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Elliptic and Parabolic Quasilinear Equations Giving Rise to a Free Boundary: The Boundary of the Support of the Solutions

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Abstract. In this survey we review some results and methods concerning the study of the support of the solutions of elliptic and parabolic quasilinear equations. In many of these equations and in contrast with the linear case, the support of the solutions does not coincide with the whole domain and thus a free boundary is generated by the boundary of that support.

1. Introduction. Let us consider the nonlinear Dirichlet problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)+|u|^{q-1}u=f \quad \text{on } \Omega, \tag{1}$$

$$u=g \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is an open set of $\mathbf{R}^N$, $N \geqslant 1$, $p>1$, $q>0$, and f, g are given functions. Equation (1) appears in many different contexts: When $p=2$ equation (1) coincides with the semilinear equation

$$-\Delta u+|u|^{q-1}u=f, \tag{3}$$

largely studied in the literature. For instance, it is well known that (3) arises in the study of a single, irreversible, isothermic reaction (see [3]). The parameter q is called the order of the reaction and its range of values determines the behaviour of the solutions. When $p \neq 1$, (1) is a quasilinear equation which becomes degenerated for $p>2$. In that case the equation is not uniformly elliptic, losing its elliptic character on the set $\{x \in \Omega\colon \nabla u(x)=0\}$. This type of equation appears, for instance, in the study of non-Newtonian fluids with a rheological

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power law. When $p > 2$ the fluids are called dilatants and for $1 < p < 2$ pseudoplastics. The case $p = 2$ corresponds to Newtonian fluids (see exact references in [37]).

Existence, uniqueness and regularity results for the Dirichlet problem (1), (2) are already well known after the important works of Ladyzhenskaya–Ural'tseva, Stampacchia, Serrin and many others (see e.g. the survey [71]). Here we are interested in putting out some qualitative properties satisfied by the solutions of such problems, which exhibit very different behaviour according to the values of p and q. More concretely, we shall fix our attention on the behaviour of the support of the solution.

A well-known fact is that when (1) is linear (i.e. $p = 2$, $q = 1$) the solution u of (1) corresponding to data, say $f \geqslant 0$ and $q \geqslant 0$, is such that $u > 0$ on Ω. This is a trivial consequence of the strong maximum principle and can also be obtained by many other arguments, e.g. the Harnack inequality.

When (1) is nonlinear, entirely different behaviour may appear. Roughly speaking, the effective power of the diffusion term $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and of the absorption term $|u|^{q-1}u$ vary with p and q, generating new phenomena. Thus, letting Ω be an unbounded open set and f and g with compact support, the support of the solution contains the whole domain Ω if $q \geqslant p - 1$, but otherwise (i.e. when $q < p - 1$) the solution u has compact support and so $u = 0$ on an unbounded region of Ω. This was first shown in [13] for equation (3) and more generally in [36, 37] for (1). The main idea in order to prove the compactness of the support, assuming $q < p - 1$, lies in the construction of adequate super and subsolutions $\bar{u}$ and $\underline{u}$ of the problem (1), (2). Such functions can be chosen with compact support and so, by a comparison argument, $\underline{u} \leqslant u \leqslant \bar{u}$ on Ω, which implies that supp u is also a compact subset.

This kind of vanishing property has, in fact, a local character and this also happens even for bounded domains Ω, in the sense that the set

$$N(u) = \{x \in \bar{\Omega}: u(x) = 0\} \qquad (N(u) = \bar{\Omega} - \operatorname{supp} u)$$

may have a positive measure. The first result in that direction seems to be the author's memoir [32], in which a general local method is proposed, and later developed in [33], [34] and [35] (see also [72, 10, 47] and [1]).

THEOREM 1. *Assume $q < p - 1$ and let u be the solution of* (1), (2). *Then*

$$N(u) \supset \{x \in N(f) \cup N(g|_{\partial\Omega}) \text{ such that } d(x, \bar{\Omega} - |N(f) \cup N(g|_{\partial\Omega})|) \geqslant (M/C)^{(p-1-q)/p}\} \tag{4}$$

with C some positive constant (explicitly known) only depending on N, p, and q, and $M = \|u\|_{L^\infty}$.

