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J I DÍAZ

Applications of symmetric rearrangement to certain nonlinear elliptic equations with a free boundary

1. INTRODUCTION

In this article we present several qualitative properties of solutions of nonlinear elliptic equations of the type

$$-Lu + f(u) = g \text{ in } \Omega, \tag{1}$$

$$u = h \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a regular bounded open set of $\mathbb{R}^N$, L is a linear elliptic second order operator,

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(b_j(x)u) + c(x)u, \tag{3}$$

where we assume

$$a_{ij},\ b_j \in W^{1,\infty}(\Omega), \quad c \in L^\infty(\Omega), \tag{4}$$

$$c + \sum_j \frac{\partial}{\partial x_j} b_j(x) \le 0, \tag{5}$$

$$\Lambda(x)\,|\xi|^2 \ge \sum_{i,j} a_{ij}(x)\xi_i\xi_j \ge \lambda(x)\,|\xi|^2 \quad \forall \xi \in \mathbb{R}^N-\{0\}, \text{ for some } \Lambda, \lambda \in L^\infty(\Omega) \text{ with } \lambda(x) > 0. \tag{6}$$

Finally, f will represent a continuous nondecreasing function such that $f(0) = 0$.

Equation (1) appears in many different contexts such as, for instance, in the study of isothermal chemical reactions [2]). There, $u \ge 0$, $L = \Delta$ (the Laplacian operator) and, usually, $g \equiv 0$, $h \equiv 1$ and $f(u) = \lambda u^q$ with $\lambda > 0$ and $q > 0$. When $0 < q < 1$, it turns out that the *null set* of u

$$N(u) = \{x \in \Omega : u(x) = 0\},$$

(there called the *dead core*) may be a positively measured set according to the value of λ and the size of Ω. In this way a *free boundary* F(u) may be generated, being $F(u) = \partial N(u) \cap \partial S(u)$, where $S(u)$ repres s the *support* of u,

$$S(u) = \overline{\{x \in \Omega: u(x) \neq 0\}}.$$

Equation (1) also appears in the study of stationary solutions of many nonlinear evolution equations such as semilinear parabolic equations or quasilinear equations (the porous media equation with absorption). Many precise references can be found, for instance, in [16], [22] and [11].

In the last few years many authors have considered the existence and properties of this free boundary $F(u)$ for solutions of (1), (2) or other formulations. The main purpose of this work is to obtain some qualitative properties on $F(u)$ by using the symmetric rearrangement of a function in the sense of Hardy and Littlewood.

We start, in Section 2, by recalling some results on the existence and location of the free boundary. Essentially, the existence of $F(u)$ is derived from the simultaneous fulfilling of the two following conditions: (i) a *diffusion-absorption balance*, given by the integral condition

$$\int_{0^+}^{\delta} \frac{ds}{F(s)^{1/2}} < +\infty, \quad F(s) = \int_0^s f(t)dt, \tag{7}$$

and (ii) a *balance between the size of the null set* $N(g)$ *and the "size" of the solution*, which, in the particular formulation of chemical reactions, reads

$$\rho \geq K, \tag{8}$$

where ρ is the radius of the largest ball contained in Ω, and K is a positive constant dependent only on λ and the dimension N. (For a more precise condition see Theorem 1.) Some location estimates on $F(u)$ are also given. In particular it is shown that, under some suitable assumptions on the data g and h, a more pathological property may occur: there is *nondiffusion of the support* of the solution u with respect to the supports of the data g and h. More precisely, if $f(u) = u^q$, $u \geq 0$, $0 < q < 1$, and we take $h \equiv 0$ and g such that, for some $K > 0$,

$$0 \leq g(x) \leq K\, d(x,\partial S(g))^{2q/(1-q)}, \text{ near } \partial S(g), \tag{9}$$

then $S(u) = S(g)$. This conclusion may be understood as an elliptic version

of the well-kn *waiting-time property* in nonlinear parabolic equations (see, e.g., [17]).

In Section 3 we use rearrangement techniques to obtain an *isoperimetric inequality* for the null set $N(u)$ of solutions of (1), (2) when $g \equiv 0$ and $h \equiv 1$: among all the linear operators satisfying (4), (5) and (6) with $\lambda(x) \equiv \lambda > 0$ and $b_i \equiv 0$, and among all the domains Ω with a prescribed measure, the measure of $N(u)$ is the greatest for the choice $a_{ij}(x) \equiv \lambda\delta_{ij}$ (δ_{ij} the Kronecker delta) $c \equiv 0$, and Ω is a ball centered at the origin. We also show that this isoperimetric inequality fails, in general, for the *exterior problem*

$$-Lu + f(u) = 0 \text{ in } \Omega\text{-}G \tag{10}$$

$$u = 1 \text{ on } \partial G, \; u = 0 \text{ on } \partial\Omega, \tag{11}$$

where G is an open set strictly contained in Ω, and that when one compares the solutions corresponding to all the domains $\Omega^* - G^*$ with $G^* \subsetneq \Omega^*$, $|G^*| = |G|$ and $|\Omega^*| = |\Omega|$.

These results are proved by means of a general comparison argument of interest in itself. Indeed, we consider v, the solution of the symmetric problem

$$-\lambda\Delta v + f(v) = g^* \text{ in } \Omega^* \tag{12}$$

$$v = 0 \qquad \text{on } \partial\Omega^* \tag{13}$$

or, respectively,

$$-\lambda\Delta v + f(v) = 0 \text{ in } \Omega^* - G^* \tag{14}$$

$$v = 1 \text{ on } \partial G^* \text{ and } v = 0 \text{ on } \partial\Omega^*, \tag{15}$$

where g^* is the symmetric decreasing rearrangement of $g \in L^1(\Omega)$ (i.e., the unique nonnegative function in $L^1(\Omega^*)$ that is radially symmetric, nonincreasing and has equimeasurable level sets with g) and Ω^*, G^* represent balls centered at the origin, of measure $|\Omega|$ and $|G|$ respectively. Then, we show that

$$\int_{B_r(0)} f(u^*)dx \leq \int_{B_r(0)} f(v)dx, \quad \text{for every } r > 0,$$

for u solution of (1), (2) with $h \equiv 0$, and v satisfying (12), (13). Moreover, we show that

$$\int_{B_r(0)} f(u^*) - C_u \le \int_{B_r(0)} f(v)dx - C_v, \text{ for every } r > 0$$

when u and v satisfy the exterior problems (10), (11) and (14), (15), respectively. Here

$$C_u = \int_{\partial G} \Sigma\, a_{ij} \frac{\partial u}{\partial x_i} n_j d\sigma \quad \text{and} \quad C_v = \lambda \int_{\partial G^*} \frac{\partial v}{\partial n} \partial\sigma \quad .$$

with $n = (n_j)$ the normal exterior vector to $\partial(\Omega - G) \cap \partial G$ or $\partial(\Omega^* - G^*) \cap \partial G^*$, respectively. Several bibliographical remarks are given at the end of this section as well as an interpretation of the above comparison in terms of its *effectiveness* in chemical reactions and of *electrostatic capacity* in the study of the electrostatic potential of a capacitor. Finally, the case of the multivalued equation

$$-Lu + \beta(u) \ni g$$

where β is a maximal monotone graph of $\mathbb{R}^2$, is also considered.

We end this introduction by noting that it is possible to offer an alternative version of all the results given here, for elliptic quasilinear equations of the type

$$-\text{div}\,(|\nabla u|^{p-2}\nabla u) + f(u) = g, \quad 1 < p < \infty .$$

This kind of equation appears, for instance, in the study of non-Newtonian fluids, and the null set of its solutions N(u), there called *quasi-solid zones*, plays a significant role (for a mathematical treatment of this problem see [11] and the references therein).

2. EXISTENCE AND LOCATION OF THE FREE BOUNDARY

Concerning the existence of the free boundary F(u), our main result is the following:

Theorem 1 Let L be the elliptic operator given by (3), (4), (5) and (6). Let f be a continuous nondecreasing function with $f(0) = 0$, and consider $g \in L^1(\Omega)$ and $h \in W^{1,1}(\Omega)$ such that $Lh \in L^1(\Omega)$. Then there exists a unique

u such that $u \in W^{1,q}(\Omega)$ for every $1 \le q < N(N-1)$, with $f(u) \in L^1(\Omega)$, weak solution of (1), (2). Moreover $u \in L^\infty_{loc}(N(g))$. In addition, if f satisfies the assumption

$$\int_0 \frac{ds}{F(s)^{1/2}} < \infty \quad , \quad F(s) = \int_0^s f(t)dt, \tag{16}$$

then the null set N(u) contains, at least, the set of $x \in N(g) \cup N(h|_{\partial\Omega})$ such that

$$d \ge \varepsilon + \psi_{\mu_0}(M(\varepsilon)), \text{ for some } \varepsilon \ge 0 \text{ and } \mu_0 > 0, \tag{17}$$

where $d = d(x, S(g) \cup S(h|_{\partial\Omega}))$, $M(\varepsilon)$ is a positive constant such that

$$\|u\|_{L^\infty(\Omega \cap B_{d-\varepsilon}(x))} \le M(\varepsilon) \tag{18}$$

and, for $\mu > 0$ and $r \in \mathbb{R}$, $\psi_\mu(r)$ is given by

$$\psi_\mu(r) = \left(\frac{1}{2\mu}\right)^{1/2} \int_0^r \frac{ds}{F(s)^{1/2}} . \tag{19}$$

In particular, if $u \in L^\infty(\Omega)$, the following estimate holds:

$$N(u) \supset \{x \in N(g) \cup N(h|_{\partial\Omega}) : d(x, S(g) \cup S(h|_{\partial\Omega})) \ge K\} \tag{20}$$

with $K = \psi_{\mu_0}(M)$, $M = \|u\|_{L^\infty(\Omega)}$.

