CMNE/CILAMCE 2007
Porto, 13 a 15 de Junho, 2007

MATHEMATICAL MODELS OF HYDROLOGICAL CYCLE

S. N. Antontsev [1,2], J.I. Díaz [2]

1: Departamento de Matemática
Universidade da Beira Interior
6201-001 Covilhã, Portugal
e-mail: anton@ubi.pt, antontsevsn@mail.ru
2: Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, Spain
e-mail: diaz.racefyn@unsde.es

Palavras chave: Simulation of the interaction between surface and ground waters, Nonlinear partial differential equations, Localization properties

Resumo. *The paper is devoted to a mathematical analysis of general models of mass transport and other interconnected physical processes developing in coupled flows of surface, soil and ground waters. Such models widely are used for forecasting (numerical simulation) of a hydrological cycle for concrete territories. The mathematical models that proved realistic are obtained by combining mathematical models of local processes.The water -exchange models take into account the following factors :water flows in confined and unconfined aquifers, vertical moisture migration with allowing earth surface evaporation, open-channel flow simulated by one-dimensional hydraulic equations,transport of contamination, etc. These models may have different levels of sophistication, ranging from systems of balance equations to systems of nonlinear partial differential equations. Investigation of the questions concerning mathematical correctness of these models, such as existence and uniqueness of solutions and the study of they qualitative properties is presented.*

1 INTRODUCTION

The paper is devoted to a mathematical analysis of general models of mass transport and other interconnected physical processes developing in coupled flows of surface, soil and ground waters.

Beginning with 70-90 years in numerous articles were offered mathematical models of mass transport in interconnected processes of surface, soil and grounds waters (see [1, 4, 10, 12],[15]-[21],[23]-[25],[29, 30] and the further references therein). Such models are called Mathematical models of hydrological cycle (MMHC). The mathematical models that proved realistic are obtained by combining mathematical models of local processes.

The water -exchange models (MMHC) take into account the following factors: water flows in confined and unconfined aquifers, vertical moisture migration with allowing earth surface evaporation, open-channel flow simulated by one-dimensional hydraulic equations, transport of contamination, etc.

These models may have different levels of sophistication, ranging from systems of balance equations to systems of nonlinear partial differential equations.

General scheme of hydrological cycle and the grid used in numerical simulation are presented on the figure 1. Vertical and horizontal section of modelling area are presented on the figure 2 and on 3, 4.

Recent activity in the study of these mathematical models and their numerical realization has lead to several very important theoretical problems for nonlinear partial differential equations.

The existing methods used to study properties of solutions to nonlinear degenerate equations do not seem to apply to the systems discussed here.

In the paper is presented investigation of the questions concerning mathematical correctness of the models, such as existence and uniqueness of solutions and the study of they qualitative properties such that asymptotic behavior with respect to time and spatial variables and stability with respect initial data and physical parameters.

MMHC are produced a composition of mathematical models of local processes and may be have different levels of complication, beginning with systems of ordinary equations until systems of partial differential equations.

In the present paper we would like to illustrate and analyze these mathematical singularities. First we demonstrate universally adopted nonlinear partial differential equations describe local hydrological process composing a general model of the interactive model.

Next we concentrate on a more simple model of simultaneous (interactive) flow of surface and ground waters (SGW).

For this model we demonstrate a scheme of the proof for existence and uniqueness theorems, convergence of iterative process splitting with respect to hydrological process. We analyze also localization properties of solutions such that finite time of localization (extinction), finite speed of propagation of disturbances from the initial data, waiting time effect, etc.

2 Mathematical Models of General Hydrological Cycle

2.1 Basic equations

We consider a bounded multiply connected modelling region $\Omega \subset \mathbf{R}^2$ with the external boundary $\Gamma = \partial\Omega = \sum_{i=1}^{n} \Gamma_i$ (see figure 4). Inside Ω there is a system of channels or rivers describes by a set of lines $\Pi = \sum_{i=1}^{l} \Pi_i$. The closed contours $\Gamma_0 = \sum_{j=1}^{m} \Gamma_{0j}$ are fixed boundaries of basins or lakes. The lines Π_i may have points of intersection: with Π (denoted as $N = \sum_{i,j=1} N_{ij}$); with the boundary Γ (denoted as $P = \sum_{i=1} P_i$); with the boundary Γ_0 (denoted as $P_0 = \sum_{i=1} P_{0i}$).

Mathematical models of hydrological cycle are based on the following subsystems (equations):

1. one-dimensional Richards equation for soil water pressure (or moisture) in unsaturated zone ([1, 26, 27, 30]),

$$\frac{\partial \vartheta(\psi)}{\partial t} = \frac{\partial}{\partial x_3}\left[K(\psi)\left(\frac{\partial \psi}{\partial x_3} + 1 \right)\right] + f_k(H,\ \vartheta, x_3, x, t), \tag{1}$$

$$H(x,t) < x_3 < H_e(x),\ x = (x_1, x_2) \in \Omega \in \mathbf{R}^2,$$

$$\vartheta = \vartheta_s / \left[1 + \left(-\frac{\psi}{a} \right)^m \right],\ \ \psi < 0,\ K = K_s\left[\ (\ \vartheta - \vartheta_r)\ /\ (\ \vartheta_s - \vartheta_r)\right]^n,$$

   where θ is the volumetric moisture content, ψ is the pressure of the soil moisture, K is the hydraulic conductivity, and x_3 is the vertical coordinate direct upward, $H(x,t)$ is the level of the ground water (elevation of the ground free surface), $H_e(x_1, x_2)$ is the given surface of the earth.

