamma| D\gamma) &= \frac{d}{dr}\left[\left|g'(r)r + g(r)\right| (g'(r)r + g(r))\right] \\ &= 2|g'(r)r + g(r)|(g''(r)r + 2g'(r)) \geq 0,\end{aligned}$$

while (3.9) implies that $\gamma \leq M$ on $\partial\Omega^\star$. Then, by the maximum principle $w \geq \gamma$ in $\Omega^\star$ and then, since $\Delta_3 w = 2(w-\gamma)$, we get $\Delta_3 w \geq 0$, which shows that $0 \leq w''(r)$ and thus $0 \leq z''(r)$. □

Note that in fact our study of the radially symmetric case did not need to use the truncation argument mentioned at the beginning of this section. Nevertheless, we can easily state a result similar to Proposition 8.

Proposition 10 *Given $g \in C(\overline{\Omega^\star})$, $g \leq 0$ in $\Omega^\star$, there exists a unique C-viscosity solution z_N to $P_N(\Omega^\star)$. Moreover*

$$\underline{z} \leq z_N \leq z_{N'} \leq 0 \quad in \;\; \Omega^\star,$$

where $\underline{z}$ is the unique C-viscosity solution to the unperturbed problem

$$P_{M-A}(\Omega^\star): \quad \begin{cases} -\det D^2 \underline{z} = g & in\ \Omega^\star \\ \underline{z} = 0 & on\ \partial\Omega^\star \\ \underline{z}\ convex\ in\ \Omega^\star \end{cases}$$

and $z_{N'}$ is the C-viscosity solution to $P_{N'}(\Omega^\star)$ for $N' > N$.

Corollary 2 *Assume g satisfies (3.9) and (3.10). Then the limit solution to problem $P(\Omega^\star)$ (constructed as in Proposition 9) coincides with the unique solution to $P(\Omega^\star)$ given in Lemma 2.*

By using the notion of rearrangement that we recalled in Sect. 2, we can prove the following result which links the asymmetry of a solution u to problem $P(\Omega)$ to the asymmetry of the datum f. This kind of results is very famous in literature and goes back to authors as celebrated as Pólya, Szegő and Weinberger. It appeared clear that symmetrization techniques are very useful to write explicit and easy to compute estimates of solutions to many variational problems (see for example [39, 40] and the references therein). The first who proved a pointwise comparison result between the Schwarz symmetrands of solutions to Poisson equations was Talenti in 1976 (see [36]). After him, many mathematicians have been interested in symmetrization techniques and have applied them to linear and quasilinear elliptic equations with lower order terms (see for example [1, 2, 40] and the references therein). The case of fully nonlinear equations is different, since preserving the measure of the level sets does not give information on the geometry of a solution. In [37] Talenti faced the Monge-Ampère equation in dimension two and recognized the opportunity of symmetrize preserving the perimeter of the level sets. Then Tso ([42], see also [41]) treated the case of Monge-Ampère equations in dimension n. For related results we refer the reader to [9, 12–14, 19].

Theorem 1 *Let $f \in C(\overline{\Omega})$ be a negative function and let u be the limit solution to problem $P(\Omega)$. Denote by $\Omega^\star$ the disc, centered at the origin, with the same perimeter L as Ω. Assume that $g(x)$ is a smooth, radially symmetric, negative function defined in $\Omega^\star$, which is increasing along the radii. Let z be the solution to the symmetrized problem $P(\Omega^\star)$. For $s \in (0, L)$ denote*

$$U(s) = \int_0^s \tilde{u}(\sigma)d\sigma, \qquad Z(s) = \int_0^s \tilde{z}(\sigma)d\sigma,$$

$$F(s) = \int_0^{s^2/4\pi} f^*(\sigma)d\sigma,\ G(s) = \int_0^{s^2/4\pi} g^*(\sigma)d\sigma.$$

Then we have

$$||(Z-U)_+||_{L^\infty(0,L)} \le ||(F-G)_+||_{L^\infty([0,L])}. \tag{3.12}$$

Proof Since $u = \lim\limits_{N\to+\infty} u_N = \sup\limits_N u_N$, it will be enough to get the conclusion by replacing u with u_N in the statement. Notice that, nevertheless, we shall not truncate the radially symmetric problem $P(\Omega^\star)$. Our first argument is that, if θ is a noncritical value for u_N (i.e. $|Du_N| \neq 0$ on $\{x \in \Omega : u_N(x) = \theta\}$), then u_N satisfies

$$-\int_{u_N<\theta} \det D^2 u_N\, dx + \int_{u_N<\theta} T_N\left(k_{u_N}\right) u_N\, dx = \int_{u_N<\theta} f\, dx. \tag{3.13}$$

By using (2.7), divergence theorem, Hölder inequality and (2.5), we obtain

$$\int_{u_N<\theta} \det D^2 u_N\, dx = \frac{1}{2}\int_{u_N=\theta} k_{u_N}|Du_N|^2 \ge \frac{4\pi^3}{\left(\lambda_N'(\theta)\right)^2}, \tag{3.14}$$

where $\lambda_N(\theta) = \lambda_{u_N}(\theta)$. Moreover, by Hardy-Littlewood inequality (2.6) and classical isoperimetric inequality (2.4) we obtain

$$\int_{u_N<\theta} (-f)\, dx \le \int_0^{\lambda_N(\theta)^2/4\pi} f^*(\sigma)\, d\sigma. \tag{3.15}$$