In order to show this result, we start by considering the main contribution, i.e., the existence of F(u) under assumptions (16) and (17). This assertion will be proved by using suitable barrier functions $\bar{u}(x{:}x_0) = \eta(|x - x_0|)$ defined on balls $B_R(x_0)$ for points x_0 of the set $N(g) \cup N(h|_{\partial\Omega})$. We shall try to find a function $\bar{u}$ (i.e., η) such that $\bar{u}$ is a supersolution (resp. subsolution) of the equation, and from this we shall derive our conclusion. To do this, we note that if, for simplicity, we take $x_0 = 0$ and call $r = |x|$, then

$$L\eta(r) = \eta''(r) \sum_{ij} a_{ij}(x) \frac{x_i x_j}{r^2} + \frac{\eta'(r)}{r} \Big[\sum_{ij} a_{ij} \frac{x_i x_j}{r^2} - \{ \sum_i a_{ii}(x)$$

$$+ \sum_{i,j} \frac{\partial}{\partial x_j} a_{ij}(x) x_i + \sum_j b_j(x) x_j \Big] + (c(x) + \Sigma \frac{\partial}{\partial x_j} b_j(x)) \eta(r).$$

Using assumptions (4), (5), (6), we have

$$-L\eta(r) \geq -\Lambda_R\, \eta''(r) - \frac{C}{r}\, \eta'(r), \tag{21}$$

where

$$\Lambda_R = \|\Lambda\|_{L^\infty(B_R(0))} \tag{22}$$

and C is a positive constant dependent only on N and the coefficients a_{ij} and b_j of L. We note that, in the particular case of $L = \Delta$, we arrive at the familiar expression

$$-L\eta(r) = -\eta''(r) - \frac{(N-1)}{r}\, \eta'(r). \tag{23}$$

As we shall see later, the existence of F(u) is related to the existence of nontrivial solutions of the homogeneous Cauchy problem

$$u'' + \frac{C_1}{r}\, u' = C_2 f(u), \tag{24}$$

$$u(0) = u'(0) = 0, \tag{25}$$

where C_1 and C_2 are given positive constants. Note that, obviously, $u \equiv 0$ is always a solution and so functions f leading to uniqueness theorems, (e.g., f Lipschitz continuous) will not allow us to obtain the required conclusion.

When $C_1 \equiv 0$ (i.e., when N = 1 in (23)), it is not difficult to show (see, e.g., Lemma 1.3 and Theorem 1.4 of [11]) that the existence of nontrivial (and nonnegative) solutions of (24), (25) is in fact *equivalent* to the integral condition (16) or, more precisely, to condition (16) at 0^+ (i.e. (7)). Here, such conditions are understood as that the integrand $F(s)^{-1/2}$ belongs to $L^1(-\varepsilon,\varepsilon)$ (condition (16) or $L^1(0,\varepsilon)$ (condition (7)), respectively, for some $\varepsilon > 0$. We also remark that, if $C_1 \equiv 0$, equation (24) is autonomous, so a uniparameter family of solutions $u_\tau(r)$ can be constructed, assumed that (7) holds. Indeed, given $\tau \geq 0$, the function

$$u_\tau(r) = \begin{cases} 0 & \text{if } 0 \leq r < \tau \\ \eta(r-\tau) & \text{if } \tau \leq r \leq \psi_\mu(\infty) + \tau \end{cases} \tag{26}$$

is a solution of (24) with $C_1 \equiv 0$ and (25), where in this case $\mu = C_2$ and $\eta = \psi_\mu^{-1}$.

The non-autonomous case $C_1 > 0$ is considerably more complicated. We first

show the existence of super and subsolutions.

<u>Lemma 1</u> Let C_1 and C_2 be positive constants, and assume that (7) holds. Let $\mathcal{L}$ be the nonlinear o.d. operator given by

$$\mathcal{L}(u) = -u'' - \frac{C_1}{r}\, u' + C_2 f(u), \tag{27}$$

and for $\mu > 0$ and $r \in [0, \psi_\mu(+\infty))$, define

$$\eta(r,\mu) = \psi_\mu^{-1}(r). \tag{28}$$

Then we have:

(i) If $\mu \geq C_2$ then $\mathcal{L}(\eta(r,\mu)) < 0$ for $r > 0$.

(ii) If $0 < \mu < \mu_0$, $\mu_0 = C_2/(C_1+1)$, then $\mathcal{L}(\eta(r,\mu)) > 0$, and $\mathcal{L}(\eta(r,\mu_0)) \geq 0$, for $r > 0$.

(iii) For every $\tau > 0$ the function $\eta_\tau(r,\mu) = \eta([r-\tau]^+,\mu)$ satisfies

$\mathcal{L}(\eta_\tau(r,\mu)) < 0$ (resp. $\mathcal{L}(\eta_\tau(r,\mu)) > 0$) if $\mu \geq C_2$ (resp. $\mu > \mu_0$) for $r > \tau$.

<u>Proof</u> By differentiating in (28) we find that $\eta' = \sqrt{2\mu}\, F(\eta)^{1/2}$. We deduce that $\eta'(0,\mu) = 0$, $\eta'(r,\mu) > 0$ if $r > 0$, and $\eta(1,\mu)$ is a convex bijection from $[0,\psi_\mu(+\infty))$ onto $[0,\infty)$. Moreover,

$$\eta'' = \mu f(\eta)$$

and then

$$\mathcal{L}(\eta) = (C_2-\mu) f(\eta) - \frac{\sqrt{2\mu}\, C_1}{r}\, F(\eta)^{1/2}.$$

So $\mathcal{L}(h(r,\mu)) < 0$ if $\mu \geq C_2$. On the other hand, if we introduce the function $\phi(r) = \sqrt{2\mu}\, F(\eta(r,\mu))^{1/2}$, it is easy to see that ϕ is convex and then $\phi(r) \leq \phi'(r) r$. In consequence,

$$\mathcal{L}(\eta) \geq (C_2-\mu) f(\eta) - C_1\phi'(r) = (C_2-\mu(C_1+1)) f(\eta),$$

which proves (ii). Finally, let $\tau > 0$ and $r \geq \tau$. Letting $s = r-\tau$, we have

$$\mathcal{L}(\eta([r-\tau]^+,\mu)) = -\frac{d^2}{ds^2}\, \eta(s,\mu) - \frac{C_1}{(s+\tau)}\, \frac{d}{ds}\, \eta(s,\mu) + C_2 f(\eta(s,\mu))$$

and again the conclusion follows from the convexity of the auxiliary function ϕ. □

An existence result for solutions of (24), (25) can be obtained from the above lemma.

<u>Lemma 2</u> Assume that C_1 and C_2 are positive constants and that condition (7) holds. Then, for every $R \in (0, \psi_{\mu_0}(+\infty))$, $\mu_0 = C_2/(C_1+1)$, there exists a solution $u_\tau(r)$ of (24), (25) such that $u_\tau(R) = \eta(R,\mu_0)$. Moreover,

$$0 \leq u_\tau(r) \leq \eta(r,\mu_0) \text{ for every } r \in (0,R)$$

and $u_\tau(r) > 0 \quad \forall r \in [\tau,R]$, with $\tau \leq R(1-1/\sqrt{1+C_1})$.