Here $N(f)$ (resp. $N(g|_{\partial\Omega})$) represents the set $\{x \in \Omega: f(x) = 0\}$ (resp. $\{x \in \partial\Omega: g(x) = 0\}$). Let us remark that conclusion (4) is not empty if Ω is unbounded and f and g have compact support (in fact the constant M in (4) can be substituted for some bound of the ess. supremum of u in the interior of the set $N(f)$). Otherwise, e.g. if Ω is bounded, Theorem 1 has implicitly the following assumption:

(5) $$\operatorname{meas}\{x \in N(f) \cup N(g|_{\partial\Omega}):\ d(x, \bar{\Omega} - N(f) \cup N(g|_{\partial\Omega})) \geqslant (M/C)^{(p-1-q)/p}\} > 0.$$

The main idea in proving Theorem 1 is to use the functions

$$v_{\pm}(x) = \pm C|x - x_0|^{p/(p-1-q)}$$

as local super and subsolutions of (1) when x_0 is adequately chosen. Then, by comparison arguments, one obtains $v_-(x) \leqslant u(x) \leqslant v_+(x)$ on some neighbourhood of x_0 and so $u(x_0) = 0$. (For details, see the paper of J. Hernandez in these Proceedings.) By a (not difficult) modification of that argument it is possible to show that, if for instance, $g = 0$, then under the assumptions of Theorem 1 a stronger conclusion holds:

(6) $$N(f) = N(u), \quad \text{ie. } \{x \in \Omega: f(x) = 0\} = \{x \in \Omega: u(x) = 0\},$$

if $f(x)$ decays to zero on $\operatorname{supp} f$ as $d(x, N(f))^{pq/(p-1-q)}$ in a tubular neighborhood of $\partial N(f)$ (see [33]).

Results such as Theorem 1 can be obtained for nonlinear equations under formulations more general than (1). That is the case of symmetric invariant equations such as

$$-\operatorname{div}\left(\frac{a(|\nabla u|)}{|\nabla u|^2}\nabla u\right) + c(u) = f,$$

where a and c are assumed to be nondecreasing and satisfying some balance condition (now given by the boundedness of an improper integral) substituting the assumption $q < p - 1$. Also nonisotropic equations

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(\frac{\partial u}{\partial x_i}\right) + c(u) = f$$

and fully nonlinear elliptic equations can be considered [37, 28, 33]. To finish this section, we remark that also for Variational Inequalities, e.g. the obstacle problem ($u \geqslant \psi$ on Ω), the method of local super and subsolutions can be applied in order to estimate the coincidence set

$$\{x \in \Omega: u(x) = \psi(x)\} \quad (\equiv N(u - \psi))$$

(see [24, 6, 77, 7, 27], and [33]). A systematic development of these results, including many others, is the subject of the book [33].

2. An energy method. An important limitation of the scope of the method commented on in §1 is its constructive character. Recently, a new method has been introduced in Antoncev [2] and developed by the author and L. Véron in [41, 42], in order to study the behaviour of the support of the solution of the general class of second order quasilinear elliptic equations

(7) $$-\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) + C(x, u) = f.$$

It also has a local character and it is based on certain estimates of some energy terms associated to the solutions of (7). The structural assumptions for the treatment of (7) are the following:

(8) $$|A(x, r, \xi)| \leqslant C_1|\xi|^{p-1} \quad \text{for some } p > 1 \text{ and } C_1 > 0,$$

(9) $$A(x, r, \xi) \cdot \xi \geqslant C_2|\xi|^p \quad \text{for some } C_2 > 0,$$

(10) $$|B(x, r, \xi)| \leqslant C_3|r|^{\alpha}|\xi|^{\beta} \quad \text{for some } C_3 \geqslant 0,\ \alpha \geqslant 0, \text{ and } \beta \geqslant 0,$$

(11) $$C(x, r)r \geqslant C_4|r|^{q+1} \quad \text{for some } q > 0 \text{ and } C_4 > 0.$$

Functions A, B and C are assumed to be Caratheodory functions on its arguments, $x \in \Omega \subset \mathbb{R}^N$, $r \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$. As in the method of local super and subsolutions, it is enough to work on the subset $N(f)$ where $f = 0$.

