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Behaviour on the boundary of solutions of parabolic equations with nonlinear boundary conditions: the evolution Signorini problem

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1. INTRODUCTION.

Let Ω be an open regular set of R^N and consider the following nonlinear parabolic boundary value problem

$$(1)\quad \begin{cases} u_t - \Delta u = f(t,x) & \text{in} \quad (0,\infty) \times \Omega = Q \\ -\frac{\partial u}{\partial n} + b(u) = g(t,x) & \text{on} \quad (0,\infty) \times \partial\Omega = \Sigma \\ u(0,.) = u_o & \text{in} \quad \Omega \end{cases}$$

where n is the outward normal to $\partial\Omega$ and b is a continuous nondecreasing function such that $b(0) = 0$. Many different results are well-known about the existence and uniqueness of solutions u under several kind of regularity assumptions of f, u_o and g (see, for instance, Friedman |8|, and Amann |2| and Alikakos |1|). It is also well-known that if $f \geq 0$, $u_o \geq 0$ and $g \geq 0$ then $u \geq 0$ in $\overline{Q}$. In fact, by the strong maximum principle, if u is non negative and $u(t,.) \not\equiv 0$ for every $t > 0$, then $u > 0$ on Q. A natural question arise. For which functions b, the behaviour of u near Σ becomes pathological, in the sense that $u(t,.)|_\Sigma$ vanishes on some subregion of Σ?. A first answer is easy because, again by the maximum principle, if $u(t,x) = 0$ on $(t_o,\infty) \times \Gamma_o$ then we arrive to the contradiction $0 > \frac{\partial u}{\partial n}(t,x) = g(t,x) - b(o) \geq 0$ on $(t_o, \infty) \times \Gamma_o$. So, this behaviour is excluded for any nondecreasing continuous function b such that $b(o) = 0$ and for which $u(t,.) \not\equiv 0$ for any $t > 0$.

The situation changes strongly if we take b in the class of the multivalued maximal monotone graphs of R^2 (that we shall denote by β). That general formulation is of interest in applications (see Duvaut-Lions | 7 |) and now the problem is

$$(2)\quad \begin{cases} u_t - \Delta u = f(t,x) & \text{in} \quad Q \\ -\frac{\partial u}{\partial n} + g(t,x) \in \beta(u) & \text{on} \quad \Sigma \\ u(0,.) = u_o & \text{in} \quad \Omega \end{cases}$$

where β is given by

(3) $\beta(r) = 0$ if $r > 0$, $\beta(o) = (-\infty, o]$ and $\beta(r)$ = empty set if $r < 0$.

This problem known as the Signorini parabolic problem. Note that in this case the boundary conditions are of "unilteral type".

$$(4)\quad u \geq 0,\ -\frac{\partial u}{\partial n} + g \leq 0 \quad \text{and} \quad u(-\frac{\partial u}{\partial n} + g) = 0 \quad \text{on } \Sigma,$$

and that the **coincidence set**, defined by

$$(5)\qquad \Sigma_o = \{(t,x) \in \Sigma : u(t,x) = 0\},$$

(i.e. the "obstacle" $\psi|_\Sigma$ 0, ≡ cf. Díaz-Jiménez |6| for eliptic case), plays an important role in the understanding of the problem. Our question now is the study of the assumptions on f, u_o and g allow the formation of the coincidence set Σ_o.

2. STATEMENT OF THE PROBLEM AND PREVIOUS RESULTS.

We consider the following evolution Signorini problem: (ESP) "Let Ω be a regular open set in R^N with boundary $\partial\Omega = \Gamma$. Given f, g, u_o and the obstacle ψ in suitable functional spaces, find u(t,x) such that

$$(6)\quad \begin{cases} u_t - \Delta u = f(t,x) & ,\ (t,x) \in Q \\ u \geq \psi,\ -\frac{\partial u}{\partial n} + g \leq 0 & \\ & ,\ (t,\xi) \in \Sigma \\ (u-\psi)(-\frac{\partial u}{\partial n} + g) = 0 & \\ \quad u(0,x) = u_o(x) & ,\ x \in \Omega. \end{cases}$$

If $u(t) \in H^2(\Omega)$ for all $t > 0$, then the (ESP) can be written in the following complementary form:

$$(7)\quad \begin{cases} u_t - \Delta u = f & ,\ Q \\ -\frac{\partial u}{\partial n} + g \in \beta(u - \psi) & ,\ \Sigma \\ u(0,x) = u_o & ,\ \Omega \end{cases}$$

Now we give some trivial lemmas that allow us to consider null obstacles ψ. Also we consider some homogeneous situations.