<u>Proof</u> Let $u_\tau(r)$ be the unique solution of the two-point problem

$$\mathcal{L}u = 0 \text{ on } (0,R)$$

$$u'(0) = 0,\ u(R) = \eta(R,\mu_0).$$

Using part (ii) of Lemma 1 and the comparison principle, we obtain that

$$0 \leq u_\tau(r) \leq \eta(r,\mu_0) \text{ for } r \in (0,R).$$

Finally, it is easy to see that by choosing $\tau_0 = R(1-1/\sqrt{1+C_1})$, $\eta(R-\tau_0,C_2) = \eta(R,\tau_0)$. Then, by part (iii) of Lemma 1 and comparison results, we deduce that

$$\eta([r-\tau_0]^+,C_1) \leq u_\tau(r) \text{ for } r \in (0,R),$$

and so the function $u_\tau(r)$ satisfies all the requirements. □

<u>Remark 1</u> In the special case of $L = \Delta$ (i.e., $C_1 = (N-1)$, $C_2 = 1$) and $f(r) = \lambda|r|^{q-1}r$ with $q > 0$, conditions (7) (and (16)) hold if and only if $q < 1$. In this case Lemma 2 can be improved, and an explicit solution $u_0(r)$ of the associated Cauchy problem can be given. More precisely, it is shown (see [12],or Lemma 1.5 of [11]) that for $C > 0$ the function

$$u(r) = Cr^{2/1-q}$$

is such that $\mathcal{L}(u) = 0$ if $C = K_{N,\lambda}$, $\mathcal{L}(u) > 0$ if $C < K_{N,\lambda}$ and $\mathcal{L}(u) < 0$ if $C > K_{N,\lambda}$, where

$$K_{N,\lambda} = \left[\frac{\lambda(1-q)^2}{2(2q+N(1-q))}\right]^{1/(1-q)} \qquad □ \tag{29}$$

Now we return to the proof of Theorem 1. The existence and uniqueness of the weak solution u is a direct consequence of the important work of Brezis and Strauss [9]. We give below these and some others of their conclusions that we shall need later.

<u>Theorem 2</u> ([9]) Assume that L, f, g and h are as in Theorem 1. Then we have:

(a) There exists a unique function u with $f(u-h) \in L^1(\Omega)$ and $u-h \in W^{1,q}(\Omega)$ for every $1 \leq q < N/(N-1)$, such that

$$\int_\Omega \{\sum_{ij} \alpha_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_j b_j u \frac{\partial v}{\partial x_j} - cuv\}dx + \int_\Omega f(u-h)v\,dx = \int_\Omega (g+Lh)v\,dx$$

for every $v \in W_0^{1,\infty}(\Omega)$.

(b) If, in addition, $g \in L^p(\Omega)$ and $Lh \in L^p(\Omega)$ for some $1 < p \leq \infty$, then $f(u-h) \in L^p(\Omega)$ and, in fact, $u-h \in W^{2,p}(\Omega)$ if $1 < p < \infty$.

(c) If $g_1 \leq g_2$ a.e. on Ω and $h_1 \leq h_2$ on $\partial\Omega$, then the corresponding solutions u_1 and u_2 satisfy $u_1 \leq u_2$, a.e. on Ω.

(d) Let $f_n(r)$ be a sequence of Lipschitz continuous and nondecreasing functions ($f_n = n(I-(I+\frac{1}{n}f)^{-1})$, and let $g_n \in L^2(\Omega)$ and h_n, with $Lh_n \in L^2(\Omega)$, such that $g_n \to g$ and $Lh_n \to Lh$ in $L^1(\Omega)$. Then $u_n \to u$ and $f(u_n-h_n) \to f(u-h)$ in $L^1(\Omega)$.

<u>Remark 2</u> Theorem 2 is, in fact, formulated in [9] (only for homogeneous boundary conditions $h \equiv 0$) as a particular application of an abstract result. Nevertheless, some easy modifications lead to the same conclusions for the general case $h \not\equiv 0$. We refer the reader to Chapter 4 of [11] for details. Also, several local L^∞-estimates for the solution u are collected there, including the regularity $u \in L^\infty_{loc}(N(g))$.

<u>Proof of Theorem 1</u> After the comments above, we only need to show that u vanishes at the points $x \in N(g) \cup N(h|_{\partial\Omega})$ satisfying (17). Let $\varepsilon > 0$,

$d = d(x, S(g) \cup S(h|_{\partial\Omega}))$ and $R = d - \varepsilon$. By Lemma 1 and (21) the function $\bar{u}(x) = \eta(|x-x_0|, \bar{\mu})$ with $\bar{\mu} = 1/(\Lambda_R + C)$ (Λ_R and C given by (21) satisfies

$$-L\bar{u} + f(\bar{u}) \geq 0 \text{ in } B_R(x_0).$$

On the other hand, let $D_\varepsilon = \Omega \cap B_R(x_0)$ and $\|u\|_{L^\infty(D_\varepsilon)} \leq M(\varepsilon)$. Then, if R is such that $R \geq \psi_{\bar{\mu}}(M(\varepsilon))$, by the definition of $\bar{u}$ and R we conclude that

$$-L\bar{u} + f(\bar{u}) \geq 0 = -Lu + f(u) \text{ in } D_\varepsilon,$$
$$\bar{u} \geq u \qquad \text{on } \partial D_\varepsilon.$$

Thus, by (c) of Theorem 2 we conclude that $u \leq \bar{u}$ a.e. in D_ε. In an analogous way, letting $\underline{u} = -u$ if f is an odd function, or alternatively by finding non-negative subsolutions of the corresponding homogeneous Cauchy problem, we arrive at the comparison $\underline{u} \leq u$ (with $\underline{u}(x_0) = 0$) and so u vanishes (in an obvious weak sense) at x_0. We note that, if $u \in L^\infty(\Omega)$, then we can take directly $R = d = d(x_0, S(g) \cup S(h|_{\partial\Omega}))$ and $M = M(\varepsilon) = \|u\|_{L^\infty(\Omega)}$, which proves the estimate (20). □

As pointed out in the introduction, the existence of the free boundary F(u) is assured, in Theorem 1, under two kinds of conditions, both of them being optimal. The integral condition (16) represents a balance between the diffusion (here related to its homogeneity number $\lambda = 2$) and the absorption (given by f or F). On the other hand, if the null set N(u) is not empty there must exist points $x \in N(g) \cup N(h|_{\partial\Omega})$ where inequality (17) is fulfilled. If, for instance, $u \in L^\infty(\Omega)$, the set of the right-hand side of (20) is not empty when condition (8) (giving the balance between the "sizes" of N(g) and u) holds with $K = \psi_{\mu_0}(M)$. The optimality of (16) is proved in [20] and that of the balance (8) in [8].

A sharper estimate on the location of the null set N(u) may be given in some cases. For instance, if we know that $u \in W^{1,\infty}(\Omega)$, using L^∞-estimates for ∇u we can obtain estimates (in the reverse sense of that given in Theorem 1) of the type

$$N(u) \subset \{x \in N(g) \cup N(h|_{\partial\Omega}) : d(x, S(g) \cup S(h|_{\partial\Omega})) \geq K^*\}$$

for some $K^* > 0$ (see [13]).

A finer estimate of the location of F(u) is also possible when some information on the decay of g or h near the boundary of its supports is known. The following result is only a particular statement for the case of $g \geq 0$, $h \equiv 0$, $L \equiv \Delta$ and $f(s) = \lambda|s|^{q-1}s$. For a more general formulation, see [11].

Theorem 3 Assume $L = \Delta$, $0 < q < 1$ and suppose $g \in L^\infty(\Omega)$ such that

$$0 \leq g(x) \leq K d(x, \partial S(g))^{2q/(1-q)} \tag{30}$$

a.e. $x \in S(g)$ with $d(x, \partial S(g)) \leq R$, for some suitable constants K and R. Then if $u \in W_0^{1,p}(\Omega)$, $1 \leq p < N/(N-1)$ satisfies (1) then $S(u) = S(g)$, i.e., support of $u \equiv$ support of g.

Proof First we claim that $u = 0$ on $\partial S(g)$. Indeed, let $x_0 \in \partial S(g) - \partial\Omega$ (recall that $u = 0$ on $\partial\Omega$), and consider the set $\Omega \cap B_R(x_0)$. Given $x \in \Omega \cap B_R(x_0)$, it is clear that $d(x, \partial S(g)) \leq |x - x_0|$, because $x_0 \in \partial S(g)$. Then, by (30), we have that

$$0 \leq g(x) \leq K|x-x_0|^{2q/(1-q)} \text{ a.e. } x \in \Omega \cap B_R(x_0)$$

(note that $g \equiv 0$ in $B_R(x_0) \cap N(g)$). Now, assume K such that there exists $C \in (0, K_{N,\lambda})$, with $\mathcal{L}(Cr^{2q/(1-q)}) \geq Kr^{2q/(1-q)}$. By Remark 1 (or Lemma 1) we conclude that the function $\bar{u}(x) = C|x-x_0|^{2/(1-q)}$ satisfies

$$-\Delta\bar{u} + f(\bar{u}) \geq g = -\Delta u + f(u) \text{ a.e. in } \Omega \cap B_R(x_0).$$

