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Contributions to nonlinear partial differential equations

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On two nonlinear parabolic equations in duality arising in thermal control: the L^∞ and L^1 semigroup approach and the asymptotic behaviour

1. INTRODUCTION

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Given $\psi \in L^2(\Omega)$ with $\psi > 0$ a.e. and $u_0 \in H_0^1(\Omega)$, we consider the problem

$$\left.\begin{array}{ll} u_t = \min\{\psi, \Delta u\} & \text{on } (0,\infty) \times \Omega \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x) & \text{on } \Omega. \end{array}\right\} \quad (1)$$

Such problems arise in heat control theory when the temporal temperature variation of a body or fluid Ω is not allowed to be greater than a given positive function ψ (called the 'obstacle'). See [10, Chapter 2].

Problem (1) can be expressed in a weak form by means of the following evolution variational inequality:

$$\left.\begin{array}{l} u_t \in K, \quad K = \{v \in H_0^1(\Omega) \mid v < \psi \text{ a.e. on } \Omega\} \\ \int_\Omega u_t(v-u_t)dx + \int_\Omega \text{grad } u \cdot \text{grad } (v-u_t)\, dx \geq 0 \quad \forall v \in K \text{ and } t > 0. \end{array}\right\} (2)$$

The existence and uniqueness of a solution of (2), for each $u_0 \in H_0^1(\Omega)$, was proved by Brézis in [5] (see also [3]). Also the asymptotic behaviour is considered in [5] by means of the abstract result on asymptotic behaviour of solutions of evolution equations. It is shown there that $u(t,x)$ converges weakly in $H_0^1(\Omega)$, when $t \to \infty$, to a function $u_\infty(x) \in H_0^1(\Omega)$ satisfying

$$\min\{\Delta u_\infty, \psi\} = 0 \text{ on } \Omega \quad (3)$$

in the sense that

$$\int_\Omega \text{grad } u_\infty \cdot \text{grad } v \quad dx \geq 0 \quad \forall v \in K.$$

Nevertheless it is neither known how the solution selects an equilibrium point among all of them nor if the convergence also holds in the strong topology of $H_0^1(\Omega)$. Both questions were proposed in [5] and they are, essentially, the main aims of this work.

Our methods for the study of the asymptotic behaviour are based on considerations made in terms of <u>strong solutions</u>, i.e. solutions which satisfy (1) a.e. Because of this we will first consider some regularity results. It is not difficult to see that if the solution u of (2) is such that $\Delta u(t,\cdot) \in L^1(\Omega)$, for $t > 0$, then u is a strong solution. Nevertheless not every solution of (2) is a strong solution (for instance, if $\psi \equiv 0$ and $\Delta u_0 > 0$ in $\mathcal{D}'(\Omega)$, then u is a strong solution iff $\Delta u_0 \in L^1(\Omega)$).

For the strong solutions, problem (1) can be equivalently formulated by

$$P \begin{cases} u_t(t,x) + \beta(x,-\Delta u(t,x)) = 0 & \text{on } (0,\infty) \times \Omega \\ u(t,x) = 0 & \text{on } (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x) & \text{on } \Omega \end{cases}$$

where

$$\beta(x,r) = -\min\{\psi(x), -r\} \quad \text{a.e. } x \in \Omega, \quad \forall r \in \mathbb{R} \quad (4)$$

To prove the existence of strong solutions we shall consider the 'dual' problem

$$P^* \begin{cases} v_t(t,x) - \Delta\beta(x,v(t,x)) = 0 & \text{on } (0,\infty) \times \Omega \\ \beta(x,v(t,x)) = 0 & \text{on } (0,\infty) \times \partial\Omega \\ v(0,x) = v_0(x) & \text{on } \Omega. \end{cases}$$

It is clear, at least formally, that the existence of solutions v of P* in $L^1(\Omega)$ (i.e. such that $v(t,\cdot) \in L^1(\Omega)$ for $t > 0$) implies the existence of strong solutions of (1) by using the relation $v = -\Delta u$.