Lemma 1. Let u be the solution of the linear problem

$$(\overline{P})\quad \begin{cases} \overline{u}_t - \overline{u} = 0 & ,\ Q \\ \overline{u} = \psi & ,\ \Sigma \\ \overline{u}(0,x) = 0 & ,\ \Omega \end{cases}$$

with u satisfying (7). Then $u^* = u - \bar{u}$ satisfies

$$(P^*)\quad \begin{cases} u^*_t - \Delta u^* = f, & Q \\ -\frac{\partial u^*}{\partial n} + g^* \in \beta(u^*), & \Sigma \\ u^*(0,x) = u_o, & \Omega. \end{cases}$$

Lemma 2. Let $\tilde{u}$ be the solution of the linear problem

$$(\tilde{P})\quad \begin{cases} \tilde{u}_t - \Delta\tilde{u} = f, & Q \\ \tilde{u} = 0, & \Sigma \\ \tilde{u}(0,x) = u_o, & \Omega \end{cases}$$

with u satisfying (P^*), then $\hat{u} = u - \tilde{u}$ satisfies

$$(\hat{P})\quad \begin{cases} \hat{u}_t - \Delta\hat{u} = 0, & Q \\ -\frac{\partial \hat{u}}{\partial n} + \hat{g} \in \beta(\hat{u}), & \Sigma \\ \hat{u}(0,x) = 0, & \Omega \end{cases}$$

where

$$(8)\qquad \hat{g}(t,\xi) = g(t,\xi) - \frac{\partial u}{\partial n}(t,\xi) \qquad \text{a.e. on } \Sigma.$$

Remark. If $u_o \equiv 0$ in ($\tilde{P}$) and $f(t,x)$ is such that $f, \nabla f \in C_b(\overline{Q})$, then $\tilde{u} \in C_b(\overline{Q})$ is the classical solution of ($\tilde{P}$) and in that case $\tilde{u}$ can be written in the form

$$(9)\qquad \tilde{u}(t,x) = \int_Q G(t-s;x,\xi)f(s,\xi)d\xi ds \quad , \quad x \in \Omega,$$

where $G(t,\tau;x,\xi)$ is the Green's function associated to the heat equation on $\Omega_{\#}$.

The following formulation of the (ESP) will be used:
Let $T > 0$,

$$(10)\quad \int_{Q_T} w_t(w-u)dxdt + \int_0^T a(u,w-u)dt \geq \int_{Q_T} f(w-u)dxdt + \int_{\Sigma_T} g(w-u)d\xi dt$$

for all $w \in L^2(0,T;H^1(\Omega))$, $w_t \in L^2(0,T;(H^1(\Omega))')$

$$(11) \qquad u(0,x) = u_o(x), \qquad x \in \Omega$$

where $Q_T \doteq (0,T) \times \Omega$ and $\Sigma_T = (0,T) \times \partial\Omega$, and $a(.,.): H^1(\Omega) \times H^1(\Omega) \to R$ is the bilinear form associated to $-\Delta$.
All of these are necessary for the next comparison result (cf. Brezis |3|).

Theorem 3. Given $u_o, \hat{u}_o \in L^2(\Omega)$, $f, \hat{f} \in L^2(0,T; (H^1(\Omega))')$, $g, \hat{g} \in L^2(\Sigma_T)$ and $u, \hat{u} \in L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega))$ such that u_o, f, g and u satisfying (10)- (11) while $\hat{u}_o$, $\hat{f}$, $\hat{g}$ and $\hat{u}$ satisfy

$$(12) \quad \int_{Q_T} w_t(w-\hat{u})dxdt + \int_0^T a(\hat{u},w-\hat{u})dt \geq \int_{Q_T} \hat{f}(w-\hat{u})dxdt + \int_{\Sigma_T} \hat{g}(w-\hat{u})d\xi dt$$

for all $w \in L^2(0,T; H^1(\Omega))$, $w_t \in L^2(0,T; (H^1(\Omega))')$.