On the other hand, if R is large enough, so that $CR^{2/1-q} \geq M$, with $M = \|u\|_{L^\infty(\Omega)}$, then $\bar{u} \geq u$ on $\partial(\Omega \cap B_R(x_0))$ and, by the comparison principle, $0 \leq u \leq \bar{u}$ in $\Omega \cap B_R(x_0)$. This proves that $u = 0$ on $\partial S(g)$ if K and R are adequate. Finally, consider the region $\tilde{\Omega} = \{x \in N(g) : d(x, S(g)) \leq \psi_{\mu_0}(M)\}$ with μ_0 given in Theorem 1. On this set, u verifies that $-\Delta u + f(u) = 0$. On the other hand, $\partial\tilde{\Omega} = \partial_1\tilde{\Omega} \cup \partial_2\tilde{\Omega} \cup \partial_3\tilde{\Omega}$, where $\partial_1\tilde{\Omega} = \partial\tilde{\Omega} \cap \partial\Omega$, $\partial_2\tilde{\Omega} = \partial S(g) \cap \tilde{\Omega}$ and $\partial_3\tilde{\Omega} = \{x \in \overline{N(g)} : x \in \partial\tilde{\Omega} - \partial\Omega$ and $d(x, \partial S(g)) = \psi_{\mu_0}(M)\}$. Then, $u = 0$ on $\partial_1\tilde{\Omega} \cup \partial_2\tilde{\Omega}$ and, by Theorem 1, $u = 0$ on $\partial_3\tilde{\Omega}$. In consequence, from the uniqueness results applied to $\tilde{\Omega}$ we deduce that $u = 0$ in $\tilde{\Omega}$, which ends the proof. □

Remark 3 The results of this section may be extended to many other situations: quasilinear equations; non-monotone absorption terms f(u) or terms depending eventually on x, f(x,u); external data g belonging merely to a measure space

(M(Ω)) or a distribution space ($H^{-1}(\Omega)$), etc. On the other hand, further studies of the free boundary F(u) concerning its regularity, behaviour of u near F(u), convexity and other topics are already available in the literature for the semilinear problem (1), (2) when simplified by taking for instance, $L = \Delta$, $g \equiv 0$ and $h \equiv 1$. For details and references we refer the reader to the monograph [11]. □

3. REARRANGEMENT AND THE INTERIOR AND EXTERIOR PROBLEMS

One of the main purposes of this section is to show the following result on the geometry of the free boundary F(u), or, more precisely, on the null set N(u).

Theorem 4 Let $g \equiv 0$, $h \equiv 1$ and let u be the solution of the problem (1), (2). Then, among all the linear operators L satisfying the assumptions (4), (5) and (6) with $\lambda(x) \equiv \lambda > 0$ and $b_j \equiv 0$ and among all the open domains Ω with same measure, the measure of the null set of the corresponding solution is the greatest for the choice $a_{ij} = \lambda\delta_{ij}$ (δ_{ij} the Kronecker delta), $c \equiv 0$, and Ω a ball centered at the origin. Moreover, if the solution of this last problem is stricttly positive, the same happens for the solution corresponding to any other choice of a_{ij}, λ and Ω. Finally, the above results are not true, in general, when the comparison is made for exterior problems such as (10), (11), when the measure of the open subset G is also prescribed.

The proof of this theorem will be obtained as one of the many consequences of a more general result involving rearrangement techniques. We first recall some definitions. Let u be a measurable function defined in a bounded open set $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with measure meas(Ω) = |Ω| . Then, the *distribution function* of u (or |u|) is the function $\mu:[0,\infty) \to \mathbb{R}^+$ defined by

$$\mu(t) = \text{meas}\ \{x \in \Omega:\ |u(x)| > t\}.$$

The *decreasing rearrangement* of u, $\tilde{u}$, is defined by

$$\tilde{u}(s) = \inf\ \{t \geq 0 : \mu(t) < s\},\quad 0 \leq s \leq |\Omega|\ .$$

Finally, the *symmetric decreasing rearrangement* of u is given by

$$u^*(x) = \tilde{u}(\omega_N|x|^N),\ \text{if}\ x \in \Omega^*,$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$ and Ω* is the ball, centered at the origin, of measure |Ω| . It will also be useful to introduce a comparison argument between two given functions φ and ψ of $L^1(\Omega)$: we say that the *concentration of* φ *is less than or equal to* that of ψ ($\phi \precsim \psi$) if

$$\int_0^t \tilde{\phi}(s)ds \leq \int_0^t \tilde{\psi}(s)ds \qquad \forall s \in [0,\ \text{meas}(\Omega)],$$

or, equivalently, if

$$\int_{B_r(0)} \phi^*(x)dx \leq \int_{B_r(0)} \psi^*(x)dx \qquad \forall r \in [0,\ \text{meas}(\Omega)].$$

Many basic properties of those functions are well known in the literature (see, e.g., [15], [23], [6] and [21]). Here we only recall a few that will be used later. If F is a Borel-measurable real function then

$$\int_\Omega F(|u|)dx \leq \int_{\Omega^*} F(u^*)dx;$$

if, moreover, F is positive and nondecreasing, $F(u^*) = F(u)^*$. If u and v belong to $L^2(\Omega)$, then

$$\int_\Omega uv dx \leq \int_{\Omega^*} u^*v^*dx$$

(Hardy-Littlewood inequality). In particular,

$$\int_D |u|dx \leq \int_0^{|D|} \tilde{u}(s)ds$$

for every measurable $D \subset \Omega$. Finally, if $u \in W_0^{1,p}(\Omega)$ then $u^* \in W_0^{1,p}(\Omega^*)$ and

$$\int_{\Omega^*} |\nabla u^*|^p\ dx \leq \int_\Omega |\nabla u|^p\ dx$$

for every $1 \leq p < \infty$.

The proof of Theorem 4 will follow from the two general comparison theorems in which, for the sake of simplicity in the statements, we limit ourselves to the case of operators L with $b_i \equiv 0$ and $\lambda(x) \equiv \lambda > 0$.

Theorem 5 Let f be a continuous nondecreasing function with f(0) = 0 and let $g_1 \in L^1(\Omega)$ and $g_2 \in L^1(\Omega^*)$ with $g_2^* = g_2$. Let $u \in W_0^{1,1}(\Omega)$ such that $-Lu + f(u) = g_1$ in Ω, and let $v \in W_0^{1,1}(\Omega)$ satisfying $-\lambda\Delta v + f(v) = g_2$ in Ω*. Then, if

$g_1^* \precsim g_2$ we have $f(u^*) \le f(v)$. In particular,

$$\int_\Omega f(u(x))dx \le \int_{\Omega^*} f(v(x))dx. \tag{31}$$

Finally, if $w \in W_0^{1,1}(\Omega)$ satisfies $-\lambda\Delta w = g_2$ in Ω^*, then

$$u^* \le w \quad \text{a.e. on } \Omega^*. \tag{32}$$

With respect to the exterior problem we have

Theorem 6 Let f be a continuous nondecreasing function with $f(0) = 0$. Let G be an open set strictly contained in Ω. Let $g_1(x) \equiv g_1 \in [0,\infty)$ and $g_2 \in L^\infty(\Omega^*-G^*)$, being a radial nonincreasing function. Let $u \in W^{2,p}(\Omega-G) \cap L^\infty(\Omega-G)$, $\forall 1 \le p < \infty$, such that

$$-Lu + f(u) = g_1 \text{ on } \Omega-G, \tag{33}$$

$$u = 1 \text{ on } \partial G,\ u = 0 \text{ on } \partial\Omega. \tag{34}$$

Let $v \in W^{2,p}(\Omega^*-G^*) \cap L^\infty(\Omega^*-G^*)$ satisfying

$$-\lambda\Delta v + f(v) = g_2 \text{ in } \Omega^* - G^* \tag{35}$$

$$v = 1 \text{ on } \partial G^*,\ v = 0 \text{ on } \partial\Omega^*. \tag{36}$$

Then, if $g_1 \precsim \bar{g}_2$ ($\bar{g} \equiv$ extension to Ω^* by $\sup|g|$ on G), we have that

$$f(u^*) - C_u \precsim f(v) - C_v \tag{37}$$

where

$$C_u = \int_{\partial G} \Sigma\, a_{ij} \frac{\partial u}{\partial x_i} n_j d\sigma \text{ and } C_v = \lambda \int_{\partial G^*} \frac{\partial v}{\partial n} d\sigma,$$

where $n = (n_j)$ is the normal unit outward vector to $\partial(\Omega-G) \cap \partial G$ or $\partial(\Omega^* - G^*) \cap \partial G^*$, respectively. Moreover, if $w \in H^1(\Omega^*-G^*)$ satisfies $-\lambda\Delta w = g_2$ in $\Omega^* - G^*$ with $w = 1$ on ∂G^* and $w = 0$ on $\partial\Omega^*$, then

$$\frac{(\bar{u})^*}{C_u} \le \frac{\bar{w}}{C_w} \quad \text{a.e. on } \Omega^* \tag{38}$$

with

$$C_w = \lambda \int_{\partial G^*} \frac{\partial w}{\partial n} d\sigma.$$

We shall prove simultaneously both theorems by using some preliminary lemmas.