2. plane filtration equations for the levels of ground waters (consequence from Boussinesq and Shchelkachev equations, [23, 25, 28, 30]),

$$\mu \frac{\partial H}{\partial t} = \mathrm{div}_{(x)}\left(M \nabla H \right) - \frac{k'}{T'}(H - H_1) + f_\Omega,\ x = (x_1, x_2) \in \Omega, t \in (0,T) \tag{2}$$

$$M = k(x)(H - H_1),\ f_\Omega = f_\Omega(H, \vartheta, x, t);$$

$$\mu_1 \frac{\partial H_1}{\partial t} = \mathrm{div}_{(x)}\left(k_1 T_1 \nabla H_1 \right) + \frac{k'}{T'}(H - H_1),\ x \in \Omega, t \in (0,T) \tag{3}$$

   where $H(x,t)$ and H_1 denote respectively the elevation of the groundwater free surface in the upper layer and the piezometric head in the lower layer, μ denotes the yield coefficient (the deficiency of saturation) and μ_1 the storage coefficient; $k,\ k_1$ and k' are the hydraulic conductivity (percolation) coefficients for the corresponding layers and $H_b(x)$ is the given confining bed height above the fixed plane , f_Ω is the a source function(see [8]). The last term in (3) characterize the rate of vertical flow from the upper layer to the one through the semipermeable intermediate layer.

3. hydraulics equation for open channels (diffusion wave approximation of the Saint Venan equations, see ([4, 17, 21, 23, 30] and the further references therein) for the water level in the channel,
$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial s}\left(\psi\phi\left(\frac{\partial u}{\partial s}\right)\right) - Q + f_\Pi,\ s \in \Pi, t \in (0,T), \tag{4}$$
$$\omega = \omega(s,u),\ \ \psi = \psi(s,u), \phi\left(\frac{\partial u}{\partial s}\right) = \left|\frac{\partial u}{\partial s}\right|^{\frac{1}{2}} \mathrm{sign}\left(\frac{\partial u}{\partial s}\right),$$
$$Q = \alpha\ u|_\Pi + \alpha_0\left[M\frac{\partial H}{\partial n}\bigg|_{\Pi_+} - M\frac{\partial H}{\partial n}\bigg|_{\Pi_-}\right];$$
where $u(s,t)$ is the water level in the channel stream , ω is the cross sectional area ($\omega_u = B$ is the width), s is the channel length measured along its axial cross-section, $\psi(s,u) = C\omega R^{\frac{2}{3}}$ is the discharge modulus, C is the coefficient Chezy , R is the hydraulic radius, f_Π is the a source function,
$$[MH_n] = \left(MH_n|_{\Pi_+} + MH_n|_{\Pi_-}\right)$$
is the total filtration inflow of ground water from the right Π_+ and left Π_-banks of the channel, and $H_n = \partial H/\partial n$ is the outer normal derivative to Π.

4. balance equations for the water level in reservoirs on the boundaries of reservoirs([4]),
$$\lambda\frac{\partial z}{\partial t} = -\oint_{\Gamma_0} M\frac{\partial H}{\partial n}ds - (\psi\phi)_{\Gamma_0}, x \in \Gamma_0, t \in (0,T). \tag{5}$$

To obtain models describing the quality of ground and surface waters we need to add the following equations of mass transfer between components (see for example [22])

1. solute transport equation in a confined aquifer
$$\frac{\partial(mC)}{\partial t} = \mathrm{div}\,(D\nabla C - vC) + \Phi(C,N) + f, \tag{6}$$
$$v = -M\nabla H,\ \ D = D_0 + \lambda|v|,\ \ \ m = m_0 + \mu(H - H_p);$$

2. kinetic equation on the skeleton of a porous medium
$$\frac{\partial N}{\partial t} = \Phi(C,N) \tag{7}$$

3. solute transport equation in open-channels (e.g., rivers)
$$\frac{\partial(\varpi S)}{\partial t} = \frac{\partial}{\partial s}(D_1\frac{\partial S}{\partial s} - v_1 S) - (q_1 C) + f, \tag{8}$$
$$D_1 = D_0^1 + \lambda_1|v_1|,\ v_1 = -\Psi(s,u)|u_s|^{1/2}sign(u_s).$$

2.2 Model of coupled flows of surface and ground waters(SGW)

2.2.1 Basic equations

Usually the basic mathematical models of hydrological cycle are a composition of above stated equations completed by corresponding initial, boundary conjugate conditions. Now we will consider the mathematical model describing interconnected process between surface (lake, channel) and ground waters, neglecting unsaturate zone and assuming that there is only one nonpressure layer (see figure 3,4). Then equations (2), (3), (4), (5) reduce to the following ones:

$$\mu\frac{\partial H}{\partial t} = \mathrm{div}_{(x)}\left(M\nabla H\right) + f_{\Omega},\ x = (x_1, x_2) \in \Omega, t \in (0, T), \tag{9}$$

$$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial s}\left(\psi\phi\left(\frac{\partial u}{\partial s}\right)\right) - Q + f,\ s \in \Pi, t \in (0, T), \tag{10}$$

$$\lambda\frac{\partial z}{\partial t} = -\oint_{\Gamma_0} M\frac{\partial H}{\partial n} ds - \left(\psi\phi\right)_{\Gamma_0}, x \in \Gamma_0,\ t \in (0, T), \tag{11}$$

for unknown functions $W(x,t) = (\ H(x,t),\ u(s,t), z(t))$ ..

2.2.2 Initial, boundary and conjugate conditions

Stated above system should be completed by the following initial, boundary and conjugate conditions

$$W(x, 0) = W_0(x),\ x \in \Omega, \tag{12}$$

$$\left(\sigma_1 M\frac{\partial H}{\partial n} + \sigma_2 H\right) = g, \quad (x,t) \in \Gamma_T = \Gamma \times [0, T], \tag{13}$$

$$\kappa_1\psi(s,u)\phi(\frac{\partial u}{\partial s}) + \kappa_2 u = g, \quad (x,t) \in P_T = P \times [0, T], \tag{14}$$

$$M\frac{\partial H}{\partial n}\bigg|_{\Pi^{\pm}} = \alpha\left(u - H_{\pm}\right) + \alpha_0\left(H_+ - H_-\right),\ (x,t) \in \Pi_T = \Pi \times [0, T], \tag{15}$$

$$u_i = u_j,\ \sum_{i=1}\psi(s, u_i)\phi(\frac{\partial u_i}{\partial s}) = 0\ \ (x,t) \in N_T = N \times [0, T], \tag{16}$$

$$H = z(t),\ x \in \Gamma_0,\ t \in (0, T). \tag{17}$$

The mathematical model is being described by equations (9)-(11), (12)-(17) we will call (SGW)-model. The mathematical model just described leads to investigation of combined-type nonlinear systems of partial differential equations (PDEs). The systems are fairly complicated.