$$(13) \qquad \hat{u}(0,x) = \hat{u}_o(x), \qquad x \in \Omega.$$

If $u_o \geq \hat{u}_o$, $f \geq \hat{f}$ and $g \geq \hat{g}$ then

$$(14) \qquad u(x,t) \geq \hat{u}(x,t) \qquad \text{a.e. on } Q_T \text{ for all } T > 0.$$

Proof. Let $u_n, \hat{u}_n \in L^2(0,T; H^1(\Omega))$, $f_n, \hat{f}_n, \in L^2(0,T; (H^1(\Omega))')$, $g_n, \hat{g}_n \in L^2(\Sigma_T)$ and $u_{on}, \hat{u}_{on} \in L^2(\Omega)$ such that

$$(15) \qquad \frac{du_n}{dt}, \ \frac{d\hat{u}_n}{dt} \in L^2(Q_T)$$

$$(16) \quad \int_\Omega \frac{du_n}{dt}(w-u_n)dx + a(u_n, w-u_n) \geq \int_\Omega f_n(w-u_n)dx + \int_\Gamma g_n(w-u_n)d\xi$$

for all $w \in H^1(\Omega)$.

$$(17) \quad \int_\Omega \frac{d\hat{u}_n}{dt}(w-\hat{u}_n)dx + a(\hat{u}_n,w-\hat{u}_n) \geq \int_\Omega \hat{f}_n(w-\hat{u}_n)dx + \int_\Gamma \hat{g}_n(w-\hat{u}_n)d\xi$$

for all $w \in H^1(\Omega)$

$$(18) \qquad u_n(0,x) = u_{on} \quad ; \quad \hat{u}_n(0,x) = \hat{u}_{on}$$

and $u_n \to u$, $\hat{u}_n \to \hat{u}$ in $L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega))$

$f_n \to f$, $\hat{f}_n \to \hat{f}$ in $L^2(0,T;(H^1(\Omega))')$

$g_n \to g$, $\hat{g}_n \to \hat{g}$ in $L^2(0,T; L^2(\Gamma))$

$u_{0n} \to u_0$, $\hat{u}_{0n} \to \hat{u}_0$ in $L^2(\Omega)$.

Let $w = \max\{u_n, \hat{u}_n\} = u_n + (\hat{u}_n - u_n)^+$ and $w = \min\{u_n, \hat{u}_n\} = \hat{u}_n - (\hat{u}_n - u_n)^+$ in (16) and (17) respectively. We obtain.

$$(19) \quad \int_\Omega \left(\frac{d\hat{u}_n}{dt} - \frac{du_n}{dt}\right)(\hat{u}_n - u_n)^+ dx + a(\hat{u}_n - u_n, (\hat{u}_n - u_n)^+) \leq \int_\Omega (\hat{f}_n - f_n)(\hat{u}_n - u_n)^+ dx + \int_\Gamma (\hat{g}_n - g_n)(\hat{u}_n - u_n)^+ d\xi.$$

Integrating on $(0,T)$ and taking limit $n \to \infty$ we observe that $(\hat{u} - u)^+ \leq 0$. Thus $u \geq \hat{u}$ a.e. on Q_T.

3. A NECESSARY CONDITION.

After these trivial considerations, it seems natural that the existence or nonexistence of the coincidence set Σ_0 will depend on the behaviour of such function.

$$\hat{g}(t,\xi) = g(t,\xi) - \frac{\partial \bar{u}}{\partial n}(t,\xi).$$

We start with a necessary condition but we consider now the forcing term $f(t,x)$.