It remains to estimate from above the second integral in the left-hand side of (3.13). To do this, we consider $\varepsilon = \varepsilon(N) > 0$ such that, denoted by x_N^m the minimum point of u_N, it holds that $B_\varepsilon := \{x \in \Omega : |x - x_N^m| > \varepsilon\} \subset \{u_N < \theta,\ k_{u_N} \le N\}$. Then, by co-area formula, we get

$$\int_{u_N<\theta} T_N(k_{u_N})\, u_N\, dx \le \int_{u_N<\theta,\, k_{u_N}\le N} k_{u_N}\, u_N\, dx \le \int_{M_\varepsilon}^{\theta} \tau\lambda_N'(\tau)\, d\tau, \tag{3.16}$$

where $M_\varepsilon = \max_{B_\varepsilon} u_N$. From (3.14) to (3.16) with $s = \lambda_N(\theta)$ we deduce the following inequality involving the rearrangement $\tilde{u}_N(s)$ of the function u_N:

$$4\pi^3 \tilde{u}_N'(s)^2 - \int_{\lambda_N(M_\varepsilon)}^{s} \tilde{u}_N(\sigma)\,d\sigma \le \int_0^{s^2/4\pi} f^*(\sigma)\,d\sigma.$$

Setting

$$U_{N,\varepsilon}(s) = \int_{\lambda_N(M_\varepsilon)}^{s} \tilde{u}_N(\sigma)\,d\sigma, \qquad s \in (\lambda_N(M_\varepsilon), L),$$

we get

$$4\pi^3 U_{N,\varepsilon}''(s)^2 - U_{N,\varepsilon}(s) \le F(s), \qquad s \in (\lambda_N(M_\varepsilon), L). \tag{3.17}$$

Reasoning in an analogous way on the solution z to the symmetrized problem $P(\Omega^\star)$, since all the inequalities become equalities, we get

$$4\pi^3 Z''(s)^2 - Z(s) = G(s), \qquad s \in (0, L). \tag{3.18}$$

Subtracting (3.18) from (3.17) we get

$$4\pi^3 \left(U_{N,\varepsilon}''(s)^2 - Z''(s)^2\right) - (U_{N,\varepsilon}(s) - Z(s)) \le F(s) - G(s), \qquad s \in (0, L),$$

where we extended $U_{N,\varepsilon}$ to zero in $(0, \lambda_N(M_\varepsilon))$. Now we observe that the operator

$$V(s) \to -4\pi^3 V''(s)^2$$

is T-accretive in $L^\infty(0, L)$. Then, by definition, there exists a finitely additive absolutely continuous positive set function Φ with total variation 1, such that

$$\int_{Z > U_{N,\varepsilon}} (Z - U_{N,\varepsilon})(s)\Phi(ds) = ||(Z - U_{N,\varepsilon})_+||_{L^\infty(0,L)}$$

and

$$\int_0^L \left(U_{N,\varepsilon}''(s)^2 - Z''(s)^2\right) \Phi(ds) \ge 0.$$

Then we easily get

$$||(Z - U_{N,\varepsilon})_+||_{L^\infty(0,L)} \le ||(F - G)_+||_{L^\infty(0,L)}.$$

Passing to the limit as ε goes to 0 and N goes to $+\infty$ we obtain (3.12). □

Remark 3 In the particular case when $g(x) = -f^\sharp(x)$, estimate (3.12) immediately gives

$$\int_0^s \tilde{u}(\sigma)\, d\sigma \geq \int_0^s \tilde{z}(\sigma)\, d\sigma, \quad s \in (0, L), \tag{3.19}$$

that can be written as

$$\int_{B(0,r)} u^\star(x)\, dx \geq \int_{B(0,r)} z(x)dx, \qquad r \in [0, R^\star],$$

where $R^\star$ is the radius of $\Omega^\star$. In the case of linear equations (and Schwarz symmetrization) the above inequality is known as "symmetrized mass comparison principle" and it is widely applied to extend estimates on the symmetric function z to the non symmetric function u. The first immediate consequence of (3.19) is the following estimate:

$$||u||_{L^p(\Omega)} \leq ||z||_{L^p(\Omega^\star)}, \quad 1 \leq p \leq +\infty.$$

Remark 4 We explicitly observe that an analogous comparison result between concentrations holds true if u and z are convex, vanishing on the boundary, solutions to the equations

$$-\det D^2 u + k_u(-u)^\alpha = f \text{ in } \Omega, \quad -\det D^2 z + |x|^{-1}(-z)^\alpha = -f^\sharp \text{ in } \Omega^\star,$$

for some $\alpha > 0$, respectively. More precisely it holds that

$$\int_0^s (-\tilde{u}(\sigma))^\alpha\, d\sigma \leq \int_0^s (-\tilde{z}(\sigma))^\alpha\, d\sigma, \quad s \in (0, L).$$

We end this section with a qualitative property of solutions to $P(\Omega)$ derived trough Theorem 1 and the consideration of this property for the symmetrized problem $P(\Omega^\star)$.