The equations that they include are defined on different sets of space variables. Equation (9) is defined in two dimension domain Ω, equation (10) on the curve Π , (11) one is an ordinary differential equation with respect to time and the right side of the one is nonlocal operator over $H, \frac{\partial H}{\partial n}$. These equations degenerate changing type or order at certain values of the solution that is sought (or its derivatives). These equations contain numerous physical parameters. The interaction between different physical processes is simulated by source functions included into differential equations, as well as by internal boundary conditions.

The solutions of such equations may exhibit behavior that cannot occur in linear models. The list of effects of this kind includes: finite time of localization (extinction), finite speed of propagation of disturbances from the initial data, waiting time effect,etc. The questions of mathematical correctness -existence, uniqueness and qualitative properties of solutions were analyzed preliminary in [8]. The mathematical singularities of such equations and its solutions will be demonstrated more intuitively in next section.

3 Mathematical questions in a simple SGW

Here we consider a simple case above stated model of coupled flows of surface and ground waters(SGW)under the following assumptions(see figure 5,6):the ground is homogeneous and isotropic, base impermeable is horizontal

$$M = H; \tag{18}$$

the area of flow cross-section of the channel is given by

$$\omega(s, u) = u; \tag{19}$$

function $\psi(s, u) = |u|^{\alpha}$, where the parameter α is defined by the geometry of channel.

Assumed also that the coefficients $\sigma_1 = \kappa_1 = 0$ and the levels of the ground waters on the left and right banks and the level of the water of the channel coincide. Last condition follows from (15) if $\alpha = \infty$ and $\alpha_0/\alpha = 0$.

3.1 Statement of the mathematical problem

3.1.1 System of Equations

Under above stated propositions equations of SGW model take the form

$$\frac{\partial H}{\partial t} = \nabla\left(H \nabla H\right) + f_{\Omega}, \quad x, t \in \Omega^{\Pi}_{T} = \Omega^{\Pi} \times (0, T), \;\; \Omega^{\Pi} = \Omega/\Pi, \tag{20}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s}\left(\mid u \mid^{\alpha} \mid \frac{\partial u}{\partial s} \mid^{-1/2} \frac{\partial u}{\partial s}\right) + \left[H \frac{\partial H}{\partial n}\right]_{\Pi} + f_{\Pi}, \;\; x, t \in \Pi_T, \tag{21}$$

3.1.2 Initial, boundary and conjugate conditions

Initial, boundary and conjugate conditions take the form

$$H(x,0)=H_0(x), \quad u(x,0)=u_0(x),\ x\in\Omega, \tag{22}$$

$$H_+ = H_- = u, \quad x,t\in\Pi_T=\Pi\times[0,T], \tag{23}$$

$$H=g, \quad x,t\in\Gamma_T,\ u=g,\ x,t\in\Pi_T \tag{24}$$

The cross-section and plan view of the modelling domain are presented on the figures 5, 6. We assume that there exist functions $H_0(x,t), u_0(x,t)$ defined on $\Omega\times(0,T)$ such that

$$H_0|_{\Gamma_T}=g, \quad u_0\,|_{\Pi_T}=g;\ \ H_0(x,0)=H_0(x), u_0\,(x,0)=u_0(x), x\in\Omega, \tag{25}$$

$$|H_0\,|\,,|u_0\,|\,,\||\nabla H_0|+|H_{0\ t}|\|_{2,\Omega_T}\,,\|u_{0s}\,\|_{3/2,\Pi_T}\,,\ \|u_{0t}\|_{2,\Pi_T}\le C<\infty. \tag{26}$$

We assume also

$$\int_0^T\left(\max_x|f_\Omega|+\max_s|f_\Pi|\right)ds\le C<\infty \tag{27}$$

3.2 Existence and uniqueness theorems

Definition 1 *Non negative bounded functions* $(H,u)=\overrightarrow{W}$ *such that*

$$0\le(H(x,t),u(s,t))\le C<\infty \tag{28}$$

$$\int_0^T\left(\int_{\Omega^\pm}H|\nabla H|^2dx+\int_\Pi\left(u^\alpha|\,u_s\,|^{\frac{3}{2}}\right)ds\right)dt\le C<\infty \tag{29}$$

are called weak solution of simple (GSW) model (20)-(24) if for every test -function η *such that*

$$\eta\in W_2^{1,1}(\Omega_T)\cap W_{3/2}^{1,1}(\Pi_T),\ \eta=0,\ (x,t)\in\Gamma_T=\Gamma\times(0,T)$$

and every $t\in[0,T]$ *the following identity holds*

$$\int_0^t\int_\Omega(-H\eta_t+H\nabla H\nabla\eta)\,dxdt+\int_\Omega H(x,\tau)\eta(x,\tau)dx|_{\tau=0}^{\tau=t} \tag{30}$$

$$+\int_0^t\int_\Pi(-u\eta_t+\psi\varphi(u_s)\eta_s)\,dsdt+\int_\Pi u(s,\tau)\eta(x,\tau)dx|_{\tau=0}^{\tau=t}$$

$$=\int_0^t\int_\Omega f_\Omega\eta dxdt+\int_0^t\int_\Pi f_\Pi\eta dsdt.$$

Theorem 1 *Let us assume that (26), (27) hold and*

$$0 \leq (\ f_\Omega, f_\Pi, H_0, u_0) \leq C_0 < \infty, 0 < \alpha < \infty. \tag{31}$$

Then the simple SGW model has at least one weak solution $W(x,t) = (H,u)$.