Let $\tilde{u}$ be the solution of $(\tilde{P})$ with $u_0(x) \equiv 0$ a.e. in Ω. We define

$$(20) \quad \tilde{f}(t,\xi) = \frac{\partial \bar{u}}{\partial n} = \int_{Q_T} \frac{\partial G}{\partial n_\xi}(t-s; x,\xi)\, f(s,\xi)\, dx ds, \quad \xi \in \Gamma.$$

Theorem 4. Let u be a (weak) solution of (ESP) and assume that

$$(21) \quad u(t,\xi) = 0 \quad \text{a.e. on } \Sigma_0 = (0,T) \times \Gamma_0$$

where Γ_0 is a "smooth" part of $\Gamma = \partial\Omega$. Then necessarily

$$(22) \quad g(t,\xi) - \tilde{f}(t,\xi) - \tilde{u}_0(0,\xi) \leq 0 \quad \text{a.e. on } \Sigma_0,$$

where

$$(23) \quad \tilde{u}_0(0,\xi) = \frac{\partial \bar{u}}{\partial n}(0,\xi), \quad \xi \in \Gamma$$

and $\tilde{u}$ be a solution of $(\tilde{P})$ with $f \equiv 0$ and $\tilde{f}$ given by (20).

The proof is based on the following lemma

Lemma 5. Every weak solutions u(t,x) of the (ESP) satisfies.

$$(24)\qquad -\int_{\Omega} u_o(x)v(0,x)dx + \int_{\Sigma_T} u\,\frac{\partial v}{\partial n}d\xi dt \geq \int_{Q_T} fvdxdt + \int_{\Sigma_T} gvd\xi dt$$

for all $v \in W^{1,2}(0,T;H^1(\Omega))$ such that $v_t + \Delta v = 0$ on Q_T, $v(T,x) = 0$ on Ω.

Proof (lemma 5)

1st part. We suppose that the data f, g and u_o are sufficiently regular such that exists u(t,x), strong solution, satisfying.

$$(25)\qquad \int_{Q_T} u_t(w-u)dxdt + \int_{Q_T} \nabla u \nabla (w-u)dxdt \geq \int_{Q_T} fvdxdt + \int_{\Sigma_T} gvd\xi dt.$$

Therefore

$$(26)\qquad \int_{\Omega} [u(T,x)v(T,x)-u(0,x)v(0,x)]dx - \int_{Q_T} u(v_t+\Delta v)dxdt +$$

$$+ \int_{\Sigma_T} u\,\frac{\partial v}{\partial n}\,d\xi dt \geq \int_{Q_T} fvdxdt + \int_{\Sigma_T} gvd\xi dt.$$

Let $v(t,x) = \hat{v}(T-t,x)$ where $\hat{v}(t,x)$ be a solution of the linear problem

$$(P_{\hat{\sigma}})\quad \begin{cases} \hat{v}_t - \Delta\hat{v} = 0, & Q_T \\ \hat{v}(t,\xi) = \hat{\sigma}(t,\xi), & \Sigma_T \\ \hat{v}(0,x) = 0, & \Omega \end{cases}$$

with $\hat{\sigma}(t,\xi) \in L^2(\Sigma_T)$, $t>0$, $\xi \in \Gamma$, being as arbitrary function. Using the Green's function $G(t,\tau;x,\xi)$ related to $(P_{\hat{\sigma}})$, we can write

$$(27)\qquad \hat{v}(t,x) = -\int_{\Sigma_t} \frac{\partial G}{\partial n_\xi}(t-\tau;x,\xi)\,\hat{\sigma}(\tau,\xi)d\xi d\tau \quad , \quad x\in\Omega$$

So v(t,x) satisfies the retrogress problem.

$$(P_T)\quad \begin{cases} v_t + \Delta v = 0, & Q_T \\ v(t,\xi) = \sigma(t,\xi), & \Sigma_T \\ v(T,x) = 0, & \Omega \end{cases}$$

where

(28) $\sigma(t,\xi) = \hat{\sigma}(T - t, \xi), \quad \xi \in \Gamma$

From (27) we obtain

$$(29) \quad v(t,x) = - \int_0^{T-t} \int_\Gamma \frac{\partial G}{\partial n_\xi}(T-t-\tau;x,\xi)\hat{\sigma}(\tau,\xi)d\xi d\tau.$$