DEFINITION. Given an open set $G \subset \mathbb{R}^N$, a function $u \in L^1_{\text{loc}}(G)$ is called a local weak solution of

(12) $$-\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) + C(x, u) = 0$$

on G if (i) $\nabla u \in L^p_{\text{loc}}(G)$, $B(x, u, \nabla u) \in L^1_{\text{loc}}(G)$, and $C(x, u) \in L^1_{\text{loc}}(G)$, and (ii) for any $\phi \in C_0^\infty(G)$ we have

$$\int_G \{A(x, u, \nabla u) \cdot \nabla\phi + (B(x, u, \nabla u) + C(x, u))\phi\}\, dx = 0.$$

Given $x_0 \in G$ such that $B\rho_0(x_0) \subset G$ and u is a local weak solution u of (12) on G, we introduce the diffusion energy on $B\rho(x_0)$, $0 < \rho < \rho_0$, by

(13) $$E(\rho) = \int_{B_\rho(x_0)} A(x, u, \nabla u) \cdot \nabla u\, dx$$

as well as the absorption energy on $B\rho(x_0)$ by

(14) $$b(\rho) = \int_{B_\rho(x_0)} |u|^{q+1}\, dx.$$

The main conclusion of the energy method is

THEOREM 2. *Assume $q < p - 1$, $\beta \leqslant p$, $\alpha = q - \beta(q + 1)/p$, and C_3 small enough. Then for every $x_0 \in G$ such that $B\rho_0(x_0) \subset G$ and for every local weak solution u of* (12) *on G, there exists a positive constant*

$$C^* = C^*(N, p, qE(\rho_0), b(\rho_0))$$

such that $u(x) = 0$ a.e. on $B\rho_1(x_0)$ where

(15) $$\rho_1 = \rho_0 - C^*.$$

Before referring to the proof of Theorem 2, we shall make some remarks about its applications. First of all we point out that no monotonicity assumptions are made on the dependence of $A(x, r, \xi)$ and $C(x, r)$ on ξ and r respectively. Hence Theorem 2 can be applied even in the absence of comparison principles, in contrast with the method in §1. In order to obtain some global consequences, consider, for instance, equation (7) on an open set $\Omega \subset \mathbb{R}^N$ with Dirichlet condition $u = 0$ on $\partial\Omega$. Assume the hypotheses of Theorem 2 and let $f \in L^{(q+1)/q}(\Omega)$. Then is not difficult to show that if $u \in W_0^{1,p}(\Omega) \cap L^{q+1}(\Omega)$ is any weak solution of (7), one has

$$\int_\Omega A(x,u,\nabla u)\cdot \nabla u\,dx + \int_\Omega |u|^{q+1}\,dx \leqslant C\int_\Omega |f|^{(q+1)/q}\,dx \tag{16}$$

for some structural constant C. Therefore we can apply Theorem 2 on the set $G = N(f)$ and then, if $x_0 \in N(f)$ and $\rho_0 = d(x_0, \overline{\Omega} - N(f))$, by (16) we have $E(\rho_0) + b(\rho_0) \leqslant C\|f\|_{(q+1)/q}^{(q+1)/q}$, which allows to us to conclude the existence of a constant $C^{**} = C^{**}(N, p, q, \|f\|_{(q+1)/q})$ such that

$$N(u) \supset \{x \in N(f) : d(x, \overline{\Omega} - N(f)) \geqslant C^{**}\}. \tag{17}$$