Proposition 11 *Let f be as in Theorem 1. Assume that $f^*(\sigma)$ is strictly monotone. Then no free boundaries (given as the boundary of the subsets where $Du = 0$, with u limit solution to $P(\Omega)$) can be formed.*

Proof Arguing as in [20] it is enough to prove the nonexistence of free boundaries for the radially symmetric solution z to $P(\Omega^\star)$ with $g = -f^\sharp$. In this case, even if the operator Δ_3 is degenerated, it is enough to observe that the right-hand side term in the equation never vanishes (see [20]). □

Remark 5 It is a curious fact that, if $g(|x|) = \frac{z_0}{|x|}$ on a suitable subset of $\Omega^\star$ with positive measure, for some $z_0 < 0$ corresponding to the minimum value of the solution z to $P(\Omega^\star)$, then the set of points where $z(|x|) = z_0$ could also have positive measure and then it could give rise to a free boundary. Anyway, we are talking about unbounded data, something which goes out of the assumptions of this paper.

4 The Evolution Problem

In this section we want to apply symmetrization techniques to the following evolution problem

$$\begin{cases} (k_u u)_t - \det D^2 u = f(x,t) & \text{in } \Omega \times (0,+\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u(\cdot,t) \text{ convex in } \Omega \end{cases} \tag{4.20}$$

where $f(x,t)$ is a smooth, negative function defined in $\Omega \times (0,+\infty)$ and $u_0(x)$ is a smooth, convex function, vanishing on the boundary of Ω. Concerning the rearrangements theory and the parabolic equations we refer the interested reader to [2, 4, 5, 21, 22, 32, 40, 43] and the references therein.

We remark that, if we proceed as in the stationary case and we integrate the equation in problem (4.20) on the subset of Ω given by $\{x \in \Omega : u(x,t) < \theta\}$ for $\theta < 0$, we obtain

$$\int_{\{x\in\Omega:u(x,t)<\theta\}} (k_u\, u)_t \, dx - \int_{\{x\in\Omega:u(x,t)<\theta\}} \det\, D^2 u \, dx = \int_{\{x\in\Omega:u(x,t)<\theta\}} f \, dx. \tag{4.21}$$

By easy calculation we may show that the second integral in the left-hand side of (4.21) can be written in terms of $U(s,t) = \int_0^s \tilde{u}(\sigma,t)\, d\sigma$, where s is the perimeter of $\{x \in \Omega : u(x,t) < \theta\}$. We would like to relate the first one with the derivative of $U(s,t)$ with respect to t. To this aim we recall a derivation formula for a function of the type

$$H(s,t) = \int_{\{x\in\Omega:u(x,t)<\tilde{u}(s,t)\}} h(x,t)\, dx$$

where u and h are smooth functions defined in $\Omega \times [0,+\infty[$ (see [30, 35], see also [3, 10]).

Proposition 12 *Let $u(x,t)$ be a smooth function in $\Omega\times[0,+\infty[$, convex with respect to x in Ω and vanishing on $\partial\Omega \times [0,+\infty[$. If $h \in C^1(\Omega \times [0,+\infty[)$, then for any*

$t \in (0, +\infty)$ *it holds true*

$$\left(\int_{\{x\in\Omega: u(x,t)<\tilde{u}(s,t)\}} h\,dx\right)_t$$

$$= \int_{\{x\in\Omega: u(x,t)<\tilde{u}(s,t)\}} h_t\,dx - \int_{\{x\in\Omega: u(x,t)=\tilde{u}(s,t)\}} \frac{h}{|Du|} \times \left\{ \frac{\displaystyle\int_{\{x\in\Omega: u(x,t)=\tilde{u}(s,t)\}} \frac{k_u}{|Du|} u_t}{\displaystyle\int_{\{x\in\Omega: u(x,t)=\tilde{u}(s,t)\}} \frac{k_u}{|Du|}} - u_t \right\}.$$

Remark 6 If $h(x,t) = k_u(x,t)u(x,t)$, then

$$\left(\int_{\{x\in\Omega: u(x,t)<\tilde{u}(s,t)\}} k_u\,u\,dx\right)_t = \int_{\{x\in\Omega: u(x,t)<\tilde{u}(s,t)\}} (k_u\,u)_t\;dx. \tag{4.22}$$

The following results dealing with comparison between rearrangements will be stated, for simplicity, for classical solutions. Nevertheless, by following the same methods used in the stationary case, the conclusions can be extended to weaker notions of solutions (see [11]).

Theorem 2 *Let u be a classical solution to problem (4.20). Denote by $\Omega^\star$ the disc, centered at the origin, with the same perimeter L as Ω. Assume that $g(x,t)$ is a smooth, negative function defined in $\Omega^\star \times (0,+\infty)$, radially symmetric with respect to the space variables, i. e. $g(x,t) = g(|x|,t)$, and $z_0(x)$ is a smooth, convex function, defined in $\Omega^\star$, vanishing on $\partial\Omega^\star$. Let z be the solution to the following parabolic problem*

$$\begin{cases} |x|^{-1} z_t - \det D^2 z = g(x,t) & in\ \Omega^\star \times (0,+\infty) \\ z = 0 & on\ \partial\Omega^\star \times (0,+\infty) \\ z(x,0) = z_0(x) & in\ \Omega^\star \\ z(\cdot,t)\ convex\ in\ \Omega^\star. \end{cases} \tag{4.23}$$

For $s \in (0,L)$ and $t > 0$ denote

$$U(s,t) = \int_0^s \tilde{u}(\sigma,t)d\sigma, \qquad Z(s,t) = \int_0^s \tilde{z}(\sigma,t)d\sigma,$$

$$F(s,t) = \int_0^{s^2/4\pi} f^*(\sigma,t)d\sigma, \quad G(s,t) = \int_0^{s^2/4\pi} g^*(\sigma,t)d\sigma.$$