Lemma 3 Let $u \in H_0^1(\Omega)$, $u \ge 0$, with $f(u) \in L^1(\Omega)$ (resp. $u \in H^1(\Omega-G) \cap L^\infty(\Omega-G)$, with $f(u) \in L^\infty(\Omega-G)$) satisfying $-Lu + f(u) = g_1$ (resp. $u \ge 0$ satisfying (33), (34)) with $g_1 \in L^2(\Omega)$ (resp. $g_1 \in L^\infty(\Omega-G)$). Then the function

$$\phi(t) = \int_{\{u>t\}} \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

is a decreasing Lipschitz continuous function of t and we have the inequality

$$0 \le -\phi'(t) \le \int_0^{\mu(t)} \tilde{g}_1(s)ds - \int_0^{\mu(t)} f(\tilde{u}(s))ds, \quad \text{a.e. } t > 0 \tag{39}$$

(resp.

$$0 \le -\phi'(t) \le \int_{\partial G} \Sigma\, a_{ij} \frac{\partial u}{\partial x_i} n_j d\sigma + \int_0^{\mu(t)} \tilde{\bar{g}}_1(s)ds$$
$$- \int_0^{\mu(t)} f(\tilde{\bar{u}}(s))ds - |G|(\|g_1\|_\infty - \|f(u)\|_\infty) \tag{40}$$
$$\text{a.e. } t \in (0, \|u\|_\infty)),$$

where $\mu(t)$ is the distribution function of u (resp. of $\bar{u}$), and $\bar{\phi}$ denotes the extension of ϕ to Ω by $\|\phi\|_\infty$ on G of any function $\phi \in L^\infty(\Omega-G)$).

Proof Given $t,h \in \mathbb{R}^+$, let $T_{t,h}:\mathbb{R}^+ \to \mathbb{R}^+$ be given by

$T_{t,h}(s) = 0$ if $0 \le s \le t$, $T_{t,h}(s) = s-t$ if $t < s < t+h$,

$T_{t,h}(s) = h$ if $s > t+h$.

Then, by well-known results (see, e.g., [14]), $T_{t,h}(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. From the information on u we have

$$-\phi(t+h)+\phi(t)=\int_{\{t<u\le t+h\}}\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx=\int_\Omega g_1(\quad))T_{t,h}(u)dx$$

$$=h\int_{\{u>t+h\}}(g_1-f(u)dx+h\int_{\{t<u\le t+h\}}(g_1-f(u))(\frac{u-t}{h})dx.$$

Hence,

$$-\Phi'(t)\le\int_{\{u>t\}}(g_1-f(u))dx\quad \text{a.e. } t>0.$$

On the other hand,

$$\int_{\{u>t\}}g_1(x)dx=\int_\Omega g_1(x)1_{\{u>t\}}dx\le\int_\Omega g_1^{*}\,1_{\{u*>t\}}=\int_0^{\mu(t)}\tilde{g}_1(s)ds.$$

Moreover,

$$\int_{\{u>t\}}f(u)dx=\int_t^\infty f(\tau)(-d\mu(\tau))=\int_0^{\mu(t)}f(\tilde{\mu}(s))ds,$$

which ends the proof of (39). To prove (40) we note that now $T_{t,h}(u)\in H^1(\Omega\text{-}G)\cap L^\infty(\Omega\text{-}G)$ with $T_{t,h}(u)=0$ on $\partial\Omega$, and so, by the Green formula,

$$-\Phi(t+h)+\Phi(t)=h\int_{\partial G}\Sigma\, a_{ij}\frac{\partial u}{\partial x_i}n_j+h\int_{\{t<u\le t+h\}}(g_1-f(u))T_{t,h}(u)dx.$$

Moreover,

$$\int_{\{u>t\}}g_1(x)dx=\int_0^{\mu(t)}\tilde{\bar{g}}_1(s)ds-\|g\|_\infty|G| \tag{41}$$

and

$$\int_{\{u>t\}}f(u)dx=\int_0^{\mu(t)}f(\tilde{\bar{u}}(s))ds-\|f(u)\|_\infty|G|. \tag{42}$$

We conclude as before. □

Using the Fleming-Rishel formula and the De Giorgi isoperimetric theorem (see details in [26] and an alternative proof, when $\mu(t)$ is continuous, in [18]), the following result is proved.

<u>Lemma 4</u> Let $z\in H_0^1(\Omega)$, $z\ge 0$. Then one has

$$N\omega_N^{1/N}\,\mu(t)^{(N-1)/N}\le(-\frac{d\mu}{dt}(t))^{\frac12}\,(-\frac{d}{dt}\int_{\{z>t\}}|\nabla z|^2dx)^{\frac12} \tag{43}$$

for a.e. $t>0$, where $\mu(t)$ is the distribution function of z.

<u>Lemma 5</u> Let $u\in H_0^1$., $u\ge 0$ and g_1 (resp. $u\in H^1(\Omega\text{-}G)\cap L^\infty(\Omega\text{-}G)$, $u\ge 0$, g_1) be as in Lemma 3 (also let g_1 be a radial on $(0,|\Omega|]$) and satisfy

$$-\frac{d\tilde{u}}{ds}(s)\le[\frac{\lambda}{N\omega_N^{1/N}s^{(N-1)/N}}]^2[\int_0^s\tilde{g}_1(\theta)d\theta-\int_0^s f(\tilde{u}(\theta))d\theta]^{\frac12}\ \text{a.e. } s\in(0,|\Omega|] \tag{44}$$

(resp.

$$-\frac{d\tilde{u}}{ds}(s)$$

$$\le[\frac{\lambda}{N\omega_N^{1/N}s^{(N-1)/N}}]^2[\int_{\partial G}\Sigma a_{ij}\frac{\partial u}{\partial x_i}n_j d\sigma+\int_0^s\tilde{\bar{g}}_1(\theta)d\theta \tag{45}$$

$$-\int_0^s f(\tilde{\bar{u}}(\theta))d\theta-|G|(\|g_1\|_\infty-\|f(u)\|_\infty)]^{\frac12}$$

for a.e. $s\in(0,|\Omega|]$).

<u>Proof</u> First we remark that, if $z\in H_0^1(\Omega)$, then

$$-\frac{d}{dt}\int_{\{z>t\}}\Sigma a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}dx\ge-\lambda\frac{d}{dt}\int_{\{z>t\}}|\nabla z|^2\ \text{a.e. } t>0,$$

where λ is given in the ellipticity condition (6) (recall that we are assuming $\lambda(x)\equiv\lambda>0$). Indeed, it suffices to integrate the inequality $\Sigma a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}\ge\lambda|\nabla z|^2$ on the set $\{t<z(x)\le t+h\}$ and then take the limit as $h\downarrow 0$. Then, using Lemmas 3 and 4, we obtain that

$$1\le\frac{-\lambda\mu'(t)}{[N\omega_N^{1/N}\mu(t)^{(N-1)/N}]^2}(\int_0^{\mu(t)}\tilde{g}_1(s)ds-\int_0^{\mu(t)}f(\tilde{u}(s))ds)\ \text{a.e. } t>0.$$

By integrating between t_1 and t_2, $0\le t_1\le t_2\le$ ess.sup u, we get

$$t_2-t_1\le\frac{\lambda}{(N\omega_N^{1/N})^2}\int_{t_1}^{t_2}\mu(t)^{-2(N-1)/N}[\int_0^{\mu(t)}\tilde{g}_1(\theta)d\theta-\int_0^{\mu(t)}f(\tilde{u}(\theta))d\theta]^{\frac12}\mu'(t)dt. \tag{46}$$

Now the formal idea is that $\mu'(t)dt\ge d\mu(t)$ and then, for every $0\le t_1\le t_2\le$ ess sup u,

$$t_2-t_1\le\frac{\lambda}{(N\omega_N^{1/N})^2}\int_{\mu(t_2)}^{\mu(t_1)}r^{-2(N-1)/N}[\int_0^r\tilde{g}_1(\theta)d\theta-\int_0^r f(\tilde{u}(\theta))d\theta]^{\frac12}dr. \tag{47}$$

A rigorous proof of this inequality needs some results of integration theory and can be found in [21] and [11]. Finally, to prove (43), let $0 < s_1 \le s_2 \le |\Omega|$ such that $\tilde{u}(t_2) < \tilde{u}(t_1)$. We have that meas $\{\theta \in (0,|\Omega|) : \tilde{u}(\theta) > \tilde{u}(s_1) - \varepsilon\} \ge s_1$. Then making $t_2 = \tilde{u}(s_2)$ and $t_1 = \tilde{u}(s_1) - \varepsilon$, and estimating the second member of (46), we deduce

$$\tilde{u}(s_1) - \varepsilon - \tilde{u}(s_2) \le \frac{\lambda}{(N\omega_N^{1/N})^2} \int_{s_1}^{s_2} r^{-2(N-1)/N}\left[\int_0^r \tilde{g}_1(\theta)d\theta - \int_0^r f(\tilde{u}(\theta))d\theta\right]^{\frac{1}{2}} dr.$$

Since ε is arbitrary, taking $s_2 = s_1 + h$, multiplying by $1/h$ and letting $h \downarrow 0$, we obtain (43).

