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# Contributions to nonlinear partial differential equations



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# Free boundary problems for some stationary reaction-diffusion systems

## 1. INTRODUCTION

We give here a brief survey of some recent results concerning the existence of free boundaries for a class of reaction-diffusion systems arising in combustion theory. Complementary results and complete proofs can be found in [11].

Here we consider a model system describing a single, irreversible non-isothermic stationary reaction

$$\left. \begin{aligned} -\Delta u + \mu^2 F(u) e^{\frac{\gamma(v-1)}{v}} &= 0 && \text{in } \Omega \\ -\Delta v - \nu \mu^2 F(u) e^{\frac{\gamma(v-1)}{v}} &= 0 && \text{in } \Omega \end{aligned} \right\} \quad (1.1)$$

$$u = v = 1 \quad \text{on } \partial\Omega \quad (1.2)$$

where  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ ,  $\mu^2$  is the Thiele number,  $\nu$  is the Prater temperature and  $\gamma$  is the Arrhenius number (cf. [2]). The function  $F$  is assumed to be increasing and such that  $F(0) = 0$ ,  $F(1) = 1$  and  $F(s) > 0$  for  $s > 0$ . The unknowns  $u$  and  $v$  are non-negative and they represent, respectively, the concentration and temperature of the reactant. Often  $F$  takes the form  $F(u) = u^p$ , where  $p > 0$  is the reaction order (cf. [2]). In the case  $p = 0$ ,  $F$  is given by  $F(0) = 0$  and  $F(s) = 1$  if  $s > 0$ . (Thus,  $F$  is discontinuous.)

Existence and uniqueness results for the parabolic problem associated with (1.1) and (1.2) were given in [1] and [3] for  $p > 1$ . Existence and, in some particular cases, uniqueness for the elliptic problem can be found in [1] or [12], again for  $p > 1$ . The case  $0 < p < 1$  is considered in [2], p. 311 (cf. also [14]) but existence theorems are not given. It is shown in [2] and [14] that for  $p = 0$  and  $\mu$  large enough, strictly positive solutions cannot exist. It is also shown, in particular examples, that the set  $\Omega_0 = \{x \in \Omega \mid u(x) = 0\}$ , which is called the *dead core*, has positive measure for

The main idea used in [14] and other papers is to reduce (1.1) and (1.2) to a nonlinear elliptic equation for  $u$  alone. Here we follow a different approach which allows us to include also the case of nonlinear boundary conditions, which cannot be handled by the preceding device.

We consider the case of discontinuous  $F$  in the framework of maximal monotone graphs (cf. [6]). As known existence theorems for elliptic reaction-diffusion systems are given for locally Lipschitz nonlinearities, which is not the case here for  $0 < p < 1$ , it is necessary to prove existence in this more general situation. This can be done by following the same lines as [12] with a fixed-point argument using coupled sub- and supersolutions and the results in [8]. (Cf. [11] for the details.)

We also study the existence and non-existence of a dead core  $\Omega_0$  where  $u = 0$  and then the existence of the free boundary  $\partial\Omega_0$  (cf. [10] for the case of a single equation). Roughly speaking, such a dead core for (1.1) and (1.2) arises when diffusion is unable to supply enough reactant from outside  $\Omega$  to reach the central region of  $\Omega$  (cf. [14]). This can occur if the reaction rate  $F(u)e^{\gamma(v-1)/v}$  remains high for small  $u$ . Thus, the existence of  $\Omega_0$  depends on three main factors: the reaction order  $p$ , the Thiele number  $\mu^2$ , and the size of  $\Omega$ . Moreover, we obtain some information about the location and size of the dead core  $\Omega_0$  and its dependence on  $\mu$ . All these results are collected in Theorem 1 of the following paragraph.

Finally, we point out that C. Bandle, R.P. Sperb and I. Stakgold have obtained independently similar results for (1.1) and (1.2) by using a different method.

## 2. MAIN RESULTS

Here we state our main theorem concerning free boundaries, together with some indications about the method of proof and the meaning of the results.