If additionally

$$0 < \delta \leq (u_0, H_0, g) \leq C_0 < \infty, \tag{32}$$

$$\int_0^T \left(\max_x |\partial f_\Omega / \partial t| + \max_s |\partial f_\Pi / \partial t| \right) ds \leq C < \infty \tag{33}$$

$$H_{0xt}, H_{0tt} \in L^2(\Omega_T^\pm), \nabla H_0 \in L^\infty(0,T; L^2(\Omega_T^\pm)), \ u_{0s} \in L^\infty(0,T; L^{\frac{3}{2}}(\Pi)), u_{0st} \in L^2(\Pi_T) \leq C, \tag{34}$$

then there exist a small $T_0 > 0$ *such that this weak solution is unique and the following estimates are valid*

$$\sup_{0 \leq t \leq T_0} \left(\int_{\Omega^\pm} | \nabla H |^2 dx + \int_\Pi | \underset{s}{u} |^{\frac{3}{2}} ds \right) \leq C < \infty \tag{35}$$

$$\int_0^{T_0} \int_{\Omega^\pm} \left(\ {H_t}^2 + | \ H_{xx} \ |^2 \right) dxdt + \int_0^T \int_\Pi \left({u_t}^2 + | \ u_s \ |^{-1/2} | \ u_{ss} \ |^2 \right) dsdt \leq C < \infty. \tag{36}$$

Proof 1 *First we assume that (32) holds. Then it follows from Definition of weak solution that*

that

$$0 < \delta \leq (\ H, u) \, .$$

The solution is constructed as the limit of the sequence of Galerkin's approximations. It assumed that the domain Ω *admits a complete system of the functions* $\Phi_k(\vec{x}) \in W_2^1(\Omega)\}$, $\Phi_k(\vec{x})_\Gamma = 0$ *which are dense in* $W_2^1(\Omega)$. *Respectively assumed that the set of the lines* Π *admits a complete system of the functions* $\Psi_k(s) \in W_{\frac{3}{2}}^1(\Pi)\}, \Psi_k(s)_{\Gamma \cup \Pi} = 0$ *which are dense in* $W_{\frac{3}{2}}^1(\Pi)\}$. *Without loss of the generality we assume that the functions* Φ_k *and* Ψ_k *are orthogonal in* $L^2(\Omega)$, $L^2(\Pi)$. *Let us consider the approximate solution in the form*

$$\vec{W}^N = (H^N, u^N) = \left(\sum_{k=1}^N H_k(t)\Phi_k(\vec{x}) + H_0, \ \sum_{k=1}^N u_k(t)\Psi_k(s) + u_0 \right), \tag{37}$$

where the functions H_0 *and* u_0 *satisfy the corresponding boundary conditions.*

We substitute last presentation into corresponding differential equations, multiply by $\Phi_j^\pm(x)$ *and* $\Psi_j(s)$ *and integrate over* $\Omega_\pm$ *and* Π.

This leads us to the Cauchy problem for a system of nonlinear ordinary differential equations

$$\frac{d\vec{Y}^N}{dt} = \vec{F}(t, \vec{Y}^N), \quad C = \vec{Y}_0^N, \quad (\vec{Y}^N = (H_1, .., H_N, u_1, .., u_N)), \tag{38}$$

with a given smooth with respect to $\vec{Y}^N$ vector function $\vec{F}$. Multiplying equations (38) by the vector $\vec{Y}^N$ and summing, we arrive at the relation

$$\frac{1}{2}\left(\int_{\Omega^\pm} \mid H^{N}(x,\tau)-H_0|^2dx+\int_\Pi \mid u^N(x,\tau)-u_0|^2ds\right)\Bigg|_{\tau=t}^{\tau=t}) \tag{39}$$

$$+\int_0^t\left(\int_{\Omega^\pm} H^{N}\nabla H^N\nabla(H^N-H_0)dx+\int_\Pi\left(|u^N|\frac{1}{\sqrt{|u_s^N|}}\frac{\partial u^N}{\partial s}(u_s^N-u_{0s})\right)ds\right)d\tau$$

$$-\int_0^t\left(\int_{\Omega^\pm} H_{0t}(H^N-H_0)dx+\int_\Pi u_{0t}(u^N-u_0)ds\right)d\tau.$$

To prove the estimates (35), (36) we differentiate (38) with respect to t and multiply by $d\vec{Y}^N/dt$. Obtained estimates permit us to pass to the limit when $N\ \to\infty$ and $\delta>0$ and next to pass to the limit when $\delta\to 0$.

Remark 1 *Notice that presentation (37) may be used as an approximative solutions if the functions Φ_j,Ψ_j may be constructed effectively.*

3.3 Splitting with respect to physical process.

In this section we propose a method to solve the mathematical model of SGW.

3.3.1 Iterative process for differential equations

The given algorithm use the splitting the initial problem into two following independent problems:

I. Plane filtration in the domain $x\in\Omega/\Pi, t\in(0,T)$,$k=1,2,...$

$$\frac{\partial H^k}{\partial t}=\nabla\left(H^k\nabla H^k\right)+f_\Omega,\qquad x\in\Omega/\Pi, \tag{40}$$

$$H^k(x,0)=H_0(x),\qquad x\in\Omega \tag{41}$$

$$H^k\frac{\partial H^k}{\partial n}\mid_{\pm}=\sigma\left(u^{k-1}-H^k{}_\pm\right),\qquad x\in\Pi \tag{42}$$

$$\left(\sigma_1H^k\frac{H^k\partial H^k}{\partial n}+\sigma_2H^k\right)=g,\qquad x\in\Gamma=\partial\Omega \tag{43}$$

II. Flow in the channel Π

$$\frac{\partial u^k}{\partial t}=\frac{\partial}{\partial s}\left(\mid u^k\mid^\alpha\mid\frac{\partial u^k}{\partial s}\mid^{-1/2}\frac{\partial u^k}{\partial s}\right)+\left[H^k\frac{\partial H^k}{\partial n}\right]_\Pi+f_\Pi, x\in\Pi \tag{44}$$

$$\left(\kappa_1\psi(s,u^k)\phi(\frac{\partial u^k}{\partial s})+\kappa_2 u^k\right)=g, x\in\Pi\cap\Gamma. \tag{45}$$

$$u^k(x,0)=u_0(x), \quad , \ x\in\Omega \tag{46}$$

Let us introduce $h^k=H^k-H^{k-1},\ z^k=u^k-u^{k-1}$.