In the case of the exterior problem we note that $u \in L^\infty(\Omega - G)$ and that, in fact, $0 \le u \le 1$. Then, if $\bar{u}$ is the extension of u to Ω by taking $\bar{u} = 1$ on G, we have that $\bar{u} \in H_0^1(\Omega)$ and so the proof of (45) is exactly as in the interior problem (note that now $\mu(t) = \mu_0(t) + |G|$, where μ and μ_0 are the distribution functions of $\bar{u}$ and u respectively). □

Lemma 6 Let $g_2 \in L^2(\Omega)$ with $g_2 = g_2^*$ (resp. $g_2 \in L^\infty(\Omega - G)$ with $\tilde{\bar{g}}_2 = \bar{g}_2$). Let $v \in H_0^1(\Omega^*)$, $v \ge 0$ satisfying $-\lambda\Delta v + f(v) = g_2$ in Ω^* (resp. $v \in H^1(\Omega^* - G^*) \cap L^\infty(\Omega^* - G^*)$, $v \ge 0$ satisfying (35), (36)). Then one has

$$-\frac{d\tilde{v}}{ds}(s) = \left[\frac{\lambda}{N\omega_N^{1/N} s^{(N-1)/N}}\right]^2 \left[\int_0^s \tilde{g}_2(\theta)d\theta - \int_0^s f(\tilde{v}(\theta))d\theta\right]^{\frac{1}{2}} \text{ a.e. } s \in (0,|\Omega|], \tag{48}$$

(resp.

$$-\frac{d\tilde{v}}{ds}(s) = \left[\frac{\lambda}{N\omega_N^{1/N} s^{(N-1)/N}}\right]^2 \left[\lambda\int_{\partial G} \frac{\partial u}{\partial n}\, d\sigma + \int_0^s \tilde{\bar{g}}_2(\theta)d\theta\right.$$

$$\left. - \int_0^s f(\tilde{\bar{v}}(\theta))d\theta - |G|(\|g_2\|_\infty - \|f(v)\|_\infty)\right]^{\frac{1}{2}} \text{ a.e. } s \in (0,|\Omega|]). \tag{49}$$

Proof By uniqueness, $v(x) = v^*(|x|) = \tilde{v}(\omega_N|x|^N)$. On the other hand, it is a routine matter to check that, after obvious changes of variable, $\tilde{v}$ satisfies (48). The proof of (49) is by analogy a direct consequence of the symmetry of v. Note that, in fact, (18) and (19) can also be proved by means of the proofs of Lemmas 3, 4 and 5 and noting that all the inequalities now become equalities. □

Proof of Theorems 5 and 6 By comparison results (Theorem 2 part (c)) we can limit ourselves to the case $g_1 \ge 0$, $g_2 \ge 0$, and so $u \ge 0$ and $v \ge 0$. On

the other hand, by Theorem 2 part (d), we can also assume $g_1 \in L^2(\Omega)$, $g_2 \in L^2(\Omega^*)$ (and then $u \in H_0^1(\Omega)$, $v \in H_0^1(\Omega^*)$) without loss of generality. Now, let us consider the set

$$J = \{t \in (0,|\Omega|] : \int_0^t f(\tilde{u}(s))ds > \int_0^t f(\tilde{v}(s))ds\}.$$

From (43), (47) and the assumption $g_1^* \precsim g_2$ we obtain that

$$-\frac{d}{ds}(\tilde{u}(s) - \tilde{v}(s)) < 0 \text{ a.e. on } J.$$

Hence, if $a = \inf\{t : t \in J\}$, it is clear that $a > 0$ and

$$\int_0^a f(\tilde{u}(s))ds = \int_0^a f(\tilde{v}(s))ds.$$

But $f(\tilde{u}(s)) - f(\tilde{v}(s))$ is a positive increasing function on J. Then $(a,|\Omega|] \subset J$, which implies that $0 = \tilde{u}(|\Omega|) > \tilde{v}(|\Omega|) = 0$, a contradiction. This proves that $f(u^*) \precsim f(v)$ which, by equimeasurability of u and u^*, gives (31). Moreover, if $w \in H_0^1(\Omega)$ satisfies $-\lambda\Delta w = g_2$ in Ω^* then, as in Lemma 6,

$$-\frac{d\tilde{w}}{ds} = \left[\frac{\lambda}{N\omega_N^{1/N} s^{(N-1)/N}}\right]^2 \left[\int_0^s \tilde{g}_2(\theta)d\theta\right]^{\frac{1}{2}}. \tag{50}$$

Hence, from (43) and (49),

$$-\frac{d}{ds}(\tilde{u}(s) - \tilde{w}(s)) \le 0 \text{ a.e. } t > 0, \tag{51}$$

and then, since $\tilde{u}(|\Omega|) = \tilde{w}(|\Omega|)$, the conclusion $u^* \le w^* = w$ follows by integrating from s to $|\Omega|$ in (51).

To show Theorem 6 we argue as before but now taking

$$J = \{t \in (0,|\Omega|] : \int_0^t f(\tilde{\bar{u}}(s))ds - C_u > \int_0^t f(\tilde{\bar{v}}(s))ds - C_v\}$$

or integrating directly in the case of inequality (38). □

Now we return to the proof of Theorem 4. We still need another auxiliary result due to Hardy-Littlewood and Polya [14] (see also [6], [11]).

Lemma 7 Let $y,z \in L^1(0,M)$ be nonnegative functions and $y(s)$ nonincreasing. Assume that

$$\int_0^t y(s)ds \le \int_0^t z(s) \quad \text{for every } t \in [0,M].$$

Then for every continuous convex function Φ we have

$$\int_0^t \Phi(y(s))ds \le \int_0^t \Phi(z(s))ds, \text{ for every } t \in [0,M].$$

Proof of Theorem 4 Let u be the solution of (1), (2) with $g \equiv 0$, $h = 1$ (and $b_i \equiv 0$). Then the function $U(x) = 1 - u(x)$ belongs to $H_0^1(\Omega)$ and satisfies

$$-LU + \hat{f}(U) = f(1) \text{ in } \Omega$$

where $\hat{f}(r) = f(1) - f(1-r)$. By analogy, if $v \in H^1(\Omega^*)$ satisfies $-\lambda\Delta v + f(v) = 0$ in Ω^*, $v = 1$ on $\partial\Omega^*$, then $V = 1 - v \in H_0^1(\Omega^*)$ and verifies

$$-\lambda\Delta V + \hat{f}(V) = f(1) \text{ in } \Omega^*.$$

Then, applying Theorem 5 to the choices $\hat{f}$, $g_1 = g_2 = f(1)$, U and V, we get, by Lemma 7, that

$$\int_\Omega \Phi(\hat{f}(U(x)))dx \le \int_{\Omega^*} \Phi(\hat{f}(V(x)))dx. \tag{52}$$

We also remark that, by the comparison principle (Theorem 2(c)), we have $0 \le U \le 1$, $0 \le V \le 1$. Now, given $\varepsilon > 0$, let $\Phi_\varepsilon(t)$ be a convex function satisfying

$$\Phi_\varepsilon(r) = 0 \text{ if } 0 \le r \le \hat{f}(1) - \varepsilon \text{ and } \Phi_\varepsilon(\hat{f}(1)) = 1.$$

Then, by (52),

$$\int_{\{\hat{f}(V)\ge\hat{f}(1)-\varepsilon\}} \Phi_\varepsilon(\hat{f}(U))dx \le \int_{\Omega^*} \Phi_\varepsilon(\hat{f}(V))dx \le \int_{\{\hat{f}(V)\ge\hat{f}(1)-\varepsilon\}} = \text{meas}\{x:\hat{f}(V(x))\ge\hat{f}(1)-\varepsilon\}.$$

Therefore

$$\text{meas }\{U = 1\} = \int_{\{U=1\}} \Phi_\varepsilon(\hat{f}(1)) \le \int_{\{\hat{f}(U)\ge\hat{f}(1)-\varepsilon\}} \Phi_\varepsilon\ (f(U))dx \le \text{meas }\{x:\hat{f}(V(x)) \ge \hat{f}(1)-\varepsilon\}.$$

Letting $\varepsilon \downarrow 0$, we obtain, in the limit, that (since $f^{-1}(0) = 0$)

$$\text{meas } N(u) = \text{meas }\{x \in \Omega:U(x) = 1\} \le \text{meas}\{x \in \Omega^*:V(x) = 1\} = \text{meas } N(v).$$