The existence of solutions of P* has been very much studied recently but the term $\beta(x,r)$ (a maximal monotone graph of $\mathbb{R}^2$ a.e. $x \in \Omega$) is always taken in the following two cases:

(a) $\beta(x,r)$ is independent of x

(b) $\beta(x,r)$ is onto a.e. $x \in \Omega$

(see [6]). Notice that the $\beta(x,r)$, given in (4), is neither in case (a) nor in case (b). Nevertheless, we shall show that P* is a 'well-posed' problem in $L^1(\Omega)$ (in the semigroup sense) when $\psi \in H^1(\Omega)$ and $(-\Delta\psi)^- \in L^2(\Omega)$.

If the obstacle ψ is assumed such that $\psi \in C^2(\bar{\Omega})$, then we shall prove that P is 'well posed' on $L^\infty(\Omega)$ and then the solution u of (2) satisfies $\Delta u(t,\cdot) \in L^\infty(\Omega)$ for $t > 0$.

Finally, using an abstract comparison result given in [1] we obtain some useful estimates allowing us to prove our main result on the asymptotic behaviour of the solutions of (1): if $\psi > 0$ and $\Delta\psi \geq 0$ a.e. on Ω, then $u(t,\cdot)$ converges strongly in $H_0^1(\Omega)$ to the equilibrium point zero. If, in addition, $\psi(x) \geq \delta > 0$ a.e. $x \in \Omega$ (for some δ) then the asymptotic behaviour is completely described in the sense that we show the solution verifies the linear heat equation $u_t = \Delta u$ on $(T_0,\infty) \times \Omega$ for an adequate finite time T_0. For other answers on the strong convergence and the selection of the equilibrium point we refer to [9].

2. THE SEMIGROUP APPROACH TO P* AND P

Problem P* can be formulated as an abstract Cauchy problem on the $L^1(\Omega)$ space

$$\left.\begin{aligned} &\frac{dv}{dt} + Av \ni 0 \text{ in } L^1(\Omega), \text{ on } (0,\infty) \\ &v(0) = v_0, \end{aligned}\right\} \qquad (5)$$

A being the operator in $L^1(\Omega)$ given by

$$\left.\begin{aligned} &D(A) = \{w \in L^1(\Omega): \beta(x,w(x)) \in W_0^{1,1}(\Omega) \text{ and } \Delta\beta(x,w(x)) \in L^1(\Omega)\} \\ &Aw = -\Delta\beta(\cdot,w(\cdot)) \text{ if } w \in D(A). \quad (\beta \text{ given in (4)}). \end{aligned}\right\} (6)$$

In order to prove that (5) is 'well posed' on $L^1(\Omega)$ we shall apply the results on evolution equations governed by accretive operators (see e.g. [7]). We have

THEOREM 1. (a) The operator A is T accretive in $L^1(\Omega)$. (b) Let $\psi \in H^1(\Omega)$ be such that $\psi \geq 0$ a.e. and $\Delta\psi$ is a measure with $(-\Delta\psi)^- \in L^2(\Omega)$. Then the operator A is m-accretive in $L^1(\Omega)$. (c) Assume ψ as in part (b). Then $\overline{D(A)}^{L^1(\Omega)} = L^1(\Omega)$.

Proof. (a) Let $[u,v]$, $[\hat{u},\hat{v}] \in A$ and consider the operator $-\Delta$ defined in $L^1(\Omega)$ by $D(-\Delta) = \{w \in W_0^{1,1}(\Omega) \mid \Delta w \in L^1(\Omega)\}$. Then $u^* = \beta(\cdot,u) - \beta(\cdot,\hat{u})$ belongs to $D(-\Delta)$, and taking

$$\alpha^*(x) = \begin{cases} 1 \text{ if } (u-\hat{u})(x) \geq 0 \text{ and } u^*(x) \geq 0 \\ \\ 0 \text{ if } (u-\hat{u})(x) \leq 0 \text{ and } u^*(x) \leq 0 \\ \quad \text{or } (u-\hat{u})(x) < 0 \text{ and } u^*(x) = 0, \end{cases}$$