Then, for every $t > 0$, we have

$$\begin{aligned} ||(Z(\cdot,t) - U(\cdot,t))_+||_{L^\infty(0,L)} &\le ||(Z(\cdot,0) - U(\cdot,0))_+||_{L^\infty(0,L)} \\ &\quad + \int_0^t ||(F(\cdot,\tau) - G(\cdot,\tau))_+||_{L^\infty([0,L])} d\tau. \end{aligned} \tag{4.24}$$

Proof We reason here as in the stationary case. Let $t > 0$ and let us consider a noncritical value $\theta < 0$ (i.e. $|D_x u| \neq 0$ on $\{x \in \Omega : u(x,t) = \theta\}$). We integrate the equation in (4.20) on the sublevel set $\{x \in \Omega : u(x,t) < \theta\}$ obtaining

$$\int_{\{x\in\Omega:u(x,t)<\theta\}} (u(x,t)\,k_u(x,t))_t\,dx - \int_{\{x\in\Omega:u(x,t)<\theta\}} \det D^2 u\,dx \qquad (4.25)$$
$$= \int_{\{x\in\Omega:u(x,t)<\theta\}} f(x,t)\,dx.$$

As in the stationary case, by using divergence theorem, Hölder inequality and (2.5), we get

$$\int_{\{x\in\Omega:u(x,t)<\theta\}} \det D^2 u(x,t)\,dx = \frac{1}{2}\int_{\{x\in\Omega:u(x,t)=\theta\}} k_u(x,t)|Du(x,t)|^2 \qquad (4.26)$$
$$\geq \frac{4\pi^3}{\left(\frac{\partial\lambda(\theta,t)}{\partial\theta}\right)^2},$$

where $\lambda(\theta,t) = \text{length}\{x \in \Omega : u(x,t) = \theta\}$. Moreover, by using (4.22) we get that the first integral in (4.25) coincides with

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{\theta} \tau \frac{\partial\lambda(\tau,t)}{\partial\tau} d\tau\right). \qquad (4.27)$$

On the other hand, by Hardy-Littlewood inequality (2.6) and the classical isoperimetric inequality (2.4) we obtain

$$\int_{\{x\in\Omega:u(x,t)<\theta\}} (-f(x,t))\,dx \leq \int_0^{\lambda(\theta,t)^2/4\pi} f^*(\sigma,t)\,d\sigma. \qquad (4.28)$$

From (4.26), (4.27) and (4.28) with $s = \lambda(\theta,t)$ we deduce the following inequality involving the rearrangement $\tilde{u}(s,t)$ of the function $u(\cdot,t)$:

$$-\frac{\partial}{\partial t}\left(\int_0^s \tilde{u}(\sigma,t)\,d\sigma\right) + 4\pi^3\left(\frac{\partial\tilde{u}(s,t)}{\partial s}\right)^2 \leq \int_0^{s^2/4\pi} f^*(\sigma,t)\,d\sigma,$$

that is

$$-U_t(s,t) + 4\pi^3 U_{ss}^2(s,t) \leq F(s,t), \qquad s \in (0,L),\ \ t > 0. \qquad (4.29)$$

Reasoning in an analogous way on the solution z to the symmetrized problem (4.23), since all the inequalities become equalities, we get

$$-Z_t(s,t) + 4\pi^3 Z_{ss}^2(s,t) = G(s,t), \qquad s \in (0,L),\ t > 0. \qquad (4.30)$$

Subtracting (4.30) from (4.29) we get

$$(Z(s,t)-U(s,t))_t+4\pi^3\left(U_{ss}^2(s,t)-Z_{ss}^2(s,t)\right)\le F(s,t)-G(s,t),\qquad s\in(0,L),t>0.$$

Now we observe that the operator

$$U(s,t)\to -4\pi^3 U_{ss}^2(s,t)$$

is T-accretive in $L^\infty(0,L)$. Then, by definition, there exists a finitely additive absolutely continuous positive set function Φ with total variation 1, such that

$$\int_{Z>U}(Z-U)(s)\Phi(ds)=||(Z-U)_+||_{L^\infty(0,L)}$$

and

$$\int_0^L\left(U_{ss}^2-Z_{ss}^2\right)\Phi(ds)\ge 0.$$

Then we easily get

$$\int_0^L(Z-U)_t\,\Phi(ds)\le\int_0^L(F-G)\Phi(ds)$$

and finally

$$\frac{d}{dt}||(Z-U)_+||_{L^\infty(0,L)}\le||(F-G)_+||_{L^\infty(0,L)}.$$

Integrating between 0 and t we get the thesis. □

Remark 7 Estimates (4.24) can be read as a continuous dependence on the data symmetry with respect to the spatial variables. Indeed, if $\Omega=\Omega^\star$, the maximal asymmetry of a solution at the time t does not exceed the sum of the asymmetry at the time 0 and the asymmetry of the datum f.

In the particular case when $g(x,t)=-f^\sharp(x,t)$ and $z_0(x)=u_0^\star(x)$, estimate (4.24) immediately implies the comparison result contained in [10, Theorem 3.1], that we state below for completeness.