2nd part. Let $f \in L^2(Q_T)$, $g \in L^2(\Sigma_T)$ and $u_0 \in L^2(\Omega)$, by Brezis [3], then $u_n \in L^2(0,T; H^1(\Omega))$, $f_n \in L^2(0,T;(H^1(\Omega))')$, $g_n \in L^2(\Sigma_T)$ and $u_{0n} \in H^1(\Omega)$ such that

$$(30) \quad \frac{du_n}{dt} \in L^2(Q_T)$$

and u_n is strong solution of $(ESP)_n$:

$$\int_{Q_T} \frac{du_n}{dt}(w-u_n)\,dxdt + \int_{Q_T} \nabla u_n \nabla (w-u_n)dxdt \geq \int_{Q_T} f_n(w-u_n dxdt + \int_{\Sigma_T} g_n(w-u_n)d\xi\, dt$$

for all $w \in W^{1,2}(0,T; H^1(\Omega))$,

$u_n(0) = u_{0n}$.

Taking the limit, we obtain (24).

Proof (Th. 4) We consider three cases

1st case: $u_0(x) \equiv 0$, $x \in \Omega$, $f(t,x) \equiv 0$ on Q_T (i.e. $\tilde{f} = u_0 \equiv 0$).

From Lemma 5,

$$(31) \int_{\Sigma_T} u \frac{\partial v}{\partial n}\, d\xi dt \geq \int_{\Sigma_T} gv d\xi dt \quad , \quad \xi \in \Gamma$$

From (29) with $\hat{\sigma}(\tau,\xi) \geq 0$ a.e. $\xi \in \Gamma_0$, $\tau \geq 0$ and $\hat{\sigma}(\tau,\xi) = 0$ a.e. $\xi \in \Gamma - \Gamma_0$, $\tau \geq 0$ we get $v(t,x) = 0$ on $\{T\} \times \Omega$; $v(t,\xi) = 0$ on $\Sigma_T - \Sigma_0$ and $v(t,\xi) \geq 0$ on Σ_0, and by the maximum principle, $v(t,x)$ attains its minimum on $(\{T\} \times \Omega) \cup (\Sigma_T - \Sigma_0)$ and from this,

$$\frac{\partial v}{\partial n} \leq 0, \quad \Sigma_T - \Sigma_0 \quad ; \quad u\frac{\partial v}{\partial n} \leq 0, \quad \Sigma_T - \Sigma_0 \quad , \quad u\frac{\partial v}{\partial n} = 0, \ \Sigma_0.$$

Therefore

$$0 \leq - \int_{\Sigma_T} g(t,\xi)v(t,\xi)d\xi dt$$

2^{nd} case: $u_o(x) \equiv 0,\ f(t,x) \not\equiv 0.$

From (24)

$$(32) \qquad \int_{\Sigma_T} u \frac{\partial v}{\partial n}\, d\xi dt \geq \int_{Q_T} fvdxdt + \int_{\Sigma_T} gvd\xi dt.$$

Similarly, we obtain

$$(33) \qquad \int_{Q_T} fvdxdt + \int_{\Sigma_T} gvd\xi dt \leq 0.$$

In the first integral in (33 we replace v(t,x) by its form given in (29) and in the second integral in (33) we replace v(t,x) by $\hat{\sigma}(T\text{-}t,\xi) \geq 0$ a.e. $(t,\xi) \in (0,T) \times \Gamma_0$, then, we obtain.

$$(34) \qquad \int_{Q_T} f(t,x) \left[- \int_0^{T-t} \int_\Gamma \frac{\partial G}{\partial n}\,(T-t-\tau;x,\xi)\ \hat{\sigma}(\tau,\xi)\ d\xi d\tau \right] dxdt +$$

$$+ \int_{\Sigma_T} g(t,\xi)\ \hat{\sigma}(T-t,\xi)\ d\xi dt \leq 0.$$

Considereing the Fubini theorem and taking $T\text{-}t = \tau$ in the second integral in (34), we obtain.