As in Theorem 1, conclusion (17) is not empty if Ω is unbounded and f has compact support. Otherwise, we need the assumption

$$\operatorname{meas}\{x \in N(f) : d(x, \overline{\Omega} - N(f)) \geqslant C^{**}\} > 0. \tag{18}$$

Hypothesis (18) has the same nature as (5) but with the important difference that no bound on $\|u\|_{L^\infty}$ is now needed. It is also interesting to remark that, when both methods may be applied, sharper estimates are obtained by using the method of local super and subsolutions. The main reason for that is the fact that comparison functions $v_\pm$ used in the proof of Theorem 1 are, in fact, exact solutions of the homogeneous equation associated to (1) and so the estimate (4) cannot be improved in some particular cases.

We now return to the proof of Theorem 2. The main ingredients in the proof are the following technical lemmas.

LEMMA 1. *Under the hypotheses of Theorem* 2, $A(\cdot,u,\nabla u)\cdot\nabla u$, $|u|^{q+1}$, $|A(\cdot,u,\nabla u)|u$, *and* $B(\cdot,u,\nabla u)u$ *belong to* $L^1(B\rho_0(x_0))$ *and, for almost every* $\rho \in (0,\rho_0)$, *we have*

$$\begin{aligned}\int_{B_\rho} A(x,u,\nabla u)\cdot\nabla u\,dx + C_4\int_{B\rho}|u|^{q+1}\,dx + \int_{B\rho} B(x,u,\nabla u)u\,dx \\ \leqslant \int_{S\rho} A(x,u,\nabla u)\cdot\vec{\nu} u\,ds,\end{aligned} \tag{19}$$

where $B\rho = B\rho(x_0)$, $S\rho = \partial B\rho$, *and* $\vec{\nu} = \vec{\nu}(x)$ *is the outward normal vector at* $x \in S\rho$.

LEMMA 2. *Let D be a bounded open set of* $\mathbb{R}^N$, $N \geqslant 1$, *with a* C^1 *boundary* ∂D. *Assume* $0 \leqslant q \leqslant p - 1 < \infty$. *Then there exists a constant* $C = C(p,q,D)$ *such that for any* $v \in W^{1,p}(D)$ *we have*

$$\|v\|_{L^p(\partial D)} \leqslant C\left(\|\nabla v\|_{L^p(D)} + \|v\|_{L^{q+1}(D)}\right)^\theta \|v\|_{L^{q+1}(D)}^{1-\theta},$$

where

$$\theta = \frac{N(p-1-q)+q+1}{N(p-1-q)+(q+1)p}.$$

If in particular $D = B\rho(x_0)$, then the inequality

$$\|v\|_{L^p(S\rho)} \leqslant C\left(\|\nabla v\|_{L^p(B\rho)} + \rho^{-\delta}\|v\|_{L^{q+1}(B\rho)}\right)^{\theta}\|v\|^{1-\theta}_{L^{q+1}(B\rho)} \tag{20}$$

holds, where

$$\theta = \frac{N(p-1-q)+(q+1)p}{p(q+1)} \quad \textit{and} \quad C = C(N,p,q).$$

Lemma 1 is proved by taking, in the definition of local weak solution, the test functions $\phi_{n,m}(x) = \psi_n(|x-x_0|)T_m(u(x))$ and passing to the limit in n and m. Here ψ_n: $[0,\rho_0] \to \mathbb{R}^+$ is such that $\psi_n(r) = 1$ if $r \in [0, \rho - 1/n]$, $\psi_n(r) = 0$ if $r \in [\rho, \rho_0]$, and $\psi_n(r) = -n(\rho - r)$ if $r \in [\rho - 1/n, \rho]$. T_m is a truncation function, such as, e.g., $T_m(r) = \operatorname{sign}(r)\min(m, |r|)$. Lemma 2 is an interpolation-trace result and is the key-stone of the proof of Theorem 2. It can be proved by using the Gagliardo–Niremberg interpolation inequalities and some trace results (see the details in [42]).