We consider the system

$$\left. \begin{aligned} -\Delta u + \alpha(u) f(v) &= 0 && \text{in } \Omega \\ -\Delta v - \beta(u) g(v) &= 0 && \text{in } \Omega \end{aligned} \right\} \quad (2.1)$$

$$u = \phi_1 \quad v = \phi_2 \quad \text{on } \partial\Omega \quad (2.2)$$

where  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . We assume

that  $\alpha$  and  $\beta$  are nondecreasing real continuous functions such that

$$\alpha(0) = \beta(0) = 0. \quad (2.3)$$

$f$  and  $g$  are  $C^1$  functions satisfying

$$f(s) > m_1 > 0 \quad \text{and} \quad 0 < g(s) < m_2 \quad \text{for } s \in \mathbb{R}. \quad (2.4)$$

$$\phi_1, \phi_2 \in C^1(\partial\Omega), \quad \phi_1, \phi_2 > 0 \quad \text{on } \partial\Omega. \quad (2.5)$$

The main theorem in this paper is the following:

**THEOREM 1.** Assume that (2.3)-(2.5) are satisfied. Then there exists at least one solution  $(u, v)$  of (2.1) and (2.2) with  $u, v \in W^{2,r}(\Omega)$  for any  $1 < r < +\infty$ , and  $v > 0$  on  $\Omega$ . Moreover, we have

(i) If  $\alpha(s) = \mu^2 |s|^{p-1} s$  and  $(u, v)$  is any solution of (2.1) and (2.2), then a dead core for  $u$  may exist only if  $0 < p < 1$ .

(ii) Let  $\alpha(s) = \mu^2 |s|^{p-1} s$  with  $0 < p < 1$  and let  $(u, v)$  be a solution of (2.1) and (2.2). For  $\lambda > 0$  let  $\Omega_\lambda = \{x \in \Omega \mid f(v(x)) > \lambda\}$ . Then

$$\Omega_0 = \left\{ x \in \Omega_\lambda \mid d(x, \partial\Omega_\lambda - (\partial\Omega - \text{supp } \phi_1)) > \left( \frac{M}{K_{\lambda, \mu}} \right)^{\frac{1-p}{2}} \right\} \quad (2.6)$$

where  $M = \|\phi_1\|_{L^\infty(\partial\Omega)}$  and

$$K_{\lambda, \mu} = \left( \frac{2N(1-p)+4p}{\lambda \mu^2 (1-p)^2} \right)^{\frac{1}{p-1}}$$

**Remark 2.1.** The case  $p = 0$  can be handled in a completely similar way by considering the graph  $\alpha(s) = \mu^2 s g(s)$  and we also obtain the estimate (2.6) for some  $M > 0$ .

**Remark 2.2.** An estimate similar to (2.6) still holds in the case of non-homogeneous nonlinear boundary conditions if  $\Omega$  has some geometrical properties and, in particular, if  $\Omega$  is convex. Cf. [11] for the details.

The above theorem is particularly interesting if  $m_1 > 0$  in (2.4) (this condition is satisfied in the case of the combustion system (1.1) and (1.2)).

$$\Omega_0 = \left\{ x \in \Omega \mid d(x, \partial\Omega) > \left( \frac{M}{K_{\lambda, \mu}} \right)^{\frac{1-p}{2}} \right\}$$

if  $\phi_1 > 0$  on  $\partial\Omega$ . We see that  $K_{\lambda, \mu} \rightarrow +\infty$  if  $\mu \rightarrow +\infty$ ; therefore the existence of a dead core can only be guaranteed by estimate (2.6) if, for example

$$\delta(\Omega) > \left( \frac{M}{K_{\lambda, \mu}} \right)^{\frac{1-p}{2}}$$

where  $\delta(\Omega)$  is the radius of the largest ball contained in  $\Omega$ . Then, for a fixed  $\Omega$ , it is clear that for  $\mu$  large enough,  $\Omega_0$  has a positive measure.

The proof of Theorem 1 can be carried out by using results for a single nonlinear equation, but not in the usual way for the combustion problem. In fact, if  $(u, v)$  is a solution of (2.1) and (2.2) with  $\alpha(s) = \mu^2 |s|^{p-1} s$ , then  $u$  satisfies

$$-\Delta u + \tilde{f}(x) \alpha(u) = F(x) \quad \text{in } \Omega \quad (2.7)$$

$$u = \phi_1 \quad \text{on } \partial\Omega \quad (2.8)$$

where  $\tilde{f}(x) = f(v(x))$  a.e. on  $\Omega$  and  $F \equiv 0$ .

Many authors have studied the subset  $\Omega_0$  for the problem (2.1) and (2.2) (cf., e.g., [4], [5], [7], [9], [10], [13], [15]) but, to the best of our knowledge, all the existing results concern the case  $f \equiv \text{constant}$ . Our results here follow the ideas in [10].

The main ingredient for the proof of Theorem 1 is the following auxiliary result.