Theorem 3 *Let u be a classical solution to problem (4.20) and let v be the solution to*

$$\begin{cases}|x|^{-1}v_t-\det D^2v=-f^\sharp(x,t) & \text{in } \Omega^\star\times(0,+\infty)\\ v=0 & \text{on } \partial\Omega^\star\times(0,+\infty)\\ v(x,0)=u_0^\star(x) & \text{in } \Omega^\star\\ v(\cdot,t)\ \text{convex in } \Omega^\star.\end{cases}$$

Then, for every $t > 0$ it holds

$$\int_0^s \tilde{u}(\sigma,t)d\sigma \geq \int_0^s \tilde{v}(\sigma,t)d\sigma, \qquad s \in (0,L). \tag{4.31}$$

As an immediate consequence of (4.31) we have the following

Proposition 13 *Under the assumptions of Theorem 3, the following estimates hold true for every $t > 0$:*

$$||u(\cdot,t)||_{L^p(\Omega)} \leq ||v(\cdot,t)||_{L^p(\Omega^\star)}, \qquad 1 \leq p \leq +\infty; \tag{4.32}$$

$$\int_\Omega u(k_u\, u)_t\, dx + \int_\Omega (-u) \det D^2 u\, dx \leq \int_{\Omega^\star} v(k_v\, v)_t\, dx + \int_{\Omega^\star} (-v) \det D^2 v\, dx. \tag{4.33}$$

Proof Estimate (4.32) can be easily deduced from (4.31) and properties of rearrangements.

Multiplying the equation in problem (4.20) by $-u$, integrating over Ω and using (4.31) yield

$$\begin{aligned}
&\int_\Omega u(k_u\, u)_t\, dx + \int_\Omega (-u) \det D^2 u\, dx \\
&\qquad = \int_\Omega f(-u)\, dx \\
&\qquad \leq \frac{1}{2\pi} \int_0^L f^*\left(\frac{s^2}{4\pi}, t\right)(-\tilde{u}(s,t))s\, ds \\
&\qquad = \frac{1}{2\pi} \int_0^L \left(\int_0^s (-\tilde{u}(\sigma,t))d\sigma\right)\left(-\frac{d}{ds}\left(f^*\left(\frac{s^2}{4\pi}, t\right) s\right)\right) ds \\
&\qquad \leq \frac{1}{2\pi} \int_0^L \left(\int_0^s (-\tilde{v}(\sigma,t))d\sigma\right)\left(-\frac{d}{ds}\left(f^*\left(\frac{s^2}{4\pi}, t\right) s\right)\right) ds \\
&\qquad = \int_{\Omega^\star} f^\sharp(x,t)(-v)\, dx = \int_{\Omega^\star} v(k_v\, v)_t\, dx + \int_{\Omega^\star} (-v) \det D^2 v\, dx,
\end{aligned}$$

that is (4.33). □

A qualitative property typical of some nonlinear models concerns the finite speed of propagation of disturbances: if the initial datum u_0 vanishes on a set of positive measure (i.e. $\operatorname{supt} u_0 \subset \Omega$), then $\operatorname{supt} u(\cdot,t) \subset \Omega$ for any $t > 0$. In our case the following estimate of the perimeter of the zero sublevel set of u can be proved. It will imply that, if $\operatorname{supt} v(\cdot,\bar{t}) = \Omega^\star$ for some $\bar{t} > 0$, then $\operatorname{supt} u(\cdot,\bar{t}) = \Omega$, which means that the equation does not satisfy the finite speed of propagation property. Finally, the following perimeter estimate shows how important is having symmetry conditions on partial differential equations in order to have solutions with small supports.

Proposition 14 *Under the assumptions of Theorem 3, if for every $t > 0$ it holds*

$$\int_{\Omega} (k_u\, u)\, dx = \int_{\Omega^\star} (|x|^{-1} v)\, dx, \tag{4.34}$$

then

$$P(\{x \in \Omega^\star :\ v(x,t) < 0\}) \le P(\{x \in \Omega :\ u(x,t) < 0\}), \qquad t > 0.$$

Proof By using co-area formula, assumption (4.34) can be written as

$$\int_{\min_\Omega u}^{0} \lambda_u(\theta, t)\, d\theta = \int_{\min_{\Omega^\star} v}^{0} \lambda_v(\theta, t)\, d\theta \tag{4.35}$$

that is in terms of rearrangements

$$\int_0^L \tilde{u}(\sigma, t)\, d\sigma = \int_0^L \tilde{v}(\sigma, t)\, d\sigma. \tag{4.36}$$

Thus, estimate (4.31) implies

$$\int_s^L \tilde{u}(\sigma, t) d\sigma \le \int_s^L \tilde{v}(\sigma, t) d\sigma.$$

Let $[0, R_u(t)]$ and $[0, R_v(t)]$ denote the support of $\tilde{u}(\cdot, t)$ and $\tilde{v}(\cdot, t)$, respectively, with $0 < R_u(t), R_v(t) \le L$. From (4.36) it immediately follows that $R_v(t) \le R_u(t)$, otherwise

$$0 = \int_{R_u(t)}^{L} \tilde{u}(\sigma, t) d\sigma \le \int_{R_u(t)}^{R_v(t)} \tilde{v}(\sigma, t) d\sigma < 0$$

which is a contradiction. □

Remark 8 When does (4.35) hold? When $t = 0$ it is clearly true since

$$\int_{\min_\Omega u}^{0} \lambda_u(\theta, 0) d\theta = \int_{\min_\Omega u_0}^{0} \lambda_{u_0}(\theta) d\theta = \int_{\min_{\Omega^\star} u_0^\star}^{0} \lambda_{u_0^\star}(\theta) d\theta = \int_{\min_{\Omega^\star} v}^{0} \lambda_v(\theta, 0) d\theta,$$

that is $\tilde{u}(\cdot, 0)$ and $\tilde{v}(\cdot, 0)$ have the same L^1 norm. Thus (4.35) is satisfied whenever $\tilde{u}(\cdot, t)$ and $\tilde{v}(\cdot, t)$ preserve the same L^1 norm for every $t > 0$. We stress that this does not mean that $u(\cdot, t)$ and $v(\cdot, t)$ have the same L^1 norm.