$$(35) \qquad - \int_{\Sigma_T} \left[\int_0^{T-t} \int_\Omega f(t,x) \frac{\partial G}{\partial n}(T-t-\tau;x,\xi)\hat{\sigma}(\tau,\xi)dxdt \right] d\xi dt +$$

$$+ \int_{\Sigma_T} g(T-\tau,\xi)\hat{\sigma}(\tau,\xi)d\xi d\tau \leq 0,$$

then

$$- \int_{\Sigma_T} \hat{\sigma}(\tau,\xi)\left[\int_0^{T-t} \int_\Omega f(t,x) \frac{\partial G}{\partial n}(T-t-\tau;x,\xi)dxdt \right] d\xi d\tau +$$

$$+ \int_{\Sigma_T} \hat{\sigma}(\tau,\xi)g(T-\tau,\xi)d\xi d\tau \leq 0,$$

that is

$$-\int_0^t \int_\Omega f(s,x) \frac{\partial G}{\partial n} (t-s;x,\xi)dxds + g(t,\xi) \leq 0$$

and from (20) $-\tilde{f}(t,\xi) + g(t,\xi) \leq 0$ a.e. $(t,\xi) \in \Sigma_o$.

3^{rd} case: $u_o(x) \neq 0, \quad f(t,x) \neq 0$

Let $\tilde{u}(t,x)$ be the solution of

$$(\tilde{P}_o) \begin{cases} \tilde{u}_t - \Delta\tilde{u} = 0 & , Q \\ \tilde{u} = 0 & , \Sigma \\ \tilde{u}(0,x) = u_o & , \Omega \end{cases}$$

that is

$$\tilde{u}(t,x) = \int_\Omega G(t,0;x,\xi)u_o(\xi)d\xi.$$

Let u be the solution of (ESP) then $u^* = u - \tilde{u}$ satisfies

$$(P^*) \begin{cases} u^*_t - \Delta u^* = f, & Q_T \\ -\frac{\partial u^*}{\partial n} + g - \frac{\partial \tilde{u}}{\partial n} \in \beta(u^*), & \Sigma_T \\ u^*(0,x) = 0 \ , \ \Omega \end{cases}$$

Defining $g^*(t,\xi) = g(t,\xi) - \frac{\partial \tilde{u}}{\partial n}(t,\xi)$, $\xi \in \Gamma$ we go back to the 2^{nd} case.

4. A SUFFICIENT CONDITION.

As in the eliptic case (see Díaz-Jiménez |6|) the condition

$$g(t,\xi) - \tilde{f}(t,\xi) - \tilde{u}_o(0,\xi) \leq 0$$

on a part of Σ is not enough to the formation of the coincidence set. Nevertheless, such a condition is "almost sufficient" as the following theorem shows.

Theorem 6. Assume, for simplicity, Ω to be convex and let f, g, and u_o be such that exist $\epsilon > 0$ such that

$$(36) \quad g(t,\xi) - \tilde{f}(t,\xi) - \tilde{u}_o(0,\xi) \leq -\epsilon \quad \text{a.e. } (t,\xi) \in (0,\infty) \times \Gamma_\epsilon = \Sigma_\epsilon,$$

then there exists $R = R(\varepsilon) > 0$ such that.

(37) $$\Sigma_0 \supset \{ (t,\xi) \in [0,\infty) \times \Gamma_\varepsilon : d(\xi, \Gamma - \Gamma_\varepsilon) \geq R\},$$

where

$$R = \begin{cases} \left(\frac{2MN}{\varepsilon(N-1)H}\right)^{1/2} & \text{if } H > 0 \\ \left(\frac{2MN}{\varepsilon}\right)^{1/2} & \text{if } H = 0 \end{cases}$$

and $\|u\|_{L^\infty(Q)} \leq M$, H is the (non negative) mean curvature of Ω.

Remark. The bound $u \in L^\infty(Q_T)$ can be obtained under the assumptions $f \in L^\infty(Q_T)$, $u_0 \in L^\infty(Q_T)$ such that

$$g^*(t,\xi) = g(t,\xi) - \tilde{f}(t,\xi), \quad \xi \in \Gamma, \quad t \geq 0$$

be the trace on Σ_T of a function

$G^*(t,x) \in BV(0,T;L^\infty(\Omega)) \cap L^2(0,T;H^1(\Omega))$ for all $T > 0$. In this case (cf Damlamian |4|, Díaz |5|, Brezis |3|) we get

$$u(t,x) \in C([0,T]; H^{-1}(\Omega)) \cap L^\infty(Q_T).$$

Moreover, for u, solution of $u_t - \Delta u = 0$, Q_T;

