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SOME RESULTS ON THE EXISTENCE OF FREE BOUNDARIES FOR PARABOLIC REACTION-DIFFUSION SYSTEMS

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ABSTRACT

We give a survey of some recent results concerning the existence, location and time evolution of free boundaries for a class of nonlinear parabolic reaction-diffusion systems arising in applications.

The case of degenerate parabolic problems is also considered. The main tool in the proofs is the use of local comparison techniques.

1. INTRODUCTION

We give a brief survey of some recent results obtained by the authors concerning the existence of free boundaries in some parabolic systems arising in applications, especially in combustion theory. The corresponding elliptic problem was treated by the authors in [5] and applications to combustion theory were also included. An extended version of these results, including full proofs and applications will appear in [6].

Now we consider the model system

$$\begin{aligned}
 u_t - \Delta u + \mu^2 F(u) e^{\frac{\gamma(v-1)}{v}} &= 0 & \text{in } Q = \Omega \times (0, \infty) \\
 \lambda v_t - \Delta v - \nu \mu^2 F(u) e^{\frac{\gamma(v-1)}{v}} &= 0 & \text{in } Q \\
 u = v = 1 & & \text{on } \Sigma = \partial\Omega \times (0, \infty) \\
 u(x, 0) = u_0(x), v(x, 0) = v_0(x) & & \text{on } \Omega
 \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain in \mathbb{R}^N , μ is the Thiele number, ν the Prater temperature, γ the Arrhenius number, λ^{-1} the Lewis number, and u and v are, respectively, the concentration and temperature of the reactant. The system (1.1) is a model for a single irreversible nonisothermic reaction (cf. the book by Aris [1]). F is a non-decreasing function such that $F(0)=0$, $F(1)=1$ and $F(s) \rightarrow 0$ for $s \rightarrow \infty$. The most common instance

of F is $F(s)=s^p$, where $p \geq 0$ is the reaction order. If $p=0$, then F is given by $F(0)=0$ and $F(s)$ if $s > 0$, and this means that in this case F is discontinuous.

A general formulation which contains (1.1) is the following

$$\begin{aligned} u_t - \Delta u + \alpha(u)f(v) &= 0 & \text{in } Q \\ v_t - \Delta v - \beta(u)g(v) &= 0 & \text{in } Q \\ u(x,t) = \psi_1, \quad v(x,t) = \psi_2, & & \text{on } \Sigma \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & & \text{in } \Omega \end{aligned} \tag{1.2}$$

where α and β are non-decreasing continuous functions, f and g are C^1 and $f(s) \geq 0, g(s) \geq 0$ for $s \geq 0$ and the boundary and initial data are smooth. The case $p=0$ can also be handled by using maximal monotone graphs.

The main interest in this paper is to study the existence of a dead core $\Omega_0(t) = \{x \in \Omega \mid u(x,t)=0\}$ and to get additional information about its size and location. The boundary of the subset $\partial\Omega_0(t)$ is a free boundary which is a priori unknown and plays an important role in applications.

We obtain some different kinds of results about $\Omega_0(t)$. Some of them seem to be new in the literature and there are also extensions to the case $f(x,t)$ nonconstant and, more important, $N > 1$, of known results. (A systematic treatment of these qualitative properties for nonlinear elliptic and parabolic problems will appear in the book [3]). Our main tools are local comparison techniques introduced in Díaz [4] and Díaz-Hernández [5].

Some of our results can be collected in the following

Theorem 1.1: Suppose that $\alpha(s) = |s|^{p-1}s, p \geq 0, \psi_1, \psi_2 \geq 0, u_0, v_0 \geq 0$.

i) If (u,v) is a solution of (1.2), then $v > 0$ on Q . If $p \geq 1$, then $u > 0$ on Q .

ii) For $0 \leq p < 1$ and (u,v) solution of (1.2) suppose that for $\lambda > 0$ there exists $\Omega_\lambda \subset \Omega$ such that

$$\Omega_\lambda \times (0, \infty) \subset \{(x,t) \in Q \mid f(v(x,t)) > \lambda\}.$$