Now we want to study the asymptotic behavior of u by proving that the stabilization to a stationary solution requires an infinite time. To this aim we need to introduce the following auxiliary eigenvalue problem

$$\begin{cases} -w'' = \frac{\lambda}{s} w & s \in (0, L) \\ w(0) = 0, w'(L) = 0. \end{cases}$$

By well-known results (see, e.g. [8]) there exists a first eigenvalue $\lambda_1 > 0$ such that the corresponding normalized eigenfunction w_1 satisfies $w_1(s) > 0$ for any $s \in (0, L)$ and $\|w_1\|_\infty = 1$. We also point out that, when $g^*(s,t) \equiv c^2$ $(c > 0)$ for any $s \in (0, L)$ in (4.23), then the problem

$$\begin{cases} -Z_t(s,t) + 4\pi^3 Z_{ss}^2(s,t) = \frac{c^2 s^2}{4\pi} & s \in (0, L),\ t > 0 \\ Z(0,t) = Z_s(L,t) = 0 & t > 0 \\ Z(s,0) = Z_0(s) & s \in (0, L) \end{cases}$$

has the unique stationary solution

$$Z_\infty(s) = \frac{c}{24\pi^2} s \left(s^2 - 3L^2\right)$$

and, in particular, if $Z_0(s) = Z_\infty(s)$ then $Z(s,t) = Z_\infty(s)$ for any $s \in [0, L]$ and $t \geq 0$.

Theorem 4 *Assume $g^*(s) \equiv c^2$ in (4.23) and $F(s,t) = G(s,t)$ for any $s \in (0, L)$, $t > 0$. If there exists $\underline{m} > 0$ such that*

$$U(s,0) \geq Z_\infty(s) + \underline{m} w_1(s) \quad \textit{for any } s \in (0, L),$$

then there exists a constant $\varepsilon > 0$ (independent of t) such that for any $t > 0$ and $s \in [0, L]$ we have

$$U(s,t) \geq Z_\infty(s) + \underline{m} w_1(s) e^{-\lambda_1 \varepsilon t}.$$

Proof Arguing as in the proof of Theorem 2, it suffices to show that the function

$$\underline{U}(s,t) := Z_\infty(s) + \underline{m} w_1(s) e^{-\lambda_1 \varepsilon t}$$

is a subsolution to the parabolic problem associated to $U(s,t)$. More precisely, we must check that we can take a constant $\varepsilon > 0$ (independent of t) such that

$$\begin{cases} -\underline{U}_t(s,t) + 4\pi^3 \underline{U}_{ss}^2(s,t) \geq \frac{c^2 s^2}{4\pi} & s \in (0, L),\ t > 0 \\ \underline{U}(0,t) = \underline{U}_s(L,t) = 0 & t > 0 \\ \underline{U}(s,0) \leq U(s,0) & s \in (0, L). \end{cases} \tag{4.37}$$

□

The above result leads to interesting consequences concerning the free boundaries raised by the solution u in the line of papers [25] and [26] (see also [24]).

Another very natural question concerning problem (4.20) is the stabilization of solutions: assumed that

$$f(\cdot,t) \to f_\infty(\cdot) \qquad \text{as} \quad t \to +\infty$$

in suitable functional spaces, is it true that $u(\cdot,t)$ tends to the solution to the associated stationary problem? In the next proposition, reasoning in an analogous way as in the proof of Theorem 4, we obtain the following asymptotic behavior of a solution u to problem (4.20) in a ball.

Proposition 15 *Let B_R be a ball with radius R and let f_∞ be a radially symmetric, negative function defined in B_R. Suppose that $f(x,t) \nearrow f_\infty(x)$ as $t \to +\infty$ for $x \in B_R$. Let u_0 be a convex function in B_R, vanishing on ∂B_R. Let u be a solution to problem (4.20) with Ω replaced by B_R and let ψ be the solution to*

$$\begin{cases} -\det D^2\psi = f_\infty & \text{in } B_R \\ \psi = 0 & \text{on } \partial B_R. \end{cases}$$

Denote

$$U(s,t) = \int_0^s \tilde{u}(\sigma,t)d\sigma, \quad \Psi(s) = \int_0^s \tilde{\psi}(\sigma)d\sigma, \quad U_0(s) = \int_0^s \tilde{u}_0(\sigma)d\sigma.$$

If $U(s,0) \ge \Psi(s)$ for $s \in (0,L)$, then $U(s,t) \ge \Psi(s)$ for every $s \in (0,L)$ and $t > 0$.