On the other hand, taking $\Phi(t) = t^p$ with $1 < p < \infty$, we deduce from (52) that

$$\|\hat{f}(U)\|_{L^p(\Omega)} \le \|\hat{f}(V)\|_{L^p(\Omega^*)}.$$

Making $p \to +\infty$ we conclude that

$$\operatorname*{ess\,sup}_\Omega U \le \operatorname*{ess\,sup}_{\Omega^*} V.$$

Then, if $v > 0$ on Ω^* we have that $0 \le V(x) < 1$ a.e. $x \in \Omega^*$ and so $U(x) < 1$ a.e. $x \in \Omega$, i.e. $u > 0$ on Ω. This, jointly with the next counterexample, completes the proof of Theorem 4. □

The fact that the null set N(u) of solutions of the exterior problem does not satisfy, in general, an isoperimetric inequality, in contrast with the interior problem, can be easily seen in the one-dimensional case. Indeed, let f satisfy assumption (16) or (7), and let $\phi(t)$ be the unique solution of

$$u'(t) = \sqrt{2}\, F(u(t))^{1/2}, \quad F(s) = \int_0^s f(\tau)d\tau$$

$$u(1) = 1.$$

By (16), it is easy to see that there exists $T_0 > 1$ such that $\phi(t) > 0$ if $1 \le t < T_0$ and $\phi(t) = 0$ if $t \ge T_0$. (Note that in fact $T_0 - 1 = \psi_1(1)$, $\phi(t) = \psi_1^{-1}(T_0 - t)$, see the proof of Theorem 1.) Now, take $\Omega = (0, T_0 + \delta)$ and $G = (\varepsilon, 1)$. From the above considerations we conclude that, if u(x) is the solution of

$$-u'' + f(u) = 0 \text{ in } (0,\varepsilon) \cup (1, T_0 + \delta)$$

$$u(0) = 0,\ u(\varepsilon) = u(1) = 1,\ u(T_0 + 1) = 0,$$

then $u = \phi$ on $(1, T_0 + \delta)$ and so $N(u) = [T_0, T_0 + \delta)$, assumed ε small enough. Nevertheless

$$\Omega^* = \left(-\frac{T_0-\delta}{2}, \frac{T_0+\delta}{2}\right),\ G^* = \left(-\frac{1+\varepsilon}{2}, \frac{1-\varepsilon}{2}\right)$$

and the corresponding v(x) satisfies

$$-v'' + f(v) = 0 \text{ in } \left(-\frac{T_0+\delta}{2}, -\frac{1+\varepsilon}{2}\right) \cup \left(\frac{1-\varepsilon}{2}, \frac{T_0+\delta}{2}\right),$$

$$v\left(-\frac{T_0-\delta}{2}\right) = v\left(\frac{T_0+\delta}{2}\right) = 0,\ v\left(-\frac{1+\varepsilon}{2}\right) = v\left(\frac{1+\varepsilon}{2}\right) = 1.$$

In particular, since the equation is autonomous, it is easy to see ([11], Th. 1.4) that, if $\frac{T_0+\delta}{2} - (\frac{1-\varepsilon}{2}) - 1 < \psi_1(1)$, then $v(x) > 0$ in Ω^*-G^* and so $N(v)$ is empty. In consequence, it suffices to take ε and δ such that $\delta-\varepsilon-2 < 2\psi_1(1)$ to show that meas $N(u) \nleq$ meas $N(v)$. In fact, in the light of the proof of Theorem 4, we can conclude that meas $N(u) - C_u \le$ meas $N(v) - C_v$, but the present counterexample shows that, in general, $C_u \ge C_v$. We shall comment later on this inequality.

It is a curious fact that if we invert the values of the boundary conditions in the exterior problem, the same isoperimetric inequality, as in the interior problem, holds for $N(u)$ in some cases.

Theorem 7 Let f be a continuous nondecreasing function with $f(0) = 0$. Let G be an open set strictly contained in Ω. Let $u \in W^{2,p}(\Omega-G) \cap L^\infty(\Omega-G)$, $1 \le p < \infty$, such that

$$-Lu + f(u) = 0 \text{ in } \Omega-G ,$$

$$u = 0 \text{ on } \partial G,\ u = 1 \text{ on } \partial\Omega.$$

Let $v \in W^{2,p}(\Omega^*-G^*) \cap L^\infty(\Omega^*-G^*)$ satisfying

$$-\lambda\Delta v + f(v) = 0 \text{ in } \Omega^* - G^*,$$

$$v = 0 \text{ on } \partial G^*,\ v = 1 \text{ on } \partial\Omega^*.$$

Assume that $d(G, S(u)) > 0$. Then

$$\text{meas } N(u) \le \text{meas } N(v).$$

Proof The functions $U(x) = 1 - u(x)$ and $V(x) = 1 - v(x)$ satisfy the exterior problems (33), (34) and (35), (36) respectively, with $\hat f(r) = f(1) - f(1-r)$ instead of $f(r)$ and $g_1 = g_2 = f(1)$. Then, by (37) and remarking that $C_U = -C_u$, $C_V = -C_v$, we have

$$\hat f(U^*) + C_u \precsim \hat f(V) + C_v.$$

But note that, by the maximum principle, $v \ge 0$ on Ω^*-G^* which implies $C_v \le 0$. On the other hand, by assumption, $C_u = 0$. Therefore, $\hat f(U^*) \precsim \hat f(V)$ and we conclude as in the proof of Theorem 4. □

Some bibliographical remarks on the results of this section are in order.

Remark 4 An earlier different proof of the Theorem 4 for the interior problem and some additional assumptions (for instance, f Hölder continuous) was given in [8]. In fact, our proof gives at the same time useful information on the quantity

$$e = \frac{1}{|\Omega|} \int_\Omega f(u(x))dx,$$

where u is as in Theorem 4, called *effectiveness* in chemical reaction theory. Indeed, the proof of Theorem 4 shows that the ball is the domain with prescribed volume having the lowest effectiveness. This gives an answer to a question raised by Aris and already considered in [1] and [8].

Remark It seems that the first application of rearrangement techniques to obtain *a priori* estimates of solutions of PDEs was given in [31] for linear equations. A sharper result, containing the comparison $u^* \le w$ of Th. 5 was proved in [26] and later in [27] for nonlinear equations (see also an alternative proof in [18]). The general comparison $f(u^*) \precsim f(v)$ of Theorem 5 was first shown in [10] and [18] for the linear case and later in [19], [20] and [21] for nonlinear equations. Our proof is inspired by earlier ones, only making some delicate points rather more precise. It is important to remark that (as already used in the proof of Theorem 4) Theorem 5, jointly with Lemma 7, allows *a priori* estimates on u to be obtained, such as, for instance $\|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^*)}$, for every $1 \le p \le \infty$ (see finer estimates in the quoted references). Other applications of rearrangement techniques to the study of symmetry and even to the obtaining of an improvement of Theorem 1 for the existence of the free boundary $F(u)$, are given in [11].

Remark 6 The results of this section may be easily extended to the case of quasilinear equations ([11]). On the other hand, the simplifications $\lambda(x) \equiv \lambda > 0$ and $b_i \equiv 0$ made throughout this section may be avoided by using the approach of the works [3], [4], [5] and [28], for linear operators involving first order terms and eventually degenerated.

Remark 7 In the particular case of $f \equiv 0$, and $g_1 = g_2 \equiv 0$ the solutions u,v of the exterior problems (33), (34) and (35), (36) represent the *electrostatic potential of a capacitor* Ω-G or Ω^*-G^*, respectively. Note that, in this case $\bar u^*(0) = \bar w(0) = 1$ and so (38) reminds one of the earliest applications of rearrangement techniques in mathematical physics: the *electrostatic capacity* of a capacitor in $\mathbb{R}^3$, given by $4\pi C_u$, is the lowest in the case of a spherical ring. This isoperimetric inequality was first proved in [25] (see also [23] and [21]). The rest of the results for the exterior problem seem to be new in the literature.

