In Proceedings of the VII CEDYA, Univ. de Santander (1985)

Uniqueness of nonnegative solutions for elliptic nonlinear diffusion equations with a general perturbation term.

J.I.Diaz(*) and J.E.Saa
Facultad de Matemáticas
Universidad Complutense de Madrid, 28040 Madrid.

The main goal of this paper is to present some uniqueness results for the solutions of some second order quasilinear elliptic equations with a possible source term, such as, for instance, the following problem:

(1) $$-\Delta_p u + f(x,u) = 0 \quad \text{in} \quad \Omega\ ,\ u=0 \quad \text{on} \quad \partial\Omega\ ,$$

where Ω is a bounded domain in $\mathbb{R}^N$ and

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) \qquad 1 < p < \infty \qquad (\text{note that } \Delta_2 u = \Delta u).$$

This kind of problems appears in many different applications (see Diaz |4|). If $p=2$, (1) is a semilinear problem and, generally, there is not uniqueness of solutions if $f(x,u)$ is non increasing in u (remenber the eigenvalue problem). However, in the semilinear case it is well-known (Keller-Cohen |7| Cohen-Laetsch |3| , Amann |1|, Lions |8| , Brezis-Oswald |2| etc) that the assumption $f(x,u)/u$ strictly increasing in u is sufficient for uniqueness. If $p \neq 2$ the problem (1) becomes quasilinear and, as far as we know, the only reference in the literature is Otani |10| where the uniqueness is shown in the presence of a source term in the equation. Otani's paper has two important limitations: it is valid only if $N=1$ and the perturbations are necessarily powers functions $f(x,u) = -u^q$ (he uses in an essential way the homogeneity of $f(x,u)$).

In this communication we present a preliminar version of the results of Diaz-Saa |5| where we generalize for quasilinear problems the known results for the semilinear case. Our main result assure the uniqueness of nonnegative solutions if the perturbation $f(x,u)$ satisfies that "$f(x,u)/u^{p-1}$ is strictly increasing in u".

(*) Partially sponsored by the proyect n°3308/83 of the CAICYT and the Joint Action 19/85 France-Spain.

(this assumption is a generalization of the one used for $p=2$ and obviously it include non increasing perturbations as, for instance, $f(x,u) = -u^q$, $0 < q < p-1$)

Theorem 1. *Assume the function* $f(x,u)$ *be such that:*

(2) $x \to f(x,u)$ *belongs to* $L^\infty(\Omega)$,

(3) $u \to f(x,u)$ *is continuous in* $[0,\infty)$ *and* $u \to f(x,u)/u^{p-1}$ *is strictly increasing in* $(0,\infty)$.

Then the problem (1) *has a unique nonnegative solution.*

A similar statement was proved for $p=2$ in the paper Brezis-Oswald |2|. In fact, our proof has its origins in a remark rised in that paper. However, the proof given there (always for semilinear case) follows a different way (see Remark 1).

The main idea of our proof comes from monotone operator theory. Indeed; it is well-known (see e.g. Diaz |4|.) that the uniqueness of solutions for nondecreasing terms $f(\cdot,u)$ is easily deduced from the fact that the realization in $L^2(\Omega)$ of $-\Delta_p u$ is a monotone operator. In a semilar way, if we could prove that $-\Delta_p u/u^{p-1}$ is monotone in $L^2(\Omega)$ then the result would follows from hypothesis (3). As in the standard case, one way to check the monotonicity is by showing the convexity of a suitable l.s.c. functional. It turns out that in our case, such a functional is given by

$$J(\rho) = \frac{1}{p}\int_\Omega |\Delta\rho^{1/p}|^p\, dx.$$

However some difficulties appear in contrast with the usual case: the convexity of $J(\rho)$ is not trivial because there is a concave expression in J, and,on the other hand, J is not Gateaux-differentiable in some directions but, at least in some of them valuables for our porpouses. Before to give the proof of Theorem 1 we proof several auxiliary lemmas:

Lemma 1. *The functional* J, $J : L^1(\Omega) \to (-\infty,+\infty]$ *defined by*

$$(4)\qquad J(\rho) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla\rho^{1/p}|^p\, dx & \textit{if } \rho \geq 0 \textit{ and } \rho^{1/p} \in W_0^{1,p}(\Omega) \\ +\infty & \textit{otherwise,} \end{cases}$$

is proper, convex and lower semicontinuous.