Theorem 2 *Let assumptions (32)- (34)) of existence theorem be valid. Then*

$$y_k(t)=||h^k||^2_{2,\Omega}+||\nabla h^k||^2_{2,\Omega_T}+||u^k||^2_{2,\Pi}+||u^k_s||^{3/2}_{3/2,\Pi_T}$$

$$\leq\frac{(Ct)^{k-1}}{(k-1)!}y_0(t)\to 0, k\to\infty,\ t\leq T.$$

Remark 2 *To numerical simulate of the independent problems I and II may be applied well-known finite-difference schemes.*

4 Localization Properties of Solutions

In this section we demonstrate the localization properties of solutions of problem SGW following to the book [3]. We start with the initial-boundary value problem for independent equation of diffusion waves EDW (equation (10) with $Q=0$).

4.1 Equation of Diffusion Waves(EDW)

Let us consider the following initial boundary value problem

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial s}\left(|u|^\alpha\left|\frac{\partial u}{\partial s}\right|^{-1/2}\frac{\partial u}{\partial s}\right)+f_\Pi, s\in[-1,1], t\in[0,T], \tag{47}$$

$$u(i,t)=u^0(t),\ i=-1,1,\ (or\ \frac{\partial u(1,t)}{\partial s}=0), \tag{48}$$

$$u(s,0)=u_0(s), t\in]0,T[, \tag{49}$$

$$0<\delta\leq(u^i(t),u_0(s))\leq C_0 \tag{50}$$

4.1.1 Finite time stabilization to a non zero state

Theorem 3 *Let conditions (50),(32), be fulfilled and $f_\Pi(s,t)\equiv 0$. Then the solution of problem (47)-(49) is constant beginning with a finite time t^*:*

$$u(s,t)\equiv u^0 \qquad fors\in[0,1], t\geq t^*.$$

If $f_\Pi\not\equiv 0$, and for some $t_f>t^*$

$$\|f_\Pi\|^{3/2}_{L^2(\Omega)} \leq \varepsilon \left(1 - \frac{t}{t_f}\right)^4_+, \tag{51}$$

with a small constant ε, then the following estimate holds:

$$\left\|u(\cdot,t) - u^0\right\|^2_{L^2(\Omega)} \leq C \left(1 - \frac{t}{t_f}\right)^{4)}_+.$$

Specifically,

$$u(s,t) \equiv u^0, s \in [-1,1], t \geq t_f.$$

In physical terms, first assertion of the theorem means that the water level in the channel becomes constant in a finite time provided the external source f_Π is absent (see figures 7 and 11).

If $f \not\equiv 0$ and condition (51) is fulfilled, one can point out the source intensity $\varepsilon > 0$, such that the water level in the channel stabilizes at the same instant t_f when the source disappears.

4.1.2 Finite speed of propagation. Waiting time phenomenon

Here as opposed to the above stated situation we consider local properties of weak solution of equation (47) with zero initial data on the interval $s \in[-\rho,\rho]$,(see figures 8, 9).

Theorem 4 (Finite speed of wetting of a dry bottom) *Let $u(s,t) \geq 0$ be a weak solution of equation (47) with $\alpha > 1/2$ and let*

$$f_\Pi = 0, \qquad u_0(s) = u(s,0) = 0 \quad for|s| \leq \rho_0, t \in (0,T). \tag{52}$$

Then

$$u(s,t) = 0 \quad for|s| \leq \rho(t), \quad \theta = \theta(\alpha) > 0. \tag{53}$$

where $\rho(t)$ is defined by the formula

$$\rho^{1+\sigma}(t) = \rho_0^{1+\sigma} - Ct^\theta$$

with constants $C = C(C_0,\alpha)$, $\theta = \theta(\alpha)$, $\sigma = \sigma(\alpha)$. If, additionally to (52),

$$\int_{-\rho}^{\rho} |u_0(s)|^2 ds + \int_0^T \int_{-\rho}^{\rho} |f_\Pi|^2 ds dt \leq \varepsilon(\rho - \rho_0)_+^{1/(1-\nu)}, \quad \rho_0 \leq \rho,$$

then there exists $t_* \in [0,T)$ such that

$$u(s,t) = 0, s \in [-\rho_0, \rho_0, t \in [0,t_*]].$$

4.2 Coupled Flow of Surface and Ground Waters(SGW)

Let us return to equations (15), (20),(21), with $\alpha_0 = 0$, describing coupled flows. We will consider weak solutions of equations of this system in the domain $B_\rho \times (0,T), B_\rho = \{x \in \Omega | |x - x_0| < \rho\}$.

4.2.1 Finite Speed of Propagation. Waiting Time Phenomenon.

Theorem 5 *Let $W = (H, u)$ be a local weak solution of equations* (15) *, (20),(21) under the assumptions*

$$H_0(x) = 0, \quad f_\Omega = 0 \quad (x,t) \in B_{\rho_0} \times [0,T),$$

$$u_0(s) = 0, \quad f_\Pi = 0 \quad (s,t) \in \Pi_{\rho_0} \times [0,T).$$

Then there exist $t_* \in (0,T)$ and $\rho(t)$ such that

$$H(x,t) = 0 \quad x \in B_{\rho(t)}, \quad u(s,t) = 0 \quad s \in \Pi_{\rho(t)}, \quad t \in [0, t_*]$$

with $\rho(t)$ defined by the formula

$$\rho^{1+\sigma}(t) = \rho_0^{1+\sigma} - Ct^\theta,$$

with some constant C. If, moreover,

$$\|H_0\|^2_{L^2(B_\rho)} + \|u_0\|^2_{L^2(\Pi_\rho)} + \int_0^T \left(\|f_\Omega\|^2_{L^2(B_\rho)} + \|f_\Pi\|^2_{L^2(\Pi_\rho)} \right) d\tau$$

$$\leq \varepsilon \left(\rho - \rho_0\right)_+^\vartheta, \quad \rho > \rho_0, \quad \vartheta(\alpha) > 0.$$

Reverting to the original physical problem we interpret the results as follows. If the domain B_{ρ_0} was dry at the initial moment i.e. the levels of the surface and ground water were zero therein, then the first assertion of the theorem gives estimates on the location of the free boundaries $H(x,t)$ and $u(s,t)$ (see figure 11). The second assertion states that whatever the flux outside B_{ρ_0}, this domain can only be swamped in a finite non-zero time.

Some presented results and formulations of the problems it is possible to find in [2]-[19].