Proof It is enough to observe that

$$\begin{aligned} &-U_t + 4\pi^3 U_{ss}^2 \le F \\ &U(s,0) = U_0(s) \text{ for } s \in (0,L) \\ &U(0,t) = U_s(L,t) = 0 \quad \text{for} \quad t > 0 \end{aligned}$$

and

$$4\pi^3 \Psi_{ss}^2 = F_\infty, \qquad \Psi(0) = \Psi_s(L) = 0,$$

where

$$F(s,t) = \int_0^{s^2/4\pi} f^*(\sigma,t)d\sigma, \quad F_\infty(s) = \int_0^{s^2/4\pi} f_\infty^*(\sigma)\, d\sigma.$$

Since by definition of rearrangement $F(s,t) \leq F_\infty(s)$ for any $t > 0$, then

$$-(U-\Psi)_t + 4\pi^3(U_{ss}^2 - \Psi_{ss}^2) < 0.$$

The thesis follows from the maximum principle. □

We end this paper with some considerations on the existence of solutions for the radially symmetric problem (4.23). As in the stationary case, the convexity condition is not always satisfied. So we shall need some extra conditions on the datum g. Without any interest in getting the more general result at all, we shall proceed under some additional assumptions. Let $\Omega^\star = B_{R^\star}(0)$. Suppose that $z_0 \in W^{2,1}(\Omega^\star) \cap W_0^{1,3}(\Omega^\star)$ is a nonpositive, convex, radially symmetric function such that

$$\Delta_3 z_0 \in L^2(\Omega^\star). \tag{4.38}$$

Suppose also that

$$g(x,t) = g(|x|\,,t), \quad g \in C([0,T] : L^\infty(\Omega^\star)), \tag{4.39}$$

and, for some $M > 0$,

$$-M(\cosh T)^2 \leq |x|\, g(|x|\,,t) \leq 0 \quad \text{for any } \ t \in [0,T], \tag{4.40}$$

$$\Delta_3 z_0(x) + |x|\, g(|x|\,,t) \geq 0 \quad \text{for a.e. } \ x \in \Omega^\star \ \text{ and for any } \ t \in [0,T]. \tag{4.41}$$

The following result holds.

Lemma 3 *Under the assumptions (4.38), (4.39), (4.40) and (4.41), there exists a unique convex solution $z \in C([0,T] : L^\infty(\Omega^\star)) \cap L^2([0,T] : W_0^{1,3}(\Omega^\star))$ to problem (4.23) with $\frac{z_t(\cdot,t)}{|x|} - \det D^2 z(\cdot,t) \in L^\infty(\Omega^\star)$ for a.e. $t \in (0,T)$.*

Proof Since the solution must be radially symmetric we know that z is given as the unique solution to the problem

$$\begin{cases} z_t - \frac{1}{2}\Delta_3 z = |x|\, g(|x|\,,t) & \text{in } \Omega^\star \times (0,T) \\ z = 0 & \text{on } \partial\Omega^\star \times (0,T) \\ z(x,0) = z_0(x) & \text{in } \Omega^\star. \end{cases}$$

From the assumptions (4.39) and (4.40), by the T-accretiveness of the operator $-\Delta_3 z$, we know that

$$\left\| [z(\cdot,t)]_+ \right\|_{L^1(\Omega^\star)} \leq \left\| [z_0]_+ \right\|_{L^1(\Omega^\star)} + \int_0^t |x|\, [g(|x|\,,s)]_+ \, ds = 0$$

so that $z(x,t) \le 0$ on $\Omega^\star \times (0,T)$. Moreover, in an analogous way, taking

$$\underline{z}(x,t) \equiv -\|z_0\|_{L^\infty(\Omega^\star)} - M \tanh t$$

we get that

$$\begin{cases} \underline{z}_t - \Delta_3 \underline{z} = \rho(t) & \text{in } \Omega^\star \times (0,T) \\ \underline{z} \le 0 & \text{on } \partial\Omega^\star \times (0,T) \\ \underline{z}(x,0) \le z_0(x) & \text{in } \Omega^\star, \end{cases}$$

with

$$\rho(t) = -\frac{M}{(\cosh t)^2}.$$

By (4.40)

$$\rho(t) \le |x|\, g(|x|\,, t) \qquad \text{for a.e. } (x,t) \in \Omega^\star \times (0,T),$$

then, by the maximum principle, $\underline{z}(x,t) \le z(x,t) \le 0$ in $\Omega^\star \times (0,T)$. Now, in order to prove the convexity of $z(\cdot,t)$ we argue as in Diaz-Kawohl [27] (see the proof of their Theorem 1). We start by pointing out that $0 \le z''(r,t)$ if and only if $\Delta_3 z(r,t) \ge 0$ so, since $|x|\, g(|x|) \le 0$ we only need to prove that $z_t \ge 0$. But this holds once we have condition (4.41) as in [27].

Since $z_0 \in D(\mathscr{C})$, where $\mathscr{C}$ is the operator on $H = L^2(\Omega^\star)$, given by $\mathscr{C}z = -\Delta_3 z$ and since $\mathscr{C}$ is the subdifferential in $L^2(\Omega^\star)$ of a convex function, we get that $z_t(\cdot,t) \in L^2(\Omega^\star)$, $\Delta_3 z(r,t) \in L^2(\Omega^\star)$ for a.e. $t \in (0,T)$ and the equation takes place for a.e. $x \in \Omega^\star$ and a.e. $t \in (0,T)$. Then by dividing by $|x|$, since we have (4.39), we get that $\frac{z_t(\cdot,t)}{|x|} - \det D^2 z(\cdot,t) \in L^\infty(\Omega^\star)$ for a.e. $t \in (0,T)$. □

Remark 9 We argue as in [15] (Lemma 3.3, p. 73) to get some extra regularity. For instance, by multiplying the equation in (4.23) by z_t we get

$$\int_{\Omega^\star} (z_t)^2 + \frac{1}{6}\frac{d}{dt}\int_{\Omega^\star} |\nabla z|^3 = \int_{\Omega^\star} z_t\, |x|\, g(|x|).$$

This shows that $z \in C([0,T] : W_0^{1,3}(\Omega^\star))$ and, by the Hardy inequality, $\frac{z(\cdot,t)}{|x|} \in L^3(\Omega^\star)$ for any $t \in [0,T]$.