**LEMMA 1.** Let  $F \in L^\infty(\Omega)$ ,  $\phi \in C^1(\partial\Omega)$  and suppose that  $u \in H^2(\Omega)$  satisfies

$$-\Delta u(x) + \mu^2 \tilde{f}(x) |u(x)|^{p-1} u(x) = F(x) \quad \text{in } \Omega \quad (2.9)$$

$$u = \phi \quad \text{on } \partial\Omega \quad (2.10)$$

where  $\tilde{f} \in L^\infty(\Omega)$ ,  $\tilde{f} > 0$  on  $\Omega$  and  $0 < p < 1$ . If we define  $\Omega_\lambda = \{x \in \Omega \mid \tilde{f}(x) > \lambda\}$ ,  $\lambda > 0$  we have the estimate

$$\Omega_0 = \{x \in \Omega \mid u(x) = 0\} \supset \{x \in \Omega_\lambda \mid d(x, \partial(\Omega_\lambda - \text{supp } F)) - (\partial\Omega - \text{supp } \phi) > \left(\frac{M}{K_{\lambda, \mu}}\right)^{\frac{1-p}{2}}\}. \quad (2.11)$$

Here  $\tilde{M} = \max \left\{ \left( \frac{\|F\|_{L^\infty(\Omega)}}{\lambda\mu^2} \right)^{\frac{1}{p}}, \|\phi\|_{L^\infty(\partial\Omega)} \right\}$  and  $K_{\lambda, \mu}$  is given by

$$K_{\lambda, \mu} = \left( \frac{2N(1-p)+4p}{\lambda\mu^2(1-p)^2} \right)^{1/(p-1)}.$$

Sketch of the proof. Simple comparison arguments allow us to consider only the case  $F > 0, \phi > 0$ . If  $u_\lambda \in H^2(\Omega)$  satisfies

$$\left. \begin{aligned} -\Delta u_\lambda + \lambda\mu^2 |u_\lambda|^p &> F && \text{in } \Omega_\lambda \\ u_\lambda &> \phi && \text{on } \partial\Omega_\lambda \cap \partial\Omega \\ u_\lambda &> \|u\|_{L^\infty(\Omega)} && \text{on } \partial\Omega_\lambda - \partial\Omega \end{aligned} \right\} \quad (2.12)$$

it is not difficult to show by using comparison results in [10] that  $0 < u(x) < u_\lambda(x)$  a.e. on  $\Omega_\lambda$  (the same argument works on any subset of  $\Omega_\lambda$ ). Therefore estimate (2.11) will follow by constructing such functions  $u_\lambda$ . We look for  $u_\lambda$  of the form  $u_\lambda(x) = h(|x-x_0|)$  for some  $x_0 \in \Omega_\lambda$ .

For  $0 < \eta < 1$  fixed, let  $h_\eta$  be a solution of the Cauchy problem

$$\left. \begin{aligned} h_\eta''(r) &= \eta\lambda\mu^2 |h_\eta(r)|^{p-1} h_\eta(r) \\ h_\eta(0) &= h_\eta'(0) = 0. \end{aligned} \right\} \quad (2.13)$$

It is easy to check that

$$h_\eta(r) = L_\eta r^{\frac{2}{1-p}},$$

where  $L_\eta$  is a constant explicitly given, is a solution of (2.13). If

$$0 < \eta < \frac{p+1}{1+p+(N-1)(1-p)},$$

then for any  $x \in \Omega_\lambda$ ,

$$-\Delta h_\eta(|x-x_0|) + \lambda\mu^2 h_\eta(|x-x_0|)^p > 0$$

and from this it follows that the function

$$u_\lambda(x) = K_{\lambda, \mu} |x - x_0|^{\frac{2}{1-p}}$$

satisfies

$$\begin{aligned} -\Delta u_\lambda + \lambda\mu^2 u_\lambda^p &> 0 = F(x) && \text{in } \tilde{\Omega} \\ u_\lambda &> 0 = \phi && \text{on } \partial\tilde{\Omega} \cap (\partial\Omega - \text{supp } \phi) \end{aligned}$$

where  $\tilde{\Omega} = \Omega_\lambda - \text{supp } F$ . Hence it is sufficient to have

$$u_\lambda > \max \{ \phi, \|u\|_{L^\infty(\Omega)} \} \text{ on } \partial\tilde{\Omega} - (\partial\tilde{\Omega} \cap (\partial\Omega - \text{supp } \phi)) \quad (2.14)$$

to obtain  $0 < u(x) < u_\lambda(x)$  on  $\tilde{\Omega}$ . But by the maximum principle  $u < \tilde{M}$  on  $\Omega$  and then (2.14) is satisfied if we choose  $x_0$  such that

$$|x - x_0| > \left( \frac{\tilde{M}}{K_{\lambda, \mu}} \right)^{\frac{1-p}{2}} \quad (2.15)$$

for any  $x \in \tilde{\Omega} - (\partial\tilde{\Omega} \cap (\partial\tilde{\Omega} - \text{supp } \phi))$ . The conclusion now follows from (2.14), (2.15) and  $u_\lambda(x_0) = 0$ .

