In Proceedings of the VII CEDYA, Univ. de Santander (1985)

Behaviour on the boundary of solutions of parabolic equations with nonlinear boundary conditions: the parabolic Signorini problem.

J.I.Diaz[(1)(*)] and R.F.Jimenez[(2)]

(1) Facultad de Matemáticas, Universidad Complutense de Madrid (SPAIN)

(2) Depto. de Matemáticas, Universidad de Concepción (CHILE).

Let Ω be an open regular set of $\mathbb{R}^N$ and consider the following nonlinear parabolic boundary value problem

$$(1)\quad \begin{cases} u_t - \Delta u = f(t,x) & \text{in} \quad \Omega \times (0,\infty) = Q \\ u(0,\cdot) = u_0 & \text{on} \quad \Omega \\ \dfrac{\partial u}{\partial n} + b(u) = g(t,x) & \text{on} \quad \partial\Omega \times (0,\infty) = \Sigma , \end{cases}$$

where n is the unit outward normal to $\partial\Omega$ and b is a continuous nondecreasing function such that $b(0) = 0$. Many different results are well-known on the existence and uniqueness of solutions u under several kind of regularity assumptions on f, u_0 and g (see, for instance , Friedman |8| Amann |2| and Alikakos |1|). It is also well-known that if $f \geq 0$, $u_0 \geq 0$ and $g \geq 0$ then $u \geq 0$ in $\overline{Q}$. In fact, by the strong maximum principle, if u is nonnegative and $u(t,\cdot) \not\equiv 0$ for every $t > 0$, then $u > 0$ on Q. A natural question arise: For which functions b, the behaviour of u near Σ becomes pathological, in the sense that $u(t,\cdot)|\Sigma$ vanishes on some subregion of Σ?. A first answer is easy because, again, by the maximum principle if $u(t,x) = 0$ on $\Gamma_0 x(t_0,\infty)$ then we arrive to the contradiction $0 > \frac{\partial u}{\partial n}(t,x) = g(t,x) - b(0) \geq 0$, on $\Gamma_0 x(t_0,\infty)$ So,this behaviour is excluded for any nondecreasing continuous function b such that $b(0) = 0$ and for which $u(t,\cdot) \neq 0$ for any $t > 0$.

The situation changes strongly if we take b in the class of the multivalued maximal monotone graphs of $\mathbb{R}^2$ (that we shall denote by β). That general formulation is of interest in the applications (see Duvaut-Lions | 7 |) and now the problem is

(*) Partially sporsored by the Proyect 3308/83 of the CAICYT (SPAIN) and the Joint Action 19/85 France/Spain.

$$P(f,u_0,g)\quad\begin{cases} u_t - \Delta u = f(t,x) & \text{in} \quad Q \\ u(0,\cdot) = u_0 & \text{on} \quad \Omega \\ -\dfrac{\partial u}{\partial n} + g(t,x) \in \beta(u) & \text{on} \quad \Sigma . \end{cases}$$

In fact, we shall pay our attention on the particular case of β given by

(2) $\beta(r) = 0$ if $r > 0$, $\beta(0) = (-\infty,0]$ and $\beta(r) = \phi$ (the empty set).

That problem is associated to the name of Signorini parabolic problem. Note that in this case the boundary conditions are of "unilateral type"

(3) $u \geq 0$, $-\dfrac{\partial u}{\partial n} + g \leq 0$ and $u(-\dfrac{\partial u}{\partial n} + g) = 0$, on Σ ,

and that the <u>coincidence set</u>, defined by $\Sigma_0 = \{(t,x) \in \Sigma : u(t,x) = 0\}$, plays an important role in the understanding of the problem. Our question now is the study of the assumptions on f , u_0 and g that allow the formation of the coincidence set Σ_0 . A first homogeneization Lemma show us that we shall may assume, without loss of generality, that $f \equiv 0$ and $u_0 \equiv 0$.

<u>Lemma</u>. *Let* u *solution of* $P(f,u_0,g)$ *and* v *satisfying* $v_t - \Delta v = f$ *in* Q, $v(0,\cdot) = u_0$ *on* Ω , *and* $v = 0$ *in* Σ . *Then the function* $U = u - v$ *satisfies* $P(0,0,\tilde{g})$, *where*

(4) $\tilde{g} \equiv g - \dfrac{\partial U}{\partial n}$.