PROOF OF THEOREM 2. *First step.* If u is a local weak solution of (12) then

$$E(\rho) + C_4 b(\rho) + \int_{B\rho} B(x,u,\nabla u)u\,dx \geqslant C_5(E(\rho)+b(\rho)) \tag{21}$$

for some constant $C_5 > 0$. Indeed, by using Young's inequality, for any $\varepsilon > 0$ and $\tau > 1$ we have

$$C_3|u|^{\alpha+1}|\nabla u|^{\beta} \leqslant \frac{\varepsilon C_3}{\tau}|u|^{\tau(\alpha+1)} + \frac{(\tau-1)}{\tau}C_3\varepsilon^{-1/(\tau-1)}|\nabla u|^{\beta\tau/(\tau-1)}.$$

If we choose $\tau = (q+1)/(\alpha+1)$, then $\beta\tau/(\tau-1) = p$. So, by (14),

$$\left|\int_{B\tau} B(x,u,\nabla u)u\,dx\right| \leqslant \varepsilon C_3\left(\frac{p-\beta}{p}\right)b(\rho) + \frac{\beta C_3}{C_2\rho}\varepsilon^{-(p-\beta)/\beta}E(\rho).$$

Hence, if

$$C_3 < C_4\left(\frac{p}{p-\beta}\right)^{(p-\beta)/p}\left(C_2\frac{p}{\beta}\right)^{\beta/p},$$

then it is possible to find an $\varepsilon > 0$ such that

$$\varepsilon C_3\left(\frac{p-\beta}{p}\right) < C_4 \quad \text{and} \quad \frac{\beta C_1}{C_2 p}\varepsilon^{-(p-\beta)/\beta} < 1$$

and (21) holds.

END OF THE PROOF. By Lemma 1 and (21) we have

$$C_5(E(\rho)+b(\rho)) \leqslant \int_{S\rho} A(x,u,\nabla u)\cdot\vec{\nu}u\,ds. \tag{22}$$

By (8) and the Holder inequality,

$$\int_{S\rho} A(x,u,\nabla u)\cdot\vec{\nu}u\,ds \leqslant C_1\left(\int_{S\rho}|\nabla u|^{p-1}|u|ds\right)$$
$$\leqslant C_1\left(\int_{S\rho}|\nabla u|^{p}\right)^{(p-1)/p}\left(\int_{S\rho}|u|^{p}\right)^{1/p}.$$

On the other hand, by using spherical coordinates (ω, r) with center x_0 we have

$$E(\rho) = \int_0^\rho \int_{S^{N-1}} A(r\omega, u, \nabla u) \cdot \nabla u r^{N-1}\, d\omega\, dr.$$

Hence E is differentiable almost everywhere and

$$\frac{dE}{d\rho}(\rho) = \int_{S\rho} A(r\omega, u, \nabla u) \cdot \nabla u\, d\omega$$

which, by (9), implies

$$\frac{dE}{d\rho}(\rho) \geqslant C_2 \int_{S\rho} |\nabla u|^p\, ds. \tag{23}$$

Then by (22), (23) and Lemma 2 we have

$$E(\rho) + b(\rho) \leqslant K\left(\frac{dE}{d\rho}\right)^{(p-1)/p} \left(E(\rho)^{Q/p} b(\rho)^{(1-Q)/(q+1)} + \rho^{-\delta Q} b(\rho)^{1/(q+1)}\right)$$

for some constant K. Then, by Young's inequality

$$E(\rho) + b(\rho) \leqslant K_1 \rho^{-\delta Q}\left(\frac{dE}{d\rho}\right)^{(p-1)/p} (E(\rho) + b(\rho))^{\omega}, \tag{24}$$

where

$$K_1 = 2K \max\left(1, b(\rho_0)^{\theta(1/(q+1)-1/p)}\right) \max\left(\rho_0^{\delta\theta}, 1\right),$$
$$\omega = \theta/p + (1-\theta)/(q+1).$$