Then we have the estimate

$$0 \leq u(x,t) \leq \left(\|u_0\|_{L^\infty(\Omega)}^{1-p} - \frac{\lambda\mu^2}{2}(1-p)t \right)^{\frac{1-p}{2}} \equiv U_{\frac{\lambda\mu^2}{2}}(t)$$

for any $t > 0$ and a.a. $x \in \Omega_\lambda$ such that $d(x, \partial\Omega_\lambda) \geq \rho_\lambda(t)$ where

$$\rho_\lambda(t) = \max \left\{ \sup_{\substack{s \in [0,t] \\ x \in \partial\Omega_\lambda \cap \partial\Omega}} \left\{ \frac{\psi_1(x,s) - U_{\frac{\lambda\mu^2}{2}}(t)}{K_{\lambda\mu^2}} \right\}, 0 \text{ (if } \partial\Omega_\lambda \subset \partial\Omega), \right. \\ \left. \sup_{s \in [0,t]} \left\{ \frac{M - U_{\frac{\lambda\mu^2}{2}}(t)}{K_{\lambda\mu^2}} \right\} \text{ (if } \partial\Omega_\lambda \cap (\mathbb{R}^N - \partial\Omega) \neq \emptyset) \right\}$$

with $M = \max \{ \|u_0\|_{L^\infty(\Omega)}, \|\psi_1\|_{L^\infty(\Sigma)} \}$

$$K_a = \left| \frac{2|1+p+(N-1)(p-1)|}{a(1-p)^2} \right|^{\frac{1}{p-1}}$$

In particular, if $\psi_1 \equiv 0, u(x,t) = 0$ for $x \in \Omega_\lambda$ and $t \geq T_0, T_0$ given by

$$T_0 = \frac{2 \|u_0\|_{L^\infty(\Omega)}^{1-p}}{\lambda\mu^2(1-p)}.$$

iii) If $0 \leq p < 1$ and if we define for $t \geq 0$

$$P(t) = \{x \in \Omega \mid u(x,t) > 0\}$$

we obtain the uniform estimate

$$\Omega_0(t) \supset \{x \in \Omega_\lambda - P(0) \mid d(x, \partial(\Omega_\lambda - P(0))) - (\partial\Omega - \text{supp } \|\psi_1(x, \cdot)\|_{L^\infty(0, \infty)})$$

$$\geq \left(\frac{M}{K_{\lambda\mu^2}} \right)^{\frac{1-p}{2}} \}$$

for $t \geq 0$ and the growth of $P(t)$ is given by

$$(P(t) \cap \Omega_\lambda) \subset (P(0) \cap \Omega_\lambda) + B(0, C(\lambda)t^{\frac{1-p}{2}})$$

where $C(\lambda)$ is a constant.

iv) If $0 < p < 1$ and $\psi_1 \equiv 0$, and if for $x_0 \in \Omega_\lambda$ we have

$$0 \leq u_0(x) \leq K_{\lambda\mu^2} |x - x_0|^{\frac{2}{1-p}} \text{ a.e. on } \Omega_\lambda$$

then $u(x_0, t) = 0$ for $t \geq 0$.

This theorem is especially meaningful if $f(s) \geq m_1 > 0$ for $s \geq 0$. In this case $\Omega_\lambda = \Omega$ for $\lambda \in (0, m_1]$ and the above estimates become completely explicit. We also point out that $K_a \rightarrow +\infty$ if $a \rightarrow +\infty$ and this implies that the dependence of T_0 and the estimate for $\Omega_0(t)$ in iii) on the "geometry" and size of Ω can be analyzed in terms of the parameter μ .

2. MAIN RESULTS

In this section we state some theorems related with Theorem 1.1. Detailed proofs, complementary results, and applications can be found in [6].

Now we want to study the existence of a "dead core". i.e., we shall show that under suitable assumptions the set $\{(x,t) \in Q \mid u(x,t) = 0\}$ with u solution of (1.2) has a positive measure. Our results follow easily from the case of a single equation. As we want to show the full power of our comparison techniques, we consider the (possibly degenerate) parabolic problem

$$\begin{aligned} u_t - \Delta \phi(u) + \tilde{f}(x,t)\alpha(u) &= 0 && \text{in } Q \\ u(x,t) &= \psi(x,t) && \text{on } \Sigma \\ u(x,0) &= u_0(x) && \text{on } \Omega, \text{ where} \end{aligned} \tag{2.1}$$

$\phi: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and non-decreasing with $\phi(0) = 0$ and α is a maximal monotone graph such that $0 \in \alpha(0)$.

It is clear that if (u,v) is a solution of (1.2), then u satisfies (2.1) for $\phi(s) = s$ and $\tilde{f}(x,t) = f(v(x,t))$. Here we do not insist on existence and we assume that (2.1) has a solution which is at least continuous on \bar{Q} .