Proof: We know that the domain of J is $D(J)=\{\rho \geq 0 \ / \ \rho^{1/p} \in W_0^{1,p}(\Omega)\}$. Then, J is proper because $D(J) \neq \emptyset$ (for instance, if $\Omega = (0,1) \subset \mathbb{R}$, $\rho(x) = x^{\lambda}(1-x)^{N}$ $\rho(x) \in D(J)$ if $\lambda > p-1$ and $N > p-1$). Now we prove the convexity of J. First of all, assume that $\phi_1, \phi_2 \in D(J)$ and let $\psi_1 = \phi_1^{1/p}$ $\psi_2 = \phi_2^{1/p}$, $\psi_3 = (t\phi_1 + (1-t)\phi_2)^{1/p}$ with $t \in [0,1]$. Our goal is to prove that

(5) $$|\nabla\psi_3|^p \leq t\,|\nabla\psi_1|^p + (1-t)|\nabla\psi_2|^p.$$

We can easily see that

$$\nabla\psi_i = \frac{1}{p}\ \psi_i^{1-p}\ \nabla\phi_i \quad \text{for} \quad i = 1,2 \quad \text{and} \quad \nabla\psi_3 = \frac{1}{p}\ \psi_3^{1-p}(t\nabla\phi_1 + (1-t)\nabla\phi_2) \text{ and}$$

$$\psi_3^{p-1}\nabla\psi_3 = (t^{(p-1)/p}\psi_1^{p-1})\ (t^{1/p}\nabla\psi_1) + ((1-t)^{(p-1)/p}\ \psi_2^{p-1})((1-t)^{1/p}\nabla\psi_2).$$

Applying Holder's inequality

$$\psi_3^{p-1}|\nabla\psi_3| \leq (t\ \psi_1^p + (1-t)\ \psi_2^p)^{(p-1)/p}(t|\nabla\psi_1|^p + (1-t)|\nabla\psi_2|^p)^{1/p};$$

and then (5) is proved, and as $\psi_3 \in W_0^{1,p}(\Omega)$ we get

$$J(t\,\phi_1 + (1-t)\phi_2) \leq t\ J(\phi_1) + (1-t)J(\phi_2).$$

Finally, we have to prove J is lower semicontinuous in $L^1(\Omega)$, so we are going to prove that if $\phi_n \to \phi_n$ in $L^1(\Omega)$ and $J(\phi_n) \leq \lambda$ then $J(\rho) \leq \lambda$. As $\rho_n^{1/p}$ is bounded in $W^{1,p}(\Omega)$ there is a subsequence of $\rho_n^{1/p}$, that we still will call $\rho_n^{1/p}$, such that $\rho_n^{1/p}$ converges weakly in $W^{1,p}(\Omega)$. Then $\nabla\rho_n^{1/p}$ converges weakly to $\nabla\rho^{1/p}$ in $L^p(\Omega)$ and since the norm is lower semicontinuous we obtain $\liminf_n J(\rho_n) \geq J(\rho)$, and hence $\lambda \geq J(\rho)$.

Now we shall study the Gateaux-differential of J at the point ρ and direction ξ:

$$J'(\rho,\xi) = \lim_{t\to 0} \frac{J(\rho+t\xi)-J(\rho)}{t}.$$

Lemma 2. *Let* J *be the functional defined by* (4) *and let* $\rho_i : \Omega \to R$, *with* $i=1,2$, *be nonnegatives functions such that*:

(6) $$\rho_i \in L^\infty(\Omega), \quad \rho_i^{1/p} \in W_0^{1,p}(\Omega), \quad \Delta_p\,\rho_i^{1/p} \in L^\infty(\Omega),$$

(7) $\rho_i/\rho_j \in L^\infty(\Omega)$ *and* $\dfrac{\rho_i}{\rho_j^{(p-1)/p}} \in W_0^{1,p}(\Omega)$ if $i \neq j$.