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# Compactness results and existence of many solutions of nonlinear elliptic problems in strip-like domains

## 0. INTRODUCTION

We shall be interested here in finding nontrivial solutions  $u$  of the following second-order nonlinear elliptic problem:

$$\left. \begin{aligned} -\Delta u &= f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \right\} \quad (1)$$

where  $f$  is a given nonlinearity, and  $\Omega$  is a strip in  $\mathbb{R}^N$  i.e.,  $\Omega = 0 \times \mathbb{R}^D$ , where  $0$  is a bounded domain of  $\mathbb{R}^m$ .

This kind of problem, which arises in many physical and mechanical problems, has been studied by many authors when  $\Omega$  is bounded. The unboundedness of  $\Omega$  makes it very difficult to find a global solution of (1), because of the lack of compactness.

Problem (1) was solved for  $\Omega = \mathbb{R}^N$  (see Berestycki and Lions [3] and Strauss [12]), and for a few more cases, but the principal fact seemed to prove a certain sort of compactness which permitted the application of global methods.

In the case of  $\Omega = \mathbb{R}^N$ , for example, the solutions were sought in the subspace of  $H^1(\mathbb{R}^N)$ , which consists of the radial symmetric functions.

When  $\Omega$  is a strip, it seemed to be important to obtain a certain compactness using the symmetries of the strip. This is the key result which enables us to obtain global existence results.

The structure of this paper is as follows. In the first section, we state our principal results and we give the fundamental compactness lemma and some other auxiliary ones. The second section applies the preceding lemmas to the proof of the existence of a positive solution or many solutions of (1). In the third section we give some alternative results for a natural eigenvalue problem related to (1). This third part of our work gives us a better idea of the situation regarding existence and quantity of solutions of (1).

For more details about these results see Esteban and Lions [7] and Esteban [6].

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## 1. MAIN RESULTS AND AUXILIARY LEMMAS

Let  $\Omega$  be the strip  $0 \times \mathbb{R}^p$ , where  $0$  is a bounded domain of  $\mathbb{R}^m$ ,  $m > 1$ , and let  $f$  be a function which satisfies the following assumptions:

$$f(0) = 0, f(t) = g(t) + vt, \text{ with } v < \lambda_1(0), \text{ where } \lambda_1(0) \text{ (2)}$$

is the first eigenvalue of  $-\Delta$  acting on  $H_0^1(0)$ ,

$$-\infty < \lim_{t \rightarrow 0^+} \frac{g(t)}{t} < \overline{\lim}_{t \rightarrow 0^+} \frac{g(t)}{t} = -k < 0; \quad (3)$$

$$\overline{\lim}_{t \rightarrow +\infty} \frac{g(t)}{t^\lambda} < 0, \text{ where } \lambda = \frac{N+2}{N-2} \text{ if } N > 2; \lambda < +\infty \text{ if } N = 2; \quad (4)$$

$$G(t) t^{-\theta} \text{ is nondecreasing for } t > 0, 0 < \overline{\lim}_{t \rightarrow +\infty} G(t) t^{-\theta} < +\infty \quad (5)$$

for some  $\theta > 2$ , where  $G(t) = \int_0^t g(s) ds$ .

Then we can prove the two following theorems:

**THEOREM 1.** Suppose that  $p > 2$  and let  $f, g$  satisfy (2)-(5), then problem (1) has a solution  $u \in W_{loc}^{2,q}(\Omega) \cap H_0^1(\Omega)$ ,  $\forall q < +\infty$ , satisfying:

- (i)  $u > 0$  in  $\Omega$ ;
- (ii)  $u$  is axially symmetric, i.e.,  $u$  has the form  $u(x_1, x^1) = u(x_1, |x^1|)$ ,  $\forall x_1 \in 0, \forall x^1 \in \mathbb{R}^p$ ; moreover,  $u$  is decreasing in  $|x^1|$ ;
- (iii) if  $f$  is locally Hölder continuous,  $u \in C^2(\bar{\Omega})$ .