$-\frac{\partial u}{\partial n} \in \beta(u)$, Σ_T; $u(0,x) = u_0$, we have

$$\|u(t,.)\|_{L^\infty(\Omega)} \leq \frac{C}{t^{N/2}} \|u_0\|_{L^1(\Omega)} \quad \text{for all } 0 < t \leq T. \#$$

Proof (Th. 6) Let $x_0 \in \Gamma_\varepsilon$ be such that $d(x_0, \Gamma-\Gamma_\varepsilon) = R$ and $D = \Omega \cap B(x_0,R)$, $\partial_1 D = \partial D \cap \Gamma$, $\partial_2 D = \partial D - \Gamma$. We consider $D_\infty = (0,\infty) \times D$ and let $\partial_1 D_\infty = (0,\infty) \times \partial_1\Gamma$ and $\partial_2 D_\infty = (0,\infty) \times \partial_2\Gamma$. By Lemma 2, it suffices to exhibit the estimate (36) for u, u being the solution of $(\hat{P})$. Following the same way of the proof of the Theorem 5 on Diaz-Jimenez |6|, let $U(x) \in H^2(\Omega)$ such that $U \geq 0$ in D, $-\Delta U \geq C$ on D, $U = 0$ on $\partial_1 D$ and

$\frac{\partial U}{\partial n} = -\varepsilon$ on $\partial_1 D$. We consider the auxiliar function

(38) $$\bar{u}(t,x) = U(x) + \frac{C}{2N}|x-x_0|^2$$

where $C = (N\text{-}1)H\varepsilon$ if $H > 0$ and $C = \frac{\varepsilon}{R}$ if $H = 0$.

We have

(39) $$\bar{u}_t - \Delta\bar{u} = -\Delta\bar{u} = -\Delta U - C \geq 0 \text{ in } D_\infty,$$

(40) $$\overline{U}(t,x)\Big|_{\partial_2 D_\infty} \geq \frac{C}{2N} R^2 \geq M \geq u(t,x)\Big|_{\partial_2 D_\infty} \quad \text{iff} \quad R \geq \left(\frac{2MN}{C}\right)^{1/2}$$

(41) $$-\frac{\partial \overline{u}}{\partial n}\Big|_{\partial_1 D_\infty} \leq \varepsilon;$$

this is because we also have

$$-\frac{\partial \overline{u}}{\partial n} = -\frac{\partial U}{\partial n} - \frac{C}{N}|x - x_\circ| \cos(n(\xi), \xi - x_\circ) \leq \varepsilon.$$

Finally, in D we have

$$\overline{u}(0,x) \geq 0 = u_\circ(x) = \tilde{u}_\circ(x)$$

By Theorem 3, $u(t,x) \leq \overline{u}(t,x)$ in $\overline{D}_\infty$; in particular $0 \leq u(t,x) \leq \frac{C}{N}|\xi - x_\circ|^2$, $\xi \in \Gamma \cap B(x_\circ,R)$, $t \geq 0$#.

A stronger result is possible: the coincidence set may occur only after a finite time. More precisely we have the following

Theorem 7. Assume Ω be convex and let g and f such that there exists $\varepsilon > 0$, $\Gamma_\varepsilon \subset \partial\Omega$ and $t_\varepsilon > 0$ such that

(42) $$g(t,\xi) - \frac{\partial F}{\partial n}(t,\xi) < -\varepsilon \quad \text{on } [t_\varepsilon,\infty) \times \Gamma_\varepsilon,$$

where $F(t,x)$ satisfies $F_t - \Delta F = f$ on Q_T, $F \equiv 0$ on Σ_T and $F(0,.)=0$ on Ω. Then, for any given initial data u_0, the solution u of (P*) satisfies the condition that there exists a finite time $T_0 \geq t_\varepsilon$ such that

(43) $$\Sigma_\circ \supset \{(t,\xi) \in [T_\circ,\infty) \times \Gamma_\varepsilon : d(\xi, \Gamma - \Gamma_\varepsilon) \geq R\}$$

for some $R = R(\varepsilon)$. In particular if $\Gamma_\varepsilon = \Gamma$, then $u \equiv 0$ on $(T_\circ, \infty) \times \Gamma$ and u becomes the solution of the homogeneous Dirichlet problem after the time $T_\circ$.