we have that $\alpha^*(x) \in \operatorname{sign}^+(u(x)-u(x)) \cap \operatorname{sign}^+ u^*(x)$ and that

$$\int_\Omega (Au-A\hat{u})\alpha^* \, dx \geq 0$$

because $-\Delta$ is a strongly T-accretive operator in $L^1(\Omega)$ (see [7]). (b) Given $f \in L^1(\Omega)$ it is easy to see that u satisfies $u + \lambda Au = f$ if and only if the function $h(x) = \beta(x,u(x))$ satisfies $h \in W_0^{1,1}(\Omega)$, $\Delta h \in L^1(\Omega)$ and

$$\left.\begin{aligned} &-\lambda\Delta h(x) + h(x) + \gamma(h(x) + \psi(x)) \ni f(x) \text{ a.e. } x \in \Omega \\ &h = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{aligned}\right\} \qquad (7)$$

being $\gamma(r)$ the maximal monotone graph of $\mathbb{R}^2$ defined by

$$\begin{aligned} &\gamma(r) = 0 \text{ if } r > 0,\ \gamma(0) = (-\infty,0] \text{ and} \\ &\gamma(r) = \emptyset \text{ (the empty set) if } r < 0. \end{aligned} \qquad (8)$$

Arguing as in [6], to prove the existence of solutions of (7) it suffices to consider f in a dense set of $L^1(\Omega)$. Actually, when $f \in L^2(\Omega)$, we can choose h as the unique solution of the variational inequality

$$\left.\begin{aligned} &h(x) \geq -\psi(x) \quad \text{a.e. } x \in \Omega \\ &-\lambda\Delta h + h \geq f \text{ a.e. on } \Omega \\ &(h+\psi)(-\lambda\Delta h+h-f)=0 \text{ a.e.on } \Omega \\ &h = 0 \text{ on } \partial\Omega \end{aligned}\right\} \qquad (9)$$

and by the regularity result of [3] we know that $h \in H^2(\Omega) \cap H_0^1(\Omega)$. Then h solves (7) and equivalently $R(I + \lambda A) = L^1(\Omega)$ for every $\lambda > 0$. (c) Take $f \in L^\infty(\Omega)$ and for each $\lambda > 0$ let $z_\lambda \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of (7). By Theorem I.1 of [3] we get

$$\|\lambda\Delta z_\lambda\|_{L^2(\Omega)} < \|f\|_{L^2(\Omega)} + C\,\|(-\lambda\Delta\psi)^-\|_{L^2(\Omega)} \quad (C \text{ indep. of } \lambda). \tag{10}$$

Then $\{\lambda z_\lambda\}$ converges weakly in $H^2(\Omega)$ and by the comparison results it is shown that $\{\lambda z_\lambda\} \to 0$ in $L^2(\Omega)$. Setting $y_\lambda(x) = f(x) + \lambda\Delta z_\lambda(x)$ it is clear that $y_\lambda \in D(A)$ and y_λ converges (weakly) to f in $L^2(\Omega)$. Finally from (10) we deduce that

$$\lim_{\lambda\to 0} \|y_\lambda\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}$$

and then y_λ converges strongly in $L^2(\Omega)$.

Remark. Problem P* is also well posed on the space $H^{-1}(\Omega)$. Indeed, from the result of [5] it is easy to see that for every $\psi \in L^2(\Omega)$, $\psi \geq 0$ a.e. on Ω, P* is governed by a maximal monotone operator on the Hilbert space $H^{-1}(\Omega)$. Such an operator can be characterized as the subdifferential of an adequate convex, l.s.c., functional on $H^{-1}(\Omega)$ in the following cases: (a) $\psi(x) \equiv \delta$ (see [4]), (b) $\psi \in H^1(\Omega)$ (unpublished result of A. Damlamian). Finally, we refer to the lecture of M.F. Bidaut-Véron for a very complete discussion about nonlinear equations with terms depending on x (such as (7)).

The existence of strong solutions of (1) is now a consequence of Theorem 1.