Acknowledge

The work of the first author was supported by the research projects DECONT, FCT/MCES (Portugal) at “Centro de Matemática”, Universidade da Beira Interior, POCI/ MAT/ 61576/ 2004, FCT (Portugal) and SAB-2005-0017 of the Secretaria de Estado de Universidades e Investigación (Spain). This paper was written when the first author was Sabbatical Professor at the Departamento de Matemática Aplicada of the University Complutense, Madrid.

REFERÊNCIAS

[1] M. Abbott, J. Bathurst, and e. a. Cunge J.A., *An introduction to the european hydrological system (she), part 2*, J. Hydrol., 87 (1986), pp. 129–148.

[2] S. N. Antontsev and J. I. Díaz, *Space and time localization in the flow of two immiscible fluids through a porous medium: energy methods applied to systems*, Nonlinear Anal., 16 (1991), pp. 299–313.

[3] S. N. Antontsev, J. I. Díaz, and S. Shmarev, *Energy Methods for Free Boundary Problems:Applications to Non-linear PDEs and Fluid Mechanics*, Bikhäuser, Boston, 2002. Progress in Nonlinear Differential Equations and Their Applications, Vol. 48.

[4] S. N. Antontsev, G. P. Epikhov, and A. A. Kashevarov, *Mathematical system modelling of water exchange processes*, "Nauka" Sibirsk. Otdel., Novosibirsk, 1986.

[5] S. N. Antontsev and A. A. Kashevarov, *Finite rate of propagation of perturbations in simultaneous flows of surface and ground water*, Dinamika Sploshn. Sredy, 57 (1982), pp. 21–27. (in Russian).

[6] ——, *Correctness of the mathematical model of joint motion of surface and ground waters*, Uspekhi Mat. Nauk, 4 (1984), p. 116. (In Russian).

[7] ——, *Splitting according to physical processes in the problem of interaction between surface and underground water*, Dokl. Akad. Nauk SSSR, 288 (1986), pp. 86–90. (English translation: Soviet Phys. Dokl, v.31, n.5, 1986, pp.381–383).

[8] ——, *Localization of solutions of nonlinear parabolic equations that are degenerate on a surface*, Dinamika Sploshn. Sredy, 111 (1996), pp. 7–14. (in Russian).

[9] ——, *Solutions localization of nonlinear parabolic equations with degeneracy on a surface.* Abstracts of the International Conference "Nonlinear Partial Differential Equations", Kiev, August 26-30 1997. National Academy of Science of Ukrain.

[10] ——, *Correctness of a hydraulic model of filtration for ground waters with an unsaturated zone*, in Mathematical models in filtration and its applications, Institute of Hydrodynamics, Novosibirsk, 1999, pp. 21–35.

[11] ——, *Mathematical models of mass transfer in interconnected processes of surface, soil and ground waters*, in Abstracts of the International conference "Modern approaches to flows in porous media", September 6–8, Moscow, 1999, pp. 165–166.

[12] S. N. Antontsev, A. A. Kashevarov, and A. M. Meĭrmanov, *Numerical modelling of simultaneous motion of surface channel and ground waters*, in Abstracts of the International conference on numerical modelling of river, channel overland flow for water resources and environmental applications, May 4–8, Bratislava, Czechoslovakia, 1981, pp. 1–11.

[13] S. N. Antontsev, A. A. Kashevarov, and A. Semenko, *An iterative method for solving a stationary problem on saturated-unsaturated filtration in a hydraulic approximation*, Filtration and Mass-Transfer through Porous Media, 90 (1989), pp. 3–15. (In Russian).

[14] S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids*, North-Holland Publishing Co., Amsterdam, 1990. Translated from original russian edition: Nauka, Novosibirsk, (1983).

[15] S. N. Antontsev and A. M. Meĭrmanov, *Mathematical models of the coupled motion of surface and ground waters*, in Mechanics, Third Congress: Theoretical and Applied Mechanics,(Varna, Sept.13-16,1977), vol.1, Acad. of Sci. of the Bulgaria, Sofia, 1977, pp. 223–228.

[16] ——, *Mathematical models of simultaneous motions of surface and ground waters*, Novosibirsk State University, Lecture Notes, 1979.

[17] ——, *Mathematical questions of the correctness of initial-boundary value problems of a hydraulic model of diffusion waves*, Dinamika Sploshnoi Sredy, 30 (1977), pp. 7–34.

[18] ——, *Mathematical questions on the correctness of a model of the simultaneous motion of surface and underground water*, Dinamika Zhidkosti so Svobod. Granitsami, 31 (1977), pp. 5–51.

[19] ——, *Questions of correctness of a model of the simultaneous motion of surface and ground waters*, Dokl. Akad. Nauk SSSR, 242 (1978), pp. 505–508.

[20] A. Cunningham and S. P.J., *Application and analysis of coupled surface and groundwater model*, J. Hydrol., 43 (1979), pp. 129–148.

[21] G. Epikhov, *Computation system modelling of water exchange in interactive regime)*, Vodnue resursy, 5 (1985), pp. 11–25.

[22] A. A. Kashevarov, *Mathematical modeling of salt transport by coupled subsurface and surface water flows*, J. Applied Mechanics and Technical Physics, 39 (1998), pp. 584–591.

[23] L. Kuchment, V. Demidov, and Y. G. Motovilov, *Formation of a River Basin*, Nauka, Moscow, 1983. Russian.

[24] L. Luckher, *Gekoppelte grundwasser-oberflächen-wassermodell(a coupled model of ground and surface waters)*, Wasserwirt.-Wassertechn.,(WWT), 29(8) (1978), pp. 276–278.

[25] J. Miles, *Modelling the interaction between aquifers and rivers*, in Advances in Water Engineering, Elsev. Appl. Scien. Publ., Ltd., London-New York, 1985, pp. 94–100.

[26] J. Philip, *Evaporation and moisture and heat fields in the soil*, J. Meteorology, 14 (1957), pp. 354–366.

[27] ——, *The theory of infiltration*, Adv. Hydrosci., 5 (1969), pp. 215–296.

[28] P. Y. Polubarinova-Kochina, *Theory of the Motion of ground Waters*, Nauka, Moscow, 1977. Russian.