Theorem 2.1: Let $m \geq 1$, $0 < p < 1$, and let u be a solution of (2.1) for $\phi(s) = |s|^m \operatorname{sg} s$ and $\alpha(s) = |s|^p \operatorname{sg} s$. Suppose that for $\lambda > 0$ and $T > 0$ there exists $\Omega_{\lambda,T} \subset \Omega$ such that

$$\Omega_{\lambda,T} \times (0,T) \subset \{(x,t) \in \Omega \times (0,T) \mid \tilde{f}(x,t) \geq \lambda\}$$

We define for $t > 0$

$$U_{\frac{\lambda}{2}}^\pm(t) = \left(\pm \|u_0\|_{L^\infty(\Omega)}^{1-p} + \frac{\lambda}{2} (p-1)t \right)^\pm \frac{1}{1-p} \tag{2.2}$$

Then we get the estimate

$$U_{\frac{\lambda}{2}}^-(t) \leq u(x,t) \leq U_{\frac{\lambda}{2}}^+(t) \tag{2.3}$$

for any $x \in \Omega_{\lambda,T}$ such that $d(x, \partial\Omega_{\lambda,T}) \geq \rho_{\lambda,T}$ with

$$\rho_{\lambda,T} = \max \{ \rho_{\lambda,T}^+, \rho_{\lambda,T}^- \}$$

$$\begin{aligned} \rho_{\lambda,T}^+ &= \max_{\substack{t \in [0,T] \\ x \in \partial\Omega_\lambda \cap \partial\Omega}} \left\{ \frac{\psi^+(x,t)^m - U_{\lambda/2}^+(t)^m}{K_{\lambda/2}} \right\}^{\frac{1-p/m}{2}} \\ &\sup_{t \in [0,T]} \left\{ \frac{M_+^m - U_{\lambda/2}^+(t)^m}{K_{\lambda/2}} \right\}^{\frac{1-p/m}{2}} \end{aligned}$$

where $M_+ = \max \{ \|u_0^+\|_\infty, \|\psi^+\|_\infty \}$. (M_- and $\rho_{\lambda,T}^-$ are defined analogously).

Remark 2.1: If there exists $m_1 > 0$ such that $\tilde{f}(x,t) \geq m_1$, then $\Omega_\lambda = \Omega$ for $0 < \lambda \leq m_1$ and the above estimates take a simpler form. If $0 \leq p < 1$ we extend some very particular results of Kersner [8] concerning extinction in finite time. If $\psi \equiv 0$ we obtain results due to Kalashnikov [7] and Veron [9]. If $1 < p < m$ and $\psi \equiv 0$ we get an estimate included in [2].

Theorem 2.2: Suppose $0 < p < 1 \leq m$, For $\lambda > 0$ and

$$\epsilon = \left(\frac{\|u_0\|_{L^\infty(\Omega)}^m}{K_{\frac{\lambda}{2}}} \right)^{\frac{1-p/m}{2}}$$

let $x_0 \in \Omega_{\lambda,\infty}$ be such that $d(x_0, \partial\Omega_{\lambda,\infty}) \geq \rho_\lambda(\infty) + \epsilon$ with

$$\rho_\lambda(\infty) = \left(\frac{\max \{ \|u_0\|_\infty, \|\psi\|_\infty \}^m - \|u_0\|^m}{K_{\frac{\lambda}{2}}} \right)^{\frac{1-p/m}{2}}$$

Then $u(x_0, t) = 0$ for $t \geq T_0(x_0, \epsilon)$, where

$$T_0(x_0, \epsilon) = \frac{\|u_0\|_{L^\infty(B(x_0, \epsilon))}}{(1-p)\frac{\lambda}{2}}$$

Theorem 2.3: Suppose $0 < p < m$. If $\lambda > 0, T > 0$. we have for $t \in [0, T]$

$$\Omega_0(t) \supset \{x \in \Omega_{\lambda, T} - P(0) \mid d(x, \partial(\Omega_{\lambda, T} - P(0))) - (\partial\Omega - \text{supp} \|\psi(x, \cdot)\|_{L^\infty(0, T)})\} \geq \left(\frac{M}{K_\lambda}\right)^{\frac{m-p}{2}}$$

with $M = \max \{\|u_0\|_\infty, \|\psi\|_\infty\}$.

Theorem 2.4: Suppose $0 \leq p < 1 = m, \lambda > 0, T > 0$.

If u is a solution such that $u_t \in L^\infty(Q)$, then for $0 < t < T$

$$P(t) \cap \Omega_{\lambda, T} \subset (P(0) \cap \Omega_{\lambda, T} + B(0, C(\lambda)t^{\frac{1-p}{2}})), C(\lambda) \text{ constant.}$$

Remark 2.2: This last result seems to be new for $0 < p < 1$.

The following theorem is related with work by Kersner [8] concerning the time evolution of the subsets $P(t)$.

Theorem 2.5: Let $p < m$ and let $x_0 \in \Omega_{\lambda, T}$ such that $u_0(x_0) = 0$. We assume

that
$$d(x_0, \partial\Omega_{\lambda, T}) \geq \epsilon = \left(\frac{\max \{\|u_0\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Sigma)}\}}{K_\lambda}\right)^{\frac{1-p/m}{2}}$$

and that for any $x \in B(x_0, \epsilon)$

$$0 \leq u_0(x) \leq K_\lambda |x - x_0|^{\frac{2}{1-p/m}}$$

Then $u(x_0, t) = 0$ for $t \in [0, T]$.