PROPOSITION 1. Assume $\psi \in H^1(\Omega)$ such that $\psi \geq 0$ on Ω and $(-\Delta\psi)^- \in L^2(\Omega)$. Let $u_0 \in H_0^1(\Omega)$ with $\Delta u_0 \in L^1(\Omega)$. Then the weak solution of (1) satisfies $\Delta u \in C([0,\infty): L^1(\Omega))$.

Proof. From Theorem 1 it is enough to show that $-\Delta u(t,\cdot)$ coincides with $v(t)$, the unique $L^1(\Omega)$-semigroup solution of P* corresponding to the initial datum $v_0 = -\Delta u_0$. Using the continuity in $L^1(\Omega)$ of the semigroup generated by A, we can suppose $u_0 \in H^1(\Omega) \cap H^2(\Omega)$. On the other hand, we recall that u is given by the solution of

$$\left.\begin{aligned} &\frac{du}{dt} + Bu \ni 0 \text{ in } H_0^1(\Omega), \text{ on } (0,\infty) \\ &u(0) = u_0, \end{aligned}\right\} \tag{11}$$

where B is the maximal monotone operator on $H_0^1(\Omega)$ defined by

$$Bu = -\partial\phi^*(-u) \tag{12}$$

where ϕ^* is the conjugate convex function of

$$\phi(z) = \begin{cases} \frac{1}{2}\int_\Omega |z|^2\,dx & \text{if } z \in K \\ +\infty & \text{if } z \notin K \end{cases} \tag{13}$$

(see [5]). Then if $v(t) = \lim_{n\to\infty} v_n(t)$ with $v_n(t) = a_k^n$ for $k\lambda_n < t < (k+1)\lambda_n$ where $a_k^n = (I + \lambda_n A)^{-K}(-\Delta u_0)$ and $\lambda_n \to 0$, it is easy to see that $a_k^n \in L^2(\Omega)$ and that the functions $b_k^n = (-\Delta)^{-1}\, a_k^n$ satisfy $b_k^n = (I + \lambda_n B)^{-K} u_0$. Then

$$-\Delta u(t,\cdot) = \lim v_n(t) = v(t).$$

(The details can be found in [9]).

To prove a further regularity result to (1), consider P formulated as the following abstract Cauchy problem:

$$\left.\begin{aligned} &\frac{d\bar{u}}{dt} + C\bar{u} \ni 0 \text{ in } L^\infty(\Omega), \text{ on } (0,\infty) \\ &\bar{u}(0) = \bar{u}_0, \end{aligned}\right\} \tag{14}$$

C being the operator on $L^\infty(\Omega)$ given by

$$\left.\begin{aligned} &D(C) = \{w \in L^\infty(\Omega) \cap H_0^1(\Omega) : \Delta w \in L^\infty(\Omega),\ \min(\psi,\Delta w) \in L^\infty(\Omega)\} \\ &Cw = -\min\{\psi,\Delta w\} \quad \text{if } w \in D(C). \end{aligned}\right\} \tag{15}$$

THEOREM 2. (i) The operator C is T-accretive in $L^\infty(\Omega)$. (ii) If $\psi \in C^2(\bar{\Omega})$ with $\psi \geq 0$ on $\bar{\Omega}$, C satisfies the range condition $R(I + \lambda C) \supseteq D(C)$, $\forall\lambda > 0$.

Proof. (i) Follows from the maximum principle and (ii) is shown by means of the Brézis-Kinderlehrer regularity result for stationary variational inequalities (see [9]).

PROPOSITION 2. Assume $\psi \in C^2(\bar{\Omega})$, $\psi \geq 0$ on Ω. Let $u_0 \in H_0^1(\Omega)$ be such that $\Delta u_0 \in L^\infty(\Omega)$. Then the weak solution u of (1) satisfies

$$u \in \dot{W}^{1,\infty}((0,\infty) \times \Omega) \cap L^\infty(0,\infty: H^2(\Omega))$$

and $\Delta u(t,\cdot) \in L^\infty(\Omega)$ a.e. $t > 0$.