In the special case when $\Omega = \Omega^\star$ and the data $f(x,t)$ and $u_0(x)$ are radially symmetric, but not necessarily decreasing along the radii, it is possible to get some information about how the corresponding solution u is becoming each time more similar to its rearrangement $u^\star$. Some results on the asymptotic stabilization to a stationary solution can be obtained trough similar results for the case of the Δ_3 operator (see, e.g. [23] or [28]).

Proposition 16 *Assume that* $z_0 \in W^{2,1}(\Omega^\star) \cap W_0^{1,3}(\Omega^\star)$ *with* $z_0(x) = z_0(|x|)$, $z_0(x) \leq 0$ *in* $\Omega^\star$, *and* $\Delta_3 z_0 \in L^2(\Omega^\star)$. *Suppose also that* $g(x,t) \in L^\infty(\Omega^\star \times (0,+\infty)) \cap W^{1,1}_{loc}((0,+\infty) : L^1(\Omega^\star))$ *satisfies*

$$g(x,t) = g(|x|\,, t)$$

with

$$\int_t^{t+1} |x| \left\| \frac{\partial}{\partial t} g(|x|, s) \right\|_{L^1(\Omega^\star)} ds \leq C \quad \textit{for any } t > 0$$

for some $C > 0$ *independent of t and*

$$-M \leq |x|\, g(|x|\,, t) \leq 0 \textit{ for a.e. } t \in (0,+\infty).$$

Suppose also that there exists $g_\infty \in L^{3/2}(\Omega^\star)$, *with* $g_\infty(x) = g_\infty(|x|)$, $g_\infty(x) \leq 0$ *in* $\Omega^\star$, *such that*

$$\int_t^{t+1} |x|\, \|g(|x|, t) - g_\infty(|x|)\|_{L^{3/2}(\Omega^\star)} \to 0 \textit{ as } t \to +\infty.$$

Then, if z is the unique strong solution to the problem

$$\begin{cases} z_t - \frac{1}{2}\Delta_3 z = |x|\, g(|x|\,, t) & \textit{in } \Omega^\star \times (0,+\infty) \\ z = 0 & \textit{on } \partial\Omega^\star \times (0,+\infty) \\ z(x,0) = z_0(x) & \textit{in } \Omega^\star, \end{cases}$$

z is also solution to the problem

$$\begin{cases} |x|^{-1} z_t - \det D_x^2 z = g(x,t) & \textit{in } \Omega^\star \times (0,+\infty) \\ z = 0 & \textit{on } \partial\Omega^\star \times (0,+\infty) \\ z(x,0) = z_0(x) & \textit{in } \Omega^\star. \end{cases} \tag{4.42}$$

Moreover, $z(.,t) \to z_\infty$ *in* $W_0^{1,3}(\Omega^\star)$, *as* $t \to +\infty$, *where* z_∞ *is the unique solution to*

$$\begin{cases} -\det D^2 z_\infty = g_\infty(x) & \textit{in } \Omega^\star \\ z_\infty = 0 & \textit{on } \partial\Omega^\star. \end{cases}$$

Proof It suffices to apply Lemma 1 and Theorem 1 in [28] to the p-Laplacian operator with $p = 3$. ◻

Finally, as a simple application of Proposition 16, if we assume for instance that

$$g(x,t) = g_\infty(x) = 0, \tag{4.43}$$

we can give an estimate about the progressive perimeter symmetrization in time of $z(.,t)$ (in a similar manner to the one in Proposition 1 of [21]).

Proposition 17 *Let $z_0(x)$ be as in Proposition 16, with $z_0 \neq z_0^\star$ and suppose that (4.43) holds true. If z is the solution to (4.42) given in Proposition 16 and ζ is the solution to the same problem with $z_0^\star$ as initial datum (always with $g(x,t) = 0$), given $r \geq 1$, for any $q > r$ we have*

$$\|z(\cdot,t) - \zeta(\cdot,t)\|_{L^q(\Omega^\star)} \leq Ct^{-\delta} \|z_0 - z_0^\star\|^{\gamma}_{L^r(\Omega^\star)}$$

with

$$\delta = \frac{2(q-r)}{q(3r+2)} \quad \text{and} \quad \gamma = \frac{r(3q+2)}{q(3r+2)}.$$

Proof Obviously $\zeta(\cdot,t) = \zeta(\cdot,t)^\star$. Then, it suffices to apply the characterization of radially symmetric solutions of the lemma and the regularizing estimate (Théorème III.4) of [44] for the p-Laplacian operator with $p = 3$. □

Remark 10 Note that, according to Proposition 16, $z(\cdot,t)$ and $\zeta(\cdot,t) \to 0$ as $t \to +\infty$ in $W_0^{1,3}(\Omega^\star)$.

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