Remark 2.3: If $\Omega_{\lambda, T} = \Omega$ and $\psi \equiv 0$ the assumption can be written

$$0 \leq u_0(x) \leq K_\lambda |x - x_0|^{\frac{2}{1-p/m}}$$

for any $x \in \Omega$. This extends Theorem 11 in [2] which is only valid for

$1 \leq p < m, \psi \equiv 0$ and $f \equiv 1$.

To give an idea of our methods we sketch the proof of Theorem 2.1.

Sketch of the proof: We define the function

$$\bar{u}(x, t) = \left(h_\lambda(x) + (U_\lambda^+(t))^m\right)^{\frac{1}{m}}$$

where $h_\lambda(x) = K_\lambda |x - x_0|^{\frac{2}{1-p/m}}$, U_λ^+ is given by (2.2), and K_λ is obtained by replacing p by p/m in the K_a of Theorem 1.1. By using lemma 2.1 in [5], the explicit expression of U_λ^+ , and the monotonicity and concavity properties of the function $\phi(s) = s^m$ and $\alpha(\phi^{-1}(s)) = s^{p/m}$ we get

$$\bar{u}_t - \Delta\phi(\bar{u}) + \lambda\alpha(\bar{u}) \geq 0.$$

On the other hand it is clear that

$$\bar{u}(x, 0) \geq U_{\lambda/2}^+(0) = \|u_0\|_{L^\infty} \geq u_0(x)$$

Now, for $t_0 > 0$ fixed, let $x_0 \in \Omega_\lambda$ be such that $|x - x_0| \geq M_\lambda^+(t_0)$, where $M_\lambda^+(t_0) = \rho_{\lambda, t_0}^+$. It is clear that $u \leq M$ on Q and

$$\begin{aligned} \bar{u}_t - \Delta\phi(\bar{u}) + \lambda\alpha(\bar{u}) &\geq u_t - \Delta\phi(u) + \lambda\alpha(u) \text{ in } \Omega_\lambda \times (0, t_0) \\ \bar{u} &\geq u \text{ on } \partial\Omega_\lambda \times (0, t_0) \\ \bar{u}(x, 0) &\geq u_0(x) \text{ on } \Omega_\lambda \end{aligned}$$

This implies, by the Maximum Principle (more precisely, by the T -accretivity in $L^1(\Omega)$ of the operator formally defined by $-\Delta\phi(u) + \alpha(u)$), that $u(x, t) \leq \bar{u}(x, t)$ on $\Omega_\lambda \times [0, t_0]$ and for $x = x_0$ this yields $u(x_0, t) \leq U_{\lambda/2}^+(t)$. The other inequality is proved in the same way.

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A NEUTRAL SYSTEM WITH STATE-DEPENDENT DELAYS

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ABSTRACT

The two-body problem of classical electrodynamics can be modeled with a system of functional differential equations of neutral type. The delays arise from the finite speed of propagation of electromagnetic effects, and hence depend on the (unknown) trajectories. A satisfactory existence and uniqueness theorem is obtained. The proof will appear elsewhere.

NATURE OF THE MODEL

Imagine two charged particles, each moving under the sole influence of the retarded fields of the other. Delays occur in the equations of motion because of the finite speed of propagation, c , of electromagnetic effects.

Specifically, let the two particles be located, with respect to some inertial reference frame, at positions $x_1(t)$ and $x_2(t)$ in R^3 at time t . Then the electromagnetic effects reaching particle i at instant t must have been produced by particle j at an earlier instant $t - r_i(t)$, where

$$cr_i(t) = |x_1(t) - x_j(t - r_i(t))| \quad (j \neq i).$$

Here, $|\cdot|$ is the Euclidean norm in R^3 .

Introduce $v_i(t) = \dot{x}_i(t)/c$ and, for economy of notation, write x_i , v_i , and r_i instead of $x_i(t)$, $v_i(t)$, and $r_i(t)$ whenever the argument is t . Then the equations of motion for particle i under the sole influence of particle j have the form

$$\dot{x}_i' = cv_i, \quad (1)$$

$$\begin{aligned} \dot{v}_i' = & (e_i e_j / m_i) G(x_i - x_j(t - r_i), v_i, v_j(t - r_i)) \\ & + (e_i e_j / m_i) G(x_i - x_j(t - r_i), v_i, v_j(t - r_i)) v_j'(t - r_i), \end{aligned} \quad (2)$$

$$cr_i = |x_i - x_j(t - r_i)|. \quad (3)$$