Proof. It is easy to show that $u_0 \in D(C)$. Then the $L^\infty(\Omega)$-semigroup solution $\bar{u}$ satisfies

$$\bar{u}(t) \in D(C)$$

and

$$\bar{u} \in W^{1,\infty}([0,\infty) \times \Omega) \cap L^\infty(0,\infty:H^2(\Omega))$$

(see [2]). Finally if $b \in D(B) \cap D(C)$, then $(I + \lambda C)^{-1}b = (I + \lambda B)^{-1}b$ for any $\lambda > 0$ and in consequence $u = \bar{u}$.

Remark. The equation of (1) can obviously be written as

$$u_t + \max\{-\Delta u, -\psi\} = 0 \tag{16}$$

and then it is similar to the so-called Bellman's equation of dynamic programming

$$u_t + \max\{L^1u-f^1, L^2u-f^2\} = 0,$$

where L^k is a second order, uniformly elliptic operator $(k = 1,2)$. Interesting regularity results for (17) can be found in [11], [12] and [13]. Nevertheless, in (16) $L^2 \equiv 0$ and the above works do not apply.

3. THE ASYMPTOTIC BEHAVIOUR

We fix our attention on the convergence of the weak solution to an equilibrium point of (1). We shall limit our attention to the case $\psi > 0$ a.e. on Ω (then it is obvious that $u_\infty = \lim_{t\to\infty} u(t,\cdot)$ is $u_\infty \equiv 0$). Other cases are discussed in [9].

If $\psi(x) \equiv \delta > 0$, by using the regularizing effects for the dual equation P* (results of Bênilan, Veron, Evans,...) it is easy to see that the weak solution u satisfies the linear heat equation after a finite time (see [8]). This kind of result is far from being so easy when ψ is a non-constant function. Indeed, due to the non-surjectivity of the graph $\beta(\cdot,r)$ given in (4), no regularizing effect is known for the problem P*. Instead we have the following result which is, with slight modifications, a particular application of the abstract result of [1]:

LEMMA 1. Let $\psi \in H^1(\Omega)$ with $\psi \geq 0$ a.e. on Ω and $(-\Delta\psi)^- \in L^2(\Omega)$. Assume $u_0 \in H_0^1(\Omega)$ such that

$$-\Delta u_0 \in \overline{D^+(A)}^{L^1}, \quad (D^+(A) = \{w \in D(A) \mid Aw \geq 0\}).$$

Then

$$h(t,x;\tilde{v}_0) \leq -\min\{\psi(x), \Delta u(t,x)\} \text{ a.e. } (t,x) \in (0,\infty) \times \Omega$$

where $\tilde{v}_0 = -\min\{\psi, \Delta u_0\}$ and $h(t,x:z)$ denotes the solution of the heat equation

$$\begin{cases} h_t = \Delta h & \text{on } (0,\infty) \times \Omega \\ h = 0 & \text{on } (0,\infty) \times \partial\Omega \\ h(0,x) = z(x) & \text{on } \Omega. \end{cases}$$

THEOREM 3. Assume $\psi \in H^2(\Omega)$ with $\psi \geq 0, \Delta\psi \geq 0$ a.e. on Ω and let $u_0 \in H_0^1(\Omega)$. Then if $\psi(x) > 0$ a.e. $x \in \Omega$, $u(t) \to 0$ (strongly) in $H_0^1(\Omega)$ when $t \to +\infty$. If in addition $\psi(x) \geq \delta$ for some $\delta > 0$, then $u_t = \Delta u$ on $(T_0,\infty) \times \Omega$ where $T_0 = (C/\delta\|\psi\|_{L^1(\Omega)})^{2/N}$ and C is a positive constant depending only on $|\Omega|$.