[29] V. Usenko and V. Zlotnik, *Mathematical models and numerical methods in the problems of interconnection of unconfined grounwater and surface water*, in Proceedings of Third Int. Symp., Kiev, 1976, Part 4, naukova Dumka, Kiev, USSR, Providence, RI, 1978, pp. 108–117.

[30] O. F. Vasiliev, *System modelling of the interaction between surface and ground waters in problems of hydrology*, Hydrological Sciences, 32 (1987), pp. 297–311.

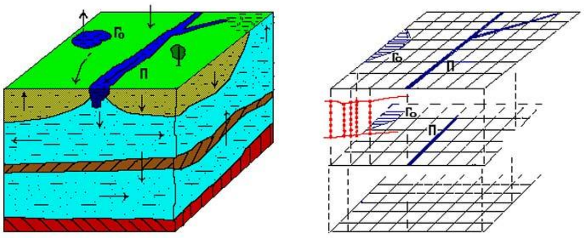

Figura 1: Scheme of interaction of underground and surface waters: a) area of modelling;b) interface of computational grids

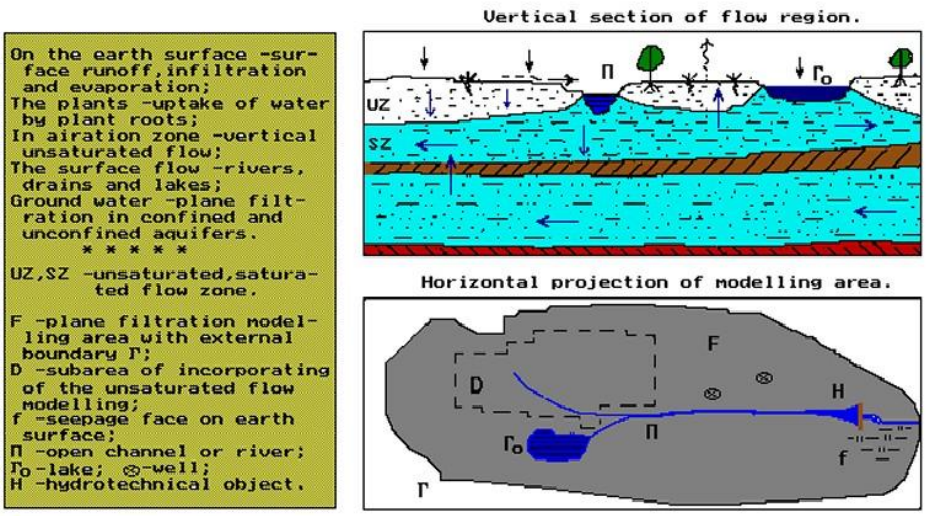

Figura 2: Vertical cross-section of the flow domain and plan view