Hence E satisfies the differential inequality

$$K_2 \rho^{-\rho\theta p/(p-1)} \frac{dE}{d\rho}(\rho) \geqslant E(\rho)^{(1-\omega)p/(p-1)}, \tag{25}$$

where $K_2 = K_1^{p/(p-1)}$. But $0 < (1-\omega)p/(p-1) < 1$ and integrating in (25) we conclude that $E(\rho_1) = 0$ if

$$\rho_1^{1+\delta\theta p/(p-1)} = \rho_0^{1+\delta\theta p/(p-1)} - \frac{K_2 E(\rho_0)^{1-(1-\omega)p/(p-1)}}{1-(1-\omega)p/(p-1)}.$$

Then, by (24), $b(\rho_1) = 0$ and this implies $u(x) = 0$ a.e. on $B\rho_1(x_0)$.

REMARKS. The constant C^* appearing in the statement of Theorem 2 can be explicitly estimated in terms of N, p, q, $E(\rho_0)$, and $b(\rho_0)$ (see [42]). We also remark that, by a careful revision of the technical lemmas and the proof of Theorem 2, this remains true in the case $q = 0$ (which arises in Variational Inequalities), and even for $-1 < q < 0$ when we consider the global Dirichlet problem (see also [23]). Finally, we send the reader to Bernis [16] and the communication of that author in this Congress for the consideration of higher order elliptic equations via another energy method. (The support of the solutions of a fourth order variational inequality is also studied in [69] and [15].)

3. Parabolic quasilinear equations. The former energy method also applies to parabolic quasilinear equations like

$$\frac{\partial}{\partial t}\beta(v) - \operatorname{div}\mathscr{A}(t, x, v, \nabla v) + \mathscr{B}(t, x, v, \nabla v) + \mathscr{C}(t, x, v) = f(t, x), \tag{26}$$

where

$$\beta(r) = |r|^{1/m} \operatorname{sign} r \quad \text{for some } m > 0, \tag{27}$$

$$|\mathscr{A}(t, x, r, \xi)| \leqslant C_1|\xi|^{p-1} \quad \text{for some } p > 1 \text{ and } C_1 > 0, \tag{28}$$

$$\mathscr{A}(t, x, r, \xi) \cdot \xi \geqslant C_2|\xi|^p \quad \text{for some } C_2 > 0, \tag{29}$$

$$|\mathscr{B}(t, x, r, \xi)| \leqslant C_3|r|^{\alpha}|\xi|^{\beta} \quad \text{for some } C_3 \geqslant 0, \alpha \geqslant 0, \text{ and } \beta \geqslant 0, \tag{30}$$

$$\mathscr{C}(t, x, r)r \geqslant C_4|r|^{q+1} \quad \text{for some } q > 0 \text{ and } C_4 \geqslant 0. \tag{31}$$

Equation (26) contains as main particular cases the generic porous media equation [12], which, for simplicity, in one dimension reads

$$u_t - (u^m)_{xx} + b_0 \cdot (u^{\lambda})_x + c_0 \cdot u^q = 0, \tag{32}$$

where we are assuming $u \geqslant 0$, $b_0 \in \mathbb{R}$, $c_0 \geqslant 0$, $m > 1$, $\lambda \geqslant 1$, and $q > 0$. Obviously (26) appears taking $v = u^m$ and $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ adequately. Equation (26) also contains the equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0, \qquad p > 1, \tag{33}$$

which, again, appears in non-Newtonian fluids [37]. As for the elliptic case, the energy method can be applied locally to show the following local version property referred to as the *finite speed of propagation property*:

(P) Let v be a weak local solution of (26) with $f = 0$ on the set $(0, \infty) \times B\rho_0(x_0)$. Assume that $v(0, x) = 0$ for $x \in B\rho_0(x_0)$. Then for every $t > 0$ there exists $\rho(t)$, $0 \leqslant \rho(t) < \rho_0$, such that $v(t, x) = 0$ on $B_{\rho(t)}(x_0)$. The main answer given in [41], [9] (see also [2]) is that in order to have such a property it suffices to have

$$m(p-1) > 1 \tag{34}$$

and no other assumption on q (about the term $\mathscr{B}$ we assume $0 \leqslant \beta \leqslant p$, $\alpha = (p - \beta(m+1))/mp$ and in fact the conclusion holds only on a finite interval of time $[0, T^*]$ if $\mathscr{B} \neq 0$ and $\beta \neq p$). Now the energies are defined by

$$E(t, \rho) = \int_0^t \int_{B\rho(x_0)} \mathscr{A}(s, x, v, \nabla v) \cdot \nabla v \, dx \, ds$$

and

$$b(t, \rho) = \operatorname*{ess\,sup}_{0 \leqslant s \leqslant t} \int_{B\rho(x_0)} |v(s, x)|^{(m+1)/m} \, dx.$$

(Here the local weak solutions are supposed to satisfy, in particular, $\nabla v \in L^p_{\text{loc}}((0, \infty) \times B\rho_0(x_0))$ and $v \in L^\infty(0, \infty: L^{(m+1)/m}(B\rho_0(x_0))).$) As in the elliptic case, it is not difficult to find global consequences. Thus, for instance, if we consider the Dirichlet problem

$$(35)\quad \begin{cases} \dfrac{\partial \beta(v)}{\partial t} - \operatorname{div} \mathscr{A}(t, x, v, \nabla v) + \mathscr{C}(t, x, v) = 0 & \text{on } (0, \infty) \times \Omega, \\ v(t, x) = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ v(0, x) = v_0(x) & \text{on } \Omega, \end{cases}$$

where Ω is an open set of $\mathbb{R}^N$, $N \geqslant 1$, and $v_0 \in L^{(m+1)/m}(\Omega)$, assuming the structural hypotheses as well as $m(p-1) > 1$, then the quantity $\rho(t)$ can be estimated independently of $x_0 \in N(v_0)$ and we find that $\rho(t) \leqslant Ct^{\mu}$ for some $\mu = \mu(N, m, p) > 0$. As a consequence, we have

$$(36)\qquad N(v(t, \cdot)) \supset \left\{ x \in N(v_0) \colon d\left(x, \overline{\Omega} - N(v_0)\right) \geqslant Ct^{\mu} \right\}.$$

In particular, if Ω is unbounded and v_0 has compact support, we find that for every $t > 0$ the support of $v(t, \cdot)$ is also compact. If in addition $N = 1$, then $\operatorname{supp} v(t, \cdot) = [\zeta_1(t), \zeta_2(t)]$ for some monotone real functions ζ_i.

Naturally, much more precise information is available for concrete formulations of (26). For instance, for the porous media equation

$$(37)\qquad u_t - \Delta\phi(u) = 0$$

with ϕ a continuous nondecreasing function such that $\phi(0) = \phi'(0) = 0$, it is known that the solution u of the Cauchy problem has compact support for each $t > 0$ (assuming $u(0, x)$ with compact support) if and only if

$$(38)\qquad \int_0^1 \frac{ds}{\phi^{-1}(s)} < +\infty$$

(see [67, 4, 68, 30, 74, 75]). In fact for the homogeneous Dirichlet problem, and even under (38), there always exists a finite time T^* such that $u(t, x) > 0$ on Ω for every $T \geqslant T^*$ [18]. The exponent μ in estimate (15) can also be optimally estimated if, for instance, $\phi(s) = |s|^m \cdot \operatorname{sign} s$ [5, 73]. For other results concerning the boundary of the support of the solution of (36), we refer to the recent survey of Berstch–Peletier [17]. For the particular equation (33) some references are [9, 36, 37, 7, 8] and [53, 56]. The former property of finite speed of propagation is essentially due to the assumption $m(p-1) > 1$, which expresses when the diffusion is "slow". Nevertheless, other different behaviours appear when the action of the absorption term $\mathscr{C}(t, x, v)$ or of the convection term $\mathscr{B}(t, x, v, \nabla v)$ is taken into account.