Proof. Let $u(t,x)$ be the solution of (P*) with $f \equiv 0$ and $g^* \equiv \tilde{g}(t,\xi) = g(t,\xi) - \tilde{f}(t,\xi)$ and let U the function of the proof of the Theorem 6.

We consider now the local-supersolution

(44) $$\overline{w}(t,x) = \frac{C}{2N}|x - x_\circ|^2 + k(T_\circ - t)^+ + U(x), \quad k > 0.$$

We have

$$\overline{w}_t - \Delta\overline{w} = -k\mathbb{1}_{\{t_\circ < t < T_\circ\}} - \Delta U - C \geq 0$$

if $-\Delta U \geq C + k = \overline{C}$ hold, where

$$\overline{C} = \begin{cases} (N-1)H\varepsilon & \text{if } H > 0 \\ \frac{\varepsilon}{R} & \text{if } H = 0. \end{cases}$$

Moreover

$$(45)\qquad -\frac{\partial \overline{W}}{\partial n} \leq -\frac{C}{N}|x-x_o|\cos(n(\xi), \xi-x_o) - \frac{\partial U}{\partial n} \leq -\varepsilon$$

on $\partial_1 D \times (t_o, \infty)$ since $\left.\frac{\partial U}{\partial n}\right|_{\partial_1 D} \geq \varepsilon$ and

$$(46)\qquad \overline{W}\Big|_{\partial_2 D \times (t_o,\infty)} \geq \frac{C}{2N} R^2 \geq M \geq u(t_o,\xi)\Big|_{\partial_2 D \times (t_o,\infty)}$$

(remember, U (ξ) = 0 on $\partial_2 D$). Finally.

$\overline{W}(t_o,x) \geq k(T_o-t_o) \geq M \geq u(t_o,x)$, $x \in D$, if $T_o \geq \frac{M}{k} + t_o$. Thaking suitable constants k, C, To we obtain the result. #

Remark. There is no dificulty in proving the following (cf. Damlamian |4|) if, by example, $g \equiv 0$ and f, u_o be such that $u_o|_\Gamma \geq 0$, $\Delta u_o \leq 0$ on Ω and $f(t) \leq 0$ a.e. $t > 0$, then $u_t \leq 0$. In particular, under these conditions

$$I_o(t_1) \subset I_o(t_2) \text{ if } t_1 \leq t_2, \text{ where}$$

$$I_o(t_o) = \{\xi \in \Gamma : u(t_o,\xi) = 0\}_\#$$

We conclude this paper by exhibiting an application of the above theorems to the study of the sign of the trace $u(t,.)|_\Sigma$ of solutions of the initial problem (1), when f, g and u_o have no constant sign on their respective domains. So, assume, for instance, that $f \equiv 0$, $u_o \equiv 0$ but $g(t,\xi)$ changes of sign on Σ. Even for the linear problem, $b(u) = u$ or $b(u) \equiv 0$ it is not an easy task to find regions of Σ where u is non positive (o non negative) when only the region of Σ where g is non positive (or non negative) is known.

Nevertheles, by a variant of the comparison principle is not difficult to show that $u_b \leq u_\beta$ (see e.g. Brezis |3|) where u_b represents the solution of (1) and u_β satisfies (ESP) with $f = u_o \equiv 0$, for β given by (3). Then, from Theorem 2 we deduce that if there exists $\varepsilon > 0$ such that $g(t,\xi) < -\varepsilon$ on a part Γ_ε of Γ and any $t > 0$, then the trace $u_b(t,.)$ on Γ is such that $u_b(t,.) \leq 0$ at least on the set

$$\{(t,\xi) \in [0,\infty) \times \Gamma_\varepsilon : d(\xi, \Gamma-\Gamma_\varepsilon) \geq R\}$$

for some $R = T(\varepsilon)$. The study of the regions of Σ where $u_b(t,.)$ is non negative may be similarly estimated by studying previously the problem (ESP) for β given by $\tilde{\beta}(r) = -\beta(-r)$ with β defined in (3)#

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