After this trivial remark, it seems natural that the existence or non-existence of the coincidence set Σ_0 will depends on the behaviour of this function $\tilde{g}$. We start with a necessary condition:

<u>Theorem 1.</u> *Let* u *be a (weak) solution of* $P(f,u_0,g)$ *for* β *given by* (2). *Assume that* $u(t,\xi) = 0$ *a.e. on* $\Sigma_0 = (0,\infty) \times \Gamma_0$, *where* Γ_0 *is a "smooth" part of* $\partial\Omega$. *Then necessarily* $\tilde{g}(t,\xi) \leq 0$ *a.e. on* Σ_0 .

<u>Idea of the proof</u>. Let $T > 0$ fixed. We first note that

(5) $-\displaystyle\int_\Omega u_0(x)v(0,x)dx + \int_0^T\int_{\partial\Omega} u \frac{\partial u}{\partial n}\, d\xi dt \geq \int_0^T\int_\Omega fv dx dt + \int_0^T\int_{\partial\Omega} gv d\xi dt$

for any $v \in W^{1,2}(0,T:H^1(\Omega))$ such that $v_t + \Delta v = 0$ in $Q_T = (0,T) \times \Omega$, $v \geq - u$ on $\partial\Omega \times (0,T)$ and $v(T,x) = 0$ on Ω. Indded, by regularizing the data f, u_0 and g, we may assume that u is a strong solution i.e. such that $u_t \in L^2$ and satisfies

$$\int_{Q_T} u_t(w-u)dxdt + \int_{Q_T} \nabla u \cdot \nabla(w-u)dxdt \geq \int_{Q_T} f(w-u)dxdt + \\ + \int_0^T \int_{\partial\Omega} g(w - u)d\xi dt$$

for every $w \in L^2(0,T : H^1(\Omega))$ with $w \geq 0$ in $\partial\Omega \times (0,T)$. Taking $w = v + u$ we obtain (5). But from the Lemma we may assume $f \equiv 0$, $u_0 \equiv 0$ and $g = \tilde{g}$ without loss of generality (note that $U = u$ on $\partial\Omega \times (0,\infty)$). Then we have that

$$\int_0^T u \frac{\partial v}{\partial n} d\xi dt \geq \int_0^T \int_{\partial\Omega} \tilde{g} v \, d\xi dt$$

for any v satisfying $v_t + \Delta v = 0$ in Q_T, $v \geq - u$ on Σ and $v(T,x) = 0$ on Ω. In particular, if we take $v(t,\xi) = \sigma(t,\xi)$ on Σ with $\sigma \geq 0$ arbitray such that $\sigma(t,\xi) = 0$ on $\Sigma - \Sigma_0$, by the strong maximum prin_ ciple we have that $\frac{\partial v}{\partial n} < 0$ on $\Sigma - \Sigma_0$ and so

$$\int_0^T \int_{\Gamma_0} \tilde{g}(t,\xi) \, \sigma(t,\xi) \, d\xi dt \leq 0$$

which gives the conclusion.#

As in the elliptic case (see Diaz-Jimenez |5|) the condition $\tilde{g}(t,\xi) \leq 0$ on a part of Σ is not enough for the formation of the coincidence set. Nevertheless, such a condition is almost sufficient as the following Theorem shows

<u>Theorem 2.</u> *Assume, for simplicity,* Ω *be convex and let* f, u_0 *and* g *such that there exists* $\varepsilon > 0$ *such that*

$$\tilde{g}(t,\xi) < - \varepsilon \qquad \textit{on a given part } \Gamma_\varepsilon \textit{ of } \partial\Omega \textit{ and for any } t > 0.$$

Then there exists $R = R(\varepsilon) > 0$ *such that*

$$\Sigma_0 \supset \{(t,\xi) \in [0,\infty) \times \Gamma_\varepsilon : d(\xi, \partial\Omega - \Gamma_\varepsilon) \geq R\},$$

i.e. u *vanishes, at least, on the part of the boundary* Σ *given by the right hand side set of the above expression.*

A stronger result is possible: the coincidence set may occur only after a finite time. More precisely we have