**THEOREM 2.** If  $p > 2$ , if  $f, g$  satisfy all the assumptions made above, and if  $g$  is odd, then problem (1) possesses an infinity of distinct solutions which are axially symmetric and which are in  $C^2(\bar{\Omega})$  whenever  $f$  is locally Hölder continuous.

Next we give two lemmas that we use in the proof of the above theorems. The first lemma, which generalizes results of Strauss [12], and Berestycki and Lions [3], is fundamental in all that follows.

**DEFINITION.** That  $u \in H_0^1(0 \times \mathbb{R}^p)$  is axially symmetric if  $u(x_1, x^1) = u(x_1, |x^1|)$  for all  $x_1 \in 0, x^1 \in \mathbb{R}^p$ .

Let us note by  $H_{0,s}^1(\Omega)$  the set of all axially symmetric functions in  $H_0^1(\Omega)$ .

Then, we can prove the following.

**LEMMA 3.** If  $p > 2$ , the Sobolev imbedding of  $H_{0,s}^1(\Omega)$  in  $L^q(\Omega)$  is compact for every  $q \in (2, \frac{2N}{N-2})$ , where  $N = m + p$ .

Furthermore, if  $\{u_n\}$  is a bounded sequence in  $H_{0,s}^1(\Omega)$ , and if  $F$  is a continuous function which satisfies:

$$F(t) = o(t^2) \text{ as } t \rightarrow 0$$

$$F(t) = o(t^{\frac{2N}{N-2}}) \text{ as } t \rightarrow +\infty,$$

then the sequence  $\{F(u_n)\}$  is relatively compact in  $L^1(\Omega)$ .

The proof of this lemma is given in detail in [7]. It uses in a fundamental way the radial symmetric with respect to the  $x^1$ -variable.

**LEMMA 4.** If  $\Omega = 0 \times \mathbb{R}^p$  and  $0 \subset \mathbb{R}^m$  is bounded, then for all  $u \in H_0^1(\Omega)$ , we have:

$$\int_{\Omega} |u|^2 dx < (\lambda_1(0))^{-1} \int_{\Omega} |\nabla u|^2 dx,$$

where  $\lambda_1(0)$  is again the first eigenvalue of  $(-\Delta)$  acting on  $H_0^1(0)$ .

We prove this lemma defining

$$\lambda = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}$$

and noting that  $\lambda$  must be actually equal to  $\lambda_1(0)$ .

## 2. PROOF OF THEOREMS 1 AND 2

**Proof of Theorem 1.** First we modify  $g$ , defining

$$\tilde{g}(t) = \begin{cases} g(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases}$$

It is clear that the positive solutions of (1) will be the same if we change  $g$  to  $\tilde{g}$ .

Then  $\alpha$  satisfies the stronger conditions

$$\lim_{t \rightarrow 0^+} \frac{|g(t)|}{t} < +\infty,$$

$$\lim_{|t| \rightarrow +\infty} \frac{|g(t)|}{|t|^{\frac{2N}{N-2}}} = 0.$$

Now we define a functional  $S$  by

$$S(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\nu}{2} \int_{\Omega} |u|^2 dx - \int_{\Omega} G(u) dx,$$

and we verify that  $S$  is a good  $C^1$ -functional on  $E = H_{0,S}^1(\Omega)$ .

The existence of a positive solution of (1) follows from the application of a critical point theorem of P. H. Rabinowitz (see Theorem 3.9 in [1]) to the functional  $S$ , for it is clear that critical points of  $S$  are solutions of (1).

Then, the proof of Theorem 1 consists in verifying that all conditions needed in the critical point theorem we want to apply are accomplished.

The most interesting point of the proof is the verification of the Palais-Smale condition in which we strongly use Lemma 4.

The decreasingness in  $|x^1|$  of the solution is proved by the application of some results of Gidas and Nirenberg [8].

Proof of Theorem 2. This is very similar to the above. The only modifications are that now we do not replace  $g$  by  $\tilde{g}$ , and that the critical point result used is another one (Theorem 3.37 in [1]) which takes into account the evenness of  $S$ .

Remark. If  $f$  is locally Hölder continuous we use a bootstrap argument to prove that  $u \in C^2(\bar{\Omega})$ .

Remark. The case  $p = 1$ ,  $\Omega = 0 \times \mathbb{R}$ , is a little different, because now we cannot apply any compactness result, and so we cannot use a global method to prove the existence of solutions of (1) in  $\Omega = 0 \times \mathbb{R}$ .