Remark 8 Another interesting extension of the results of this section concerns the multivalued equation

$$- Lu + \beta(u) \ni g \text{ in } \Omega, \tag{53}$$

where now β represents a maximal monotone graph of $\mathbb{R}^2$ with $0 \in \beta(0)$. The interest of this general formulation comes from the fact that equation (53) includes the special cases of the *obstacle problem* ($\beta(s) = \emptyset$, the empty set, if $s < 0$, $\beta(0) = (-\infty, 0]$ and $\beta(s) = \{1\}$), *zeroth-order reactions* ($\beta(s) = \{0\}$ if $s \le 0$, $\beta(0) = [0,1]$ and $\beta(s) = \{1\}$ if $s > 0$), as well as equation (1) ($\beta(s) = \{f(s)\}$). We recall that, with obvious changes, Theorem 2 was proved in [9] for the general equation (53). On the other hand, it is a trivial fact to extend Theorem 1 to that equation (now F needs to be replaced by the adequate convex function j such that $\partial j = \beta$). Moreover, if β is multivalued at the origin ($\beta(0) = [\beta^+(0), \beta^-(0)]$, the free boundary $F(u)$ does exist even for data g not necessarily vanishing on a subset of Ω. If, for instance, we know that $u \ge 0$, then a sufficient condition for the existence of $F(u)$ is

$$g(x) \le \beta^+(0) - \varepsilon, \tag{54}$$

for some $\varepsilon > 0$ and x belonging to a large enough subset of Ω (see e.g., [11]). With respect to the application of rearrangement to these problems, we point out that all the theorems remain true by replacing $f(u)$ by $b_u = g_1 + Lu$ and $f(v)$ by $b_v = g_2 + \lambda\Delta v$. Moreover, another interesting result is available when we take, in the proofs of Theorems 5 and 6, the *signed decreasing rearrangement* of the data g_1 and g_2, respectively. In this definition,

$$\tilde{u}(s) = \inf \{t \in \mathbb{R} : \text{meas } \{x \in \Omega : u(x) > t\} > s\}$$

and, finally,

$$u^*(s) = \tilde{u}(\omega_N |x|^N)$$

for a given function $u \in L^1(\Omega)$. Now the version similar to Theorem 5 is

Theorem 8 Let β be a maximal monotone graph of $\mathbb{R}^2$ with $\beta(0) = [\beta^-(0), \beta^+(0)]$, $-\infty \le \beta^-(0) \le 0 \le \beta^+(0) \ge \infty$. Let $g_1 \in L^1(\Omega)$ and $g_2 \in L^1(\Omega^*)$, g_2 such that its signed decreasing rearrangement g_2^* coincides with g_2. Let $u \in W_0^{1,1}(\Omega)$ with $u \ge 0$ on Ω, $Lu \in L^1(\Omega)$ satisfying (53). Then, if the signed rearrangement of g_1 is such that $g_1^* \precsim g_2$, we conclude that

$$u^* \le v,$$

with $v \in W_0^{1,1}(\Omega^*)$, $v \ge 0$ on Ω^* and $-\lambda\Delta v \in L^1(\Omega^*)$ satisfying

$$-\lambda\Delta v + \beta^{\#}(v) \ni g_1 \text{ on } \Omega^*,$$

where $\beta^{\#}$ is the maximal monotone graph of $\mathbb{R}^2$ given by

$$\beta^{\#}(r) = \{\beta^+(0)\} \text{ if } r > 0,\ \beta^{\#}(0) = (-\infty, \beta^+(0)],\ \beta^{\#}(r) = \emptyset \text{ if } r < 0.$$

Note that, if $\beta(r) = \{f(r)\}$ where f is a continuous nondecreasing function with $f(0) = 0$, then $\beta^{\#}(r) = \{0\}$ for $r > 0$ and thus Theorem 8 gives the comparison $u^* \le w$ of Theorem 5, assumed $g_1 \ge 0$. Nevertheless, if β is a multivalued graph at the origin, i.e., if $\beta^-(0) < \beta^+(0)$, then the important hypothesis $u \ge 0$ (or $v \ge 0$) may be compatible, with data g_1 (or g_2) eventually taking negative values. (This is the usual case in the obstacle problem.) Repeating the proof of Theorem 4, it is easy to see that, if $u \ge 0$ verifies (43), then the condition

$$\int_\Omega (g_1(x) - \beta^+(0))dx < 0$$

implies that the null set $N(u)$ has a positive measure, and that, in addition, meas $\{x : g_1(x) - \beta^+(0) > 0\} > 0$; then the existence of the free boundary $F(u)$ is assured (compare this with (54)). These results were proved in [7] for the obstacle problem (see also [20]) and in [11] for a general maximal monotone graph.

References

[1] Amundson, N.R. and Luss, D., On a conjecture of Aris: proof and remarks, *A.I.Ch.E.H.* *13* (1967), 759.

[2] Aris, R., *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, Clarendon Press, Oxford (1975).

[3] Alvino A. and Trombetti, G., Sulle migliori costanti di maggiorazione per una classe di equazions ellittiche degeneri. *Richerche di Mat.* *27* (1978) 193-212.

[4] Alvino, A. and Trombetti, G., Equazioni elllitiche con termini di ordine inferiore e riordinamenti. *Rend. Acc. Naz. Lincei* (8) *66* (1979), 1-7.

[5] Alvino, A. and Trombetti, G., Sulle migliori constanti di maggiorazione per una classe di equazioni ellittiche degeneri e non. *Ric. di Mat.* *30* (1981), 15-33.

[6] Bandle, C. *Isoperimetric Inequalities and Applications*, Pitman, London (1980).

[7] Bandle, C. and Mossino, J., Application du rearrangement à une inéquation variationelle. *C.R. Acad. Sc. Paris*, *296* (1983), 501-504.

[8] Bandle, C., Sperb, R.P. and Stakgold, I., Diffusion and reaction with monotone kinetics. *Nonlinear Anslysis Th., Meth. and Appl.* *8* (1984) 321-333.

[9] Brezis, H. and Strauss, W. Semilinear second-order elliptic equations in L^1. *J. Math. Soc. Japan*, *25* (1973), 565-590.

[10] Chiti, G., Norme di Ovlicz delle soluzioni di una classe di equazioni ellittiche, *B.U.M.I.* (5) *16* (1979), 178-185.

[11] Diaz, J.I., *Nonlinear Partial Equations and Free Boundaries. Vol.1. Elliptic Equations.* Research Notes in Mathematics No. 106, Pitman, London (to be published).

[12] Diaz, J.I. and Hernandez, J., On the existence of a free boundary for a class of reaction-diffusion systems, *SIAM J. Math. Anal.* *5* (1984), 670-685.

[13] Friedman, A. and Phillips, D., The free boundary of a semilinear elliptic equation. *Trans. Amer. Math. Soc.* *282* (1984), 153-182.

[14] Hardy, G.H., Littlewood, J.E. and Polya, G., Some simple inequalities satisfied by convex functions, *Messenger Math.*, *58* (1929), 145-152.

[15] Hardy, G.H., Littlewood, J.E. and Polya, G. *Inequalities*, Cambridge Univ. Press (1964).

[16] Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. No. 840, Springer-Verlag, Berlin (1981).

[17] Knerr, B., The porous medium equation in one dimension. *Trans. Amer. Math. Soc.* *234* (1977), 381-415.

[18] Lions, P.L., Quelques remarques sur la symétrisation de Schwarz, in *Nonlinear PDE and their Applications, College de France Seminar, Vol.I* H. Brezis and J.L. Lions (editors). Research Notes in Mathematics No. 53 Pitman, London (1981).

[19] Maderna, C., Optimal problems for a certain class of nonlinear Dirichlet problems. *Supp. B.U.M.I. I.* (1980), 31-34.

[20] Maderna, C. and Salsa, S., Some special properties of solutions to Obstacle Problems. *Rend. Sem. Mat. Univ. Padova*, *71* (1984), 121-129.

[21] Mossino, J. *Inegalités isopérimétriques et applications en physique*, Hermann, Paris (1984).

[22] Peletier, L.A., The porous media equation, in *Applications of Nonlinear Analysis in Physical Sciences.* H. Amann et al. (editors), Pitman, London (1981).

[23] Polya, G. and Szegö, C., *Isoperimetric Inequalities in Mathematical Physics.* Princeton Univ. Press (1951).

[24] Stampacchia, G., *Équations elliptiques du second ordre à coefficients discontinues. Sem. de Math. Sup. No. 16*, Les Press de L'Univ. de Montreal (1966).

[25] Szegö, C., Ueber einige Extremalavfgaten der Potentialtheorie, *Math. Z.*, *31* (1930), 583-593.

[26] Talenti, G., Elliptic equations and rearrangements, *Ann. Scuola Norm. Sup. Pisa* (4) *3* (1976), 697-718.

[27] Talenti, G., Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces, *Annali Mat. Pura Appl.* (IV) *120* (1979), 159-184.

[28] Talenti, G., Linear elliptic PDEs, level sets, rearrangements, *B.U.M.I* (to appear).

[29] Vazquez, J.L., Symétrisation pour $u_t = \Delta\phi(u)$ et applications, *C.R. Acad. Sci. Paris*, *295* (1982), 71-74.

[30] Vazquez, J.L., A strong maximum principle for some quasilinear elliptic equations (to appear).

[31] Weinberger, H., Symmetrization in uniformly elliptic problems, *Studies in Math. Anal*, Stanford Univ. Press (1962), 424-428.

J.I. Diaz
Facultad de Matematicas
Universidad Complutense de Madrid
28040 Madrid,
Spain