Theorem 3. *Assume* Ω *convex and let* g *and* f *such that there exists* $\varepsilon > 0$, Γ_ε $\partial\Omega$ *and* $t_\varepsilon > 0$ *such that*

$$g(t,\xi) - \frac{\partial F}{\partial n}(t,\zeta) < -\varepsilon \text{ on } [t_\varepsilon,\infty) \times \Gamma_\varepsilon ,$$

where $F(t,x)$ *satisfies* $F_t - \Delta F = f$ *in* Q_T, $F \equiv 0$ *on* $\partial\Omega \times (0,\infty)$ *and* $F(0,\cdot) = 0$ *on* Ω . *Then, for any given initial data* u_0, *the solution* u *of* $P(f,u_0,g)$ *satisfies that there exist a finite time* $T_0 \geq t_\varepsilon$ *such that*

$$\Sigma_0 \supset \{(t,\xi) \in [T_0,\infty) \times \Gamma_\varepsilon : d(\xi, \partial\Omega - \Gamma_\varepsilon) \geq R\}$$

for some $R = R(\varepsilon)$. *In particular if* $\Gamma_\varepsilon = \partial\Omega$ *then* $u \equiv 0$ *on* $(T_0,\infty) \times \partial\Omega$ *and* u *becomes the solution of the homogeneous Dirichlet problem after the time* T_0.

The proof of the above results is obtained through the construction of suitable local-supersolutions inspired in the elliptic case (Diaz-Jimenez |5|). For details, see Diaz-Jimenez |6| where the constants $R(\varepsilon)$ and T_0 are estimated.

We shall end this communication by giving an application of the above theorems to the study of the sign of the trace $u(t,\cdot)|_\Sigma$ of solutions of the initial problem (1) when f,g and u_0 don't have constant sign on their respective domains. So, assume, for instance, that $f \equiv 0$, $u_0 \equiv 0$ but $g(t,\xi)$ changes of sign on Σ . Even for the linear problem, $b(u) = u$, or $b(u) \equiv 0$ it is not an easy task to find regions of Σ where u is nonpositive (or nonnegative) when only the region of Σ where g is nonpositive (or nonnegative) is known. Nevertheless, by a variant of the comparison principle is not difficult to show that $u_b \leq u_\beta$ (see e.g.

Brezis |4|), where u_b represents the solution of (1) and u_β satisfies $P(0,0,g)$ for β given by (2). In consequence, from Theorem 2 we deduce that if there exists $\varepsilon > 0$ such that $g(t,x) < -\varepsilon$ on a part Γ_ε of $\partial\Omega$ and any $t > 0$, then the trace $u_b(t,\cdot)$ on $\partial\Omega$ is such that $u_b(t,\cdot) \leq 0$ at least on the set $\{(t,\xi) \in [0,\infty) \times \Gamma_\varepsilon : d(\xi,\partial\Omega\cdot\Gamma_\varepsilon) \geq R\}$ for some $R = R(\varepsilon)$. The study of the regions of Σ where $u_b(t,\cdot)$ is nonnegative may be similarly estimated by studying previously the problem $P(f,u_0,g)$ for β given by $\tilde{\beta}(r) = -\beta(-r)$ with β defined in (2).

References.

|1| N.D.Alikakos : Regularity and Asymptotic Behaviour for the Second Order Parabolic Equation with Nonlinear Boundary Conditions. Journ. of Diff. Eqs. 39 (1981).

|2| H.Amann : Invariant Sets and Existence Theorems for Semilinear Parabolic and Elliptic Systems. Journ. Math. Analy. and Applic. 65 (1978) , 432-465.

|3| I,Athanasopoulus : Regularity of the solution of an evoulition problem with inequalities on the boundary, Comm. in PDE 7(12), (1982) , 1453-1465.

|4| H.Brezis : Problemes unilateraux. J.Math. Pures Appl. 51 (1972), 1-168.

|5| J.I.Diaz and R.F.Jimenez : Comportamiento en el contorno de la solución del problema de Signorini, Actas V CEDYA, Univ. de Zaragoza, (1984), 308-314.

|6| J.I.Diaz and R.F.Jimenez : Boundary behaviour of solutions of elliptic and parabolic problems with inequalities on the boundary (to appear).

|7| G.Duvaut and J.L.Lions : Les Inequalitions en Mecanique et en Physique . Dunos, Paris (1972).

|8| A,Friedman : Partial Differential Equations of Parabolic Type. Prentice Hall, (1964).