Proof. Step 1. Assume $\psi \in C^2(\bar{\Omega})$, $\psi \geq 0$, $\Delta\psi \geq 0$ and $u_0 \in H_0^1(\Omega)$ such that $h = -\Delta u_0 \in L^\infty(\Omega)$. Set $u_{0,+}$ and $u_{0,-}$ belonging to $H_0^1(\Omega)$ such that $-\Delta u_{0,+} = h^+$ and $-\Delta u_{0,-} = -h^-$. Let u_+ and u_- be the weak solutions of (1) corresponding to the initial data $u_{0,+}$ and $u_{0,-}$ respectively. By Proposition 2 and the T-accretiveness of A we know that

$$-\Delta u_-(t) \leq -\Delta u(t) \leq -\Delta u_+(t) \text{ in } L^\infty(\Omega), \text{ a.e. } t > 0.$$

It is easy to see that $-\Delta u_+(t) \geq 0$ a.e. Then u_+ satisfies the linear heat equation, and so $-\Delta u_+(t) \to 0$ in $L^\infty(\Omega)$ when $t \to +\infty$. On the other hand, it is possible to find a $\hat{u}_0 \in H_0^1(\Omega)$ with $\Delta\hat{u}_0 \in L^\infty(\Omega)$ and such that $-\Delta\hat{u}_0 \leq -\Delta u_{0,-}$ a.e. on Ω as well as $-\Delta\hat{u}_0 \in \overline{D^+(A)}^{L^1(\Omega)}$. Indeed, we can choose $\hat{v}_0 \in L^\infty(\Omega)$ with $\hat{v}_0 \leq \min\{-\psi, -\Delta u_{0,-}\}$ and then $\hat{u}_0 = (-\Delta)^{-1}\hat{v}_0$. (We remark in this case

$\min\{\psi, \Delta\hat{u}_0\} = \psi$, so $A(-\Delta\hat{u}_0) = \Delta\psi \geq 0$). Therefore Lemma 1 shows that

$$h(t,x:-\psi(x)) \leq -\min\{\psi(x),\Delta\hat{u}(t,x)\} \leq -\min\{\psi(x),\Delta u(t,x)\} \leq 0,$$

where $\hat{u}$ is the weak solution of (1) corresponding to the initial datum $\hat{u}_0$. From the results on the asymptotic behaviour for the linear heat equation it is well known that there exists a positive constant C (only depending on $|\Omega|$) such that

$$\frac{-C}{t^{N/2}} \|\psi\|_{L^1(\Omega)} \leq h(t,x;-\psi(x)) \leq 0 \text{ a.e. } (t,x) \in (0,\infty) \times \Omega.$$

Thus, the conclusion follows easily.

Step 2. Take $\psi \in C^2(\Omega)$ with $\psi \geq 0$ and $\Delta\psi \geq 0$ a.e. on Ω. Let $u_0 \in H_0^1(\Omega)$. Consider $u_{0,n} \in H_0^1(\Omega)$ with $-\Delta u_{0,n} \in L^\infty(\Omega)$ and $u_{0,n} \to u_0$ in H_0^1 when $n \to \infty$. Then, if $u_n(t)$ is the weak solution of (1) of initial datum $u_{0,n}$, $u_n(t) \to u(t)$ in $H_0^1(\Omega)$ when $n \to \infty$ and so the first assertion follows from the first step. The second assertion can be shown by using the 'exponential formula' and the first step (see [9]).

Step 3. Let $\psi \in H_0^1(\Omega)$ with $\psi \geq 0$, $\Delta\psi \geq 0$ a.e. on Ω and $u_0 \in H_0^1(\Omega)$. Consider $\psi_n \in C^2(\bar{\Omega})$ with $\psi_n \geq 0$, $\Delta\psi_n \geq 0$, $\|\psi_n\|_{L^1(\Omega)} \leq \|\psi\|_{L^1(\Omega)}$, and such that $\psi_n \to \psi$ in $H^2(\Omega)$ when $n \to \infty$. Thanks to some convergence results for variational inequalities it can be shown that

$$(I + \lambda B_n)^{-1}z \to (I + \lambda B)^{-1}z \text{ when } n \to \infty,\ \lambda > 0,\ z \in D(B) \cap D(B_n)$$

(B_n designates the operator B corresponding to the obstacle ψ_n). The conclusion follows from the abstract results on convergence of maximal monotone operators.

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