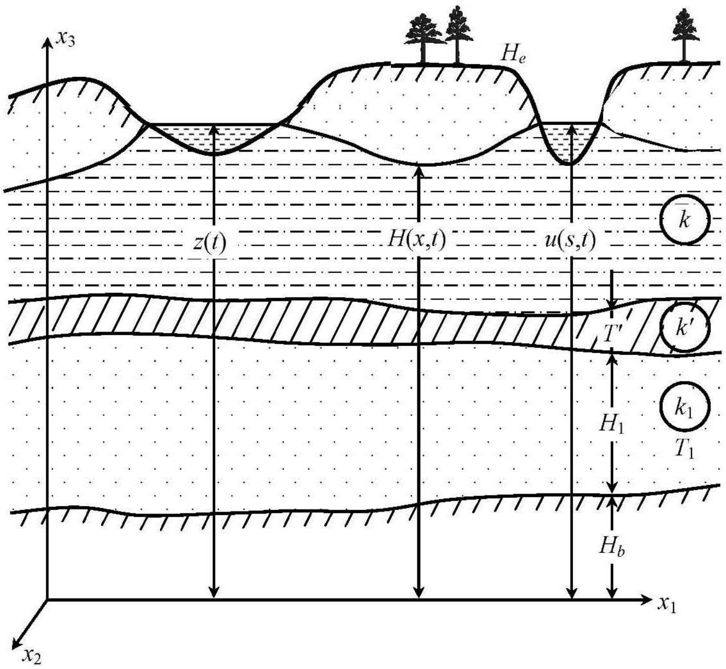

Figura 3: Vertical cross-section of the flow domain

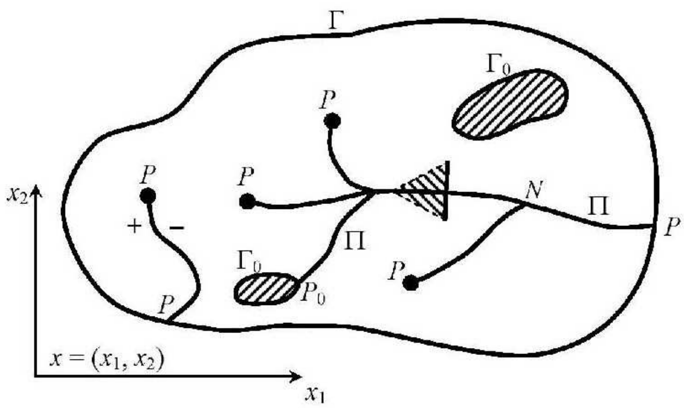

Figura 4: Plan view

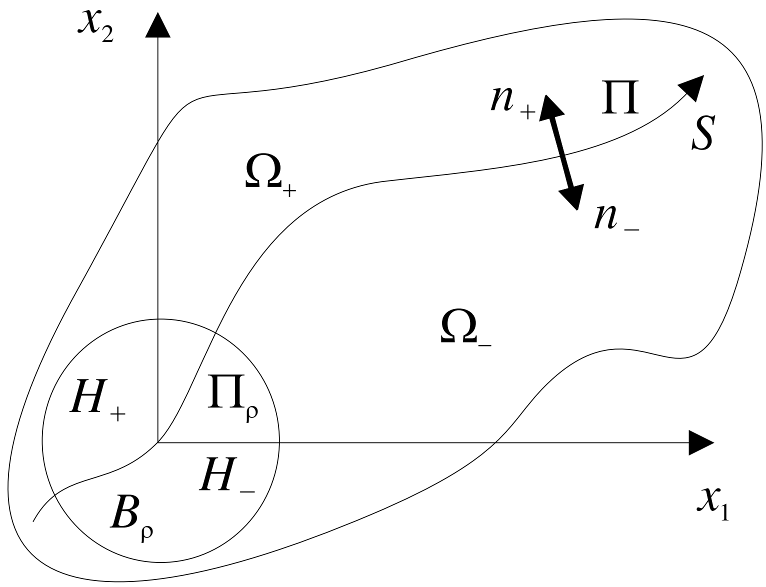

Figura 5: Plan view

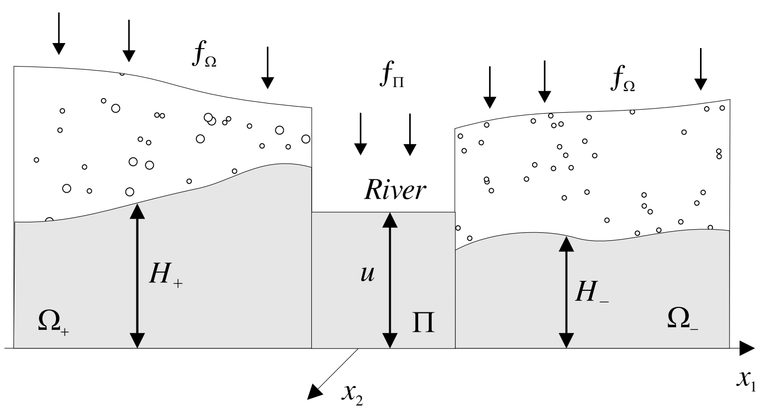

Figura 6: Vertical cross-section of the flow domain

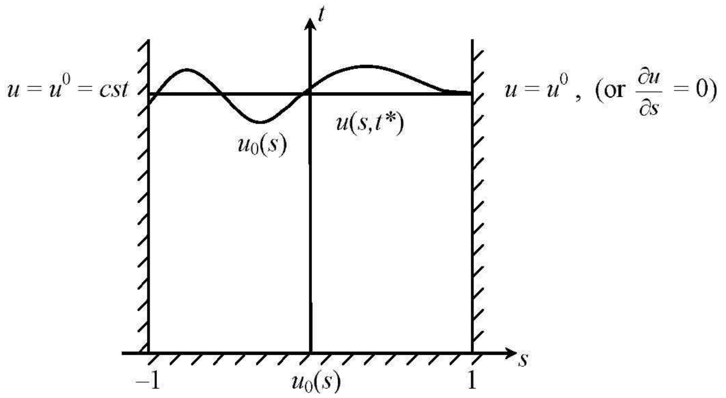

Figura 7: Stabilization to a stationary state(EDW)

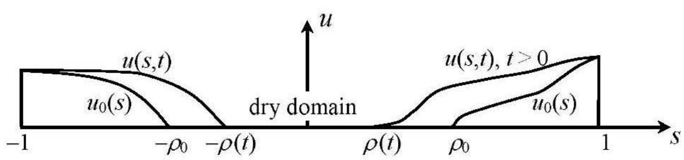

Figura 8: Finite speed of wetting(EDW)

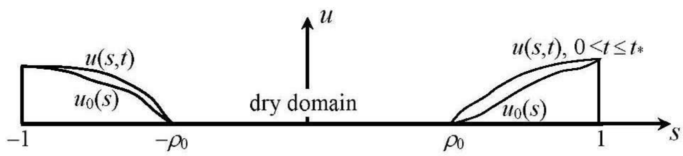

Figura 9: Waiting time of wetting(EDW)

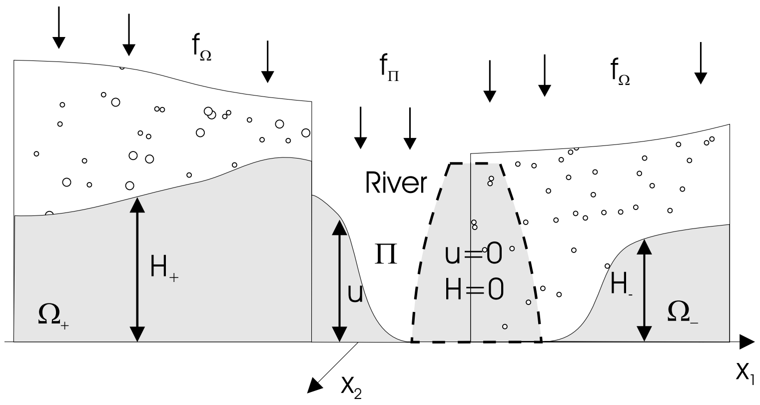

Figura 10: Finite speed of wetting(CF)

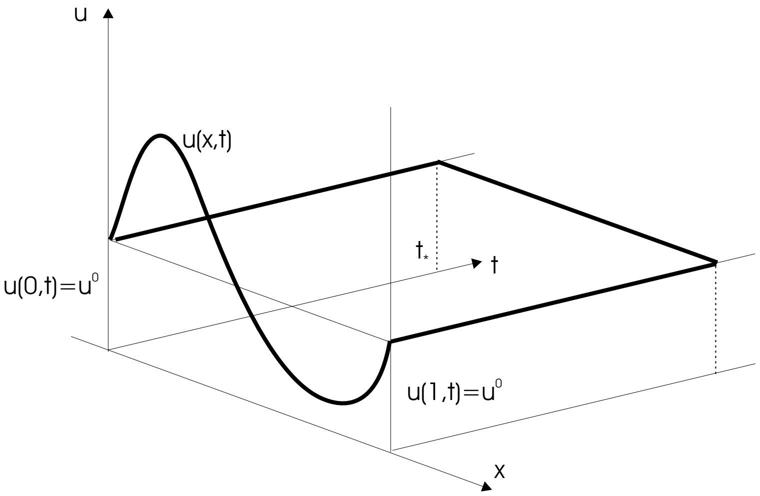

Figura 11: Stabilization to a stationary state(EDW)