Then if $\xi = \rho_1 - \rho_2$ *we have*

(8) $$J'(\rho_i,\xi) = \int_\Omega \frac{-\Delta_p \rho_i^{1/p}}{p\,\rho_i^{(p-1)/p}}\,\xi \quad \textit{for} \quad i = 1 \textit{ and } 2.$$

Proof. For the sake of the notation, let us denote by ρ to ρ_i for $i = 1$ and 2. Define

$$\phi(t) = \int_\Omega \frac{|\nabla(\rho+t\xi)^{1/p}|^p}{p}\,dx,$$

then $J'(\rho_i\xi)$ is the right derivative of ϕ at 0. Using assumptions (6) and (7) we have

$$J'(\rho_i\xi) = \int_\Omega \frac{1-p}{p}|\nabla\rho^{1/p}|^p \xi/\rho + \int_\Omega |\nabla\rho^{1/p}|^{p-2}\nabla\rho^{1/p}\nabla\xi^{1/p}(\xi/\rho)^{\frac{p-1}{p}} =$$
$$\int_\Omega |\nabla\rho^{1/p}|^{p-2}\ \nabla\rho^{1/p}\ (\nabla\xi^{1/p}(\xi/\rho)^{\frac{p-1}{p}} + \frac{1-p}{p}\nabla\rho^{1/p}(\xi/\rho))$$

Due to the regularity on ρ_i we may apply Green's equality and get

$$J'(\rho_i,\xi) = \frac{1}{p}\int_\Omega -\Delta_p\,\rho^{1/p}\,\frac{\xi}{\rho^{\frac{p-1}{p}}}\,dx$$

which proves the result.

Lemma 3. *Let* u *and* v *be two nonnegative solutions of* (1) *with* f *satisfying* (2) *and* (3). *Then there is an* $\varepsilon > 0$ *such that* $u(x) \geq \varepsilon v(x)$ $\forall x \in \Omega$. (the proof of this lemma is not difficult and uses the fact that the solutions of (1) are C^1, as well as the maximum principle: see Diaz-Saa |5| for details).

Proof of theorem 1. Suppose there are two nonnegative solutions u_1 and u_2 of problem (1). The assumption (2) and the equations allow us to write

(9) $$\int_\Omega \left(\frac{-\Delta_p u_1}{u_1^{p-1}} + \frac{\Delta_p u_2}{u_2^{p-1}}\right)(u_1^p - u_2^p) = \int_\Omega \left(\frac{-f(x,u_1)}{u_1} + \frac{f(x,u_2)}{u_2}\right)(u_1^p - u_2^p).$$

From assumption (3) the left term is not positive. Then if $\xi = u_1^p - u_2^p$ and $\rho_i = u_i^p$, and if we are in conditions to apply lemma 2 we would get

$$\int_\Omega \left(\frac{-\Delta_p u}{u_1^{p-1}} + \frac{\Delta_p u_2}{u_2^{p-1}} \right)(u_1^p - u_2^p) \geq p(J'(\rho_1;\rho_1-\rho_2)-J'(\rho_2;\rho_1-\rho_2)).$$

But this term would be not negative because J is convex. So

$$\int_\Omega \left(\frac{-f(x,u_1)}{u_1} + \frac{f(x,u_2)}{u_2}\right)(u_1^p - u_2^p) = 0$$

and we would obtain that $u_1 = u_2$ in Ω from assumption (3) . Finally in order to show that (9) has a sense and that the conditions of lemma 2 are fullyfuled it is enough to use the fact that $u_1/u_2 \in L^\infty(\Omega)$ and $u_2/u_1 \in L^\infty(\Omega)$, which is a consequence of lemma 3.