In this case we must use a 'local' method which consists in solving (1) in  $\Omega_R = 0 \times (-R, R)$ . Then we find for every  $R > R_0$  a positive solution  $u_R$  of (1) in  $\Omega_R$ .

Then we obtain a priori uniform estimates on the  $u_R$  and we prove the symmetry of  $u_R$  with respect to the  $\mathbb{R}$ -variable, and the decreasingness of  $u_R(x_1, y)$  for  $y > 0$ .

Next we apply a version of Lemma 3 which provides us with the compactness we need. This lemma states the following:

LEMMA 5. Let  $p = 1$ ,  $q \in (2, \frac{2N}{N-2})$ ,  $N = m+1$ . Then, if  $\Omega = 0 \times \mathbb{R}$ ,  $0 \subset \mathbb{R}^m$  bounded, and if we denote by  $K$  the cone of  $H_0^1(\Omega)$  defined by

$$K = \{u \in H_0^1(\Omega) \mid u > 0 \text{ in } \Omega; u(x, y) \text{ non-increasing in } y \text{ for } x \in 0, y > 0; u(x, y) \text{ non-decreasing in } y \text{ for } x \in 0, y < 0\}.$$

Then, the Sobolev imbedding from  $H_0^1(\Omega)$  into  $L^q(\Omega)$  maps bounded closed sets in  $K$  into compact sets.

The application of this lemma enables us to pass to the limit as  $R \rightarrow +\infty$ , and to find a positive solution of (1) in  $\Omega = 0 \times \mathbb{R}$ .

### 3. AN EIGENVALUE PROBLEM RELATED TO (1)

Let us now focus our attention on the following problem:

$$\left. \begin{aligned} \Delta u + \lambda f(u) &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \right\} \quad (6)$$

where  $\Omega = 0 \times \mathbb{R}^p$ ,  $0 \subset \mathbb{R}^m$  is bounded,  $p, m > 1$ , and where  $f$  satisfies (3), (4) and

$$f(0) = 0, f(t) = 0 \quad \forall t < 0 \quad (7)$$

$$\exists \zeta > 0 \text{ such that } F(\zeta) > 0, \quad (8)$$

where  $F(t) = \int_0^t f(s) ds.$

To solve (6) we consider the following minimization problem:

$$\text{Minimize } J(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx \text{ over } K_{\eta}, \quad (9)$$

where  $K_\eta = \{v \in H_0^1(\Omega) \mid \int_\Omega F(v) dx > \eta\}$ .

Under the above assumptions we can state the following.

**THEOREM 6.** For every  $\eta > 0$  problem (9) has a solution  $u \in H_0^1(\Omega) \cap W_{loc}^{2,q}(\Omega)$  ( $\forall q < +\infty$ ), which is positive and axially symmetric, i.e.  $u \in H_{0,s}^1(\Omega)$ ,  $u(x_1, x^1)$  is decreasing with respect to  $|x^1|$  and  $u$  satisfies  $\int_\Omega F(u) dx = \eta$ .

Moreover, there exists a Lagrange multiplier  $\lambda > 0$  for which  $(u, \lambda)$  is a solution of (6).

#### Remarks

(1) This theorem gives an extension of Pohozaev's results [11] for the case of strip-like domains.

(2) The fact that we can solve (6) without making assumption (5) on  $f$  suggests to us that (5) is no more than a technical hypothesis, which can probably be disregarded.

**Proof.** First we prove that for every  $\eta > 0$ , the minimizing set  $K_\eta$  is not empty. Then we take a minimizing sequence, and we consider the Steiner symmetrizations of its elements. This sequence will be another minimizing sequence, but now it will be in  $H_{0,s}^1(\Omega)$ . Then we can apply Lemma 3 to find a solution of problem (9).

To conclude we see that for this  $u \in H_{0,s}^1(\Omega) \cap W_{loc}^{2,q}(\Omega)$  ( $\forall q < +\infty$ ) there exists  $\lambda > 0$  such that  $(u, \lambda)$  is a solution of (6).

Finally, the positiveness of  $u$  follows from the strong maximum principle.

#### CONCLUSION

The results we give in this paper answer the question of the existence of solutions of (1). As we have already pointed out, it seems to us that the hypotheses made to solve (1) are not the best. Assumptions (2)-(4) seem to be optimal but it should be possible to find a different proof which would not use assumption (5), or that would consider a weakened version of it.

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