Thus, when the absorption is large with respect to the diffusion, the support of $v(t, \cdot)$ remains in a compact region for every $t \in [0, \infty)$. That property, usually referred to as *localization*, appears, for instance, for the equation

$$(39)\qquad u_t - \Delta u^m + u^s = 0$$

when $m > s$ (see [54, 61, 19, 20, 63]). This can be proven by global comparison functions when the domain is unbounded and more generally for the method of local super and subsolutions [35]. Also, the energy method allows us to find such a behaviour (even locally) for the solutions of the general equation (26) assuming $\max(q, 1/m) < p - 1$ [42].

A stronger property appears when the absorption is larger. For instance, if we assume $0 < s < 1$ in (39), then there exists a finite time $T_0 < +\infty$ such that the set $N(u(t, \cdot))$ has positive measure even for strictly positive initial data and nonhomogeneous Dirichlet boundary conditions [35, 11]. Moreover, if the Dirichlet conditions are homogeneous or we are concerned with the Cauchy problem associated to (39), then, in fact, $u(t, x) \equiv 0$ for every $t \geqslant T_0$ and a.e. in x [54, 76, 43].

With respect to the balance between diffusion and convection the situation is quite different. When the convection is large with respect to the diffusion, then there is a kind of localization property but only in some directions according to the equation. For instance, for the one-dimensional equation

$$u_t - (u^m)_{xx} + b_0 \cdot (u^\lambda)_x = 0, \tag{40}$$

where $m > 1$, $\lambda > 0$, and $b_0 \in \mathbb{R} - \{0\}$, if u is the solution of the Cauchy problem associated to (40) and if $\operatorname{supp} u(0, x) = [a, b]$, then, for every $t > 0$, $\operatorname{supp} u(t, \cdot) = [\zeta_1(t), \zeta_2(t)]$ with $\zeta_1(0) = a$, $\zeta_2(0) = b$ and $\zeta_1(t) \geqslant a - \varepsilon$ (resp. $\zeta_2(t) \leqslant b - \varepsilon$) $\forall t \in [0, \infty)$, assuming $m > \lambda$ and $b_0 < 0$ (resp. $b_0 < 0$) (see [55, 49, 50]). When the convection is larger, i.e. when $0 < \lambda < 1$, then there is only one (localized) interface: $\operatorname{supp} u(t, \cdot) = [\zeta_1(t), +\infty)$ (resp. $\operatorname{supp} u(t, \cdot) = (-\infty, \zeta_2(t)]$) when $b_0 > 0$ (resp. $b_0 < 0$) (see [39]). We also remark that the presence of convection and absorption terms in a nonlinear diffusion does not produce any new behaviour [44, 58].

Also for "fast" diffusion equations the boundary of the support of the solution can be considered as a free boundary. For instance for the equation

$$u_t - \Delta\phi(u) = 0 \tag{41}$$

it is well known that if $\phi(s) = |s|^m \cdot \operatorname{sign} s$ with $0 < m < 1$ [22], or more generally

$$\int_0^1 \frac{ds}{\phi(s)} < +\infty$$

[70, 29], then for every initial datum there exists a finite T_0 such that the solution u of the homogenous Dirichlet problem associated to (41) is such that $u(t, \cdot) = 0$ $\forall t \geqslant T_0$. In fact it is also known that $u(t, x) \neq 0$ a.e. on x if $t \in (0, T_0)$, and so $\Omega \times \{T_0\}$ is a free boundary. This behaviour also appears for the equation (33) when $1 < p < 2$ (see [9, 52 and 2]).

As in the elliptic case, there is also a large literature about the support of the solution of some parabolic variational inequalities ([14, 25, 31, 43] and [76]).

Detailed results as well as other qualitative properties of the solutions of quasilinear parabolic equations will be available in [33].

Finally, we remark that there are also some references about the study of the boundary of the support for the solutions of first order quasilinear hyperbolic equations

$$u_t - \sum_{i=1}^{N} \phi_i(u)_{x_i} + \beta(u) = f$$

(see [**65**, **64**, **26**] and more recently [**40**]).

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