Remark 1.

Under the special assumption "$f(x,u)/u$ strictly in u" instead of assumption (3), we get an easier proof of the uniqueness. Indeed, in this case the monotonicity of $\Delta p\, u/u$ can be shown by elementary algebra (note that if $p \neq 2$ this assumption is different from (3)). The proof of this fact is a generalization of the proof for $p = 2$ made in Brezis-Oswald |2|. Suppose there two nonnegative solutions u_1 and u_2 of (1), we know that

$$\int_\Omega \left(\frac{-\Delta_p u_1}{u_1} + \frac{\Delta_p u_2}{u_2}\right)(u_1^2 - u_2^2) = \int_\Omega \left(\frac{f(x,u_2)}{u_2} - \frac{f(x,u_1)}{u_1}\right)(u_1^2 - u_2^2)$$

Applying Green's equality

$$\int_\Omega \Big(|\nabla u_1|^{p-2}(\nabla u_2 - \frac{u_2}{u_1}\nabla u_1)^2 + |\nabla u_2|^{p-2}(\nabla u_1 - \frac{u_1}{u_2}\nabla u_2)^2 + |\nabla u_1|^p - |\nabla u_1|^{p-2}|\nabla u_2|^2 - |\nabla u_2|^{p-2}|\nabla u_1|^2 +$$
$$+|\nabla u_2|^p\Big) = \int_\Omega \left(\frac{f(x,u_2)}{u_2} - \frac{f(x,u_1)}{u_1}\right)(u_1^2 - u_2^2)$$

As $f(x,y) = x^p - x^{p-2}y - y^{p-2}x + y^p$ is positive in $x > 0, y > 0$, we obtain that the first term of the equality is positive and since $f(x,u)/u$ is strictly increasing the second term is negative. In consequence $u_1 = u_2$.

Using other kind of ideas, Theorem 1 is generalized in Diaz-Saa |5| to two more general contexts:

(a) case of perturbation terms satisfying that "there is $\alpha \in [0,p-1)$ such that $f(x,u)/u^\alpha$ is strictly increasing"(this assumption involve new functions even in the semilineal case, for instance, $f(x,u) = u^q \ln u$, $q \in [0,p-1)$).

(b) Case of nonlinear differential operators not necessarily in divergence form.

The proofs are by means of a transformation of the equation in another one given in terms of an accretive operator in the space $X = L^\infty(\Omega)$ and an increasing perturbation.

Bibliography

|1| H.Amann. On the existence of positive solutions of nonlinear elliptic boundary problems, Indiana Univ. Math. J 21 , 125-146, (1971)

|2| H.Brezis and L.Oswald. Remarks on sublinear elliptic equations (to appear)

|3| D.Cohen and T.Laetsch. Nonlinear boundary value problems suggest by chemical reactor theory, J.Diff. Eq. 7 , 217-226 , (1970)

|4| J,I.Diaz. Nonlinear partial differential equations and free boundaries. Vol. I Elliptic Equations. Research Notes 106. Pitman (London),(1985)

|5| J.I.Diaz and J.E.Saa. Uniqueness of nonnegative solutions for second order quasilinear equations with a possible source term (to appear)

|6| A.Friedman and D.Phillips. The free boundary of a semilinear elliptic equation , Amer.Math. Society 282 , 153-163,(1984).

|7| H.Keller and D.Cohen. Some positne problems suggested by nonlinear heat generation, J.Math. Mech. 16, 1361-1376, (1967).

|8| P.L.Lions. On the existence of positive solutions of semilinear elliptic equations, SIAM Review 24, 441-467, (1982).

|9| P.Hess. On uniqueness of positive solutions of nonlinear elliptic boundary value problems, Math. 154, 17-18, (1977).

|10| M.Otani. Sur certaines equations elliptiques differentielles ordinaires du second order associeés aux inegalites du type Sobolev-Poincaré. D.R.A.S. Paris 296 (1983), 415-418.