H Brezis, M G Crandall & F Kappel (Editors)

Université Pierre et Marie Curie / University of Wisconsin-Madison / Karl-Franzens-Universität Graz

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J I DIAZ

Qualitative properties of solutions of some nonlinear diffusion equations via a duality argument

1. Introduction

The main goal of this work is to present several qualitative properties of solutions of the two following nonlinear parabolic problems:

$$\left.\begin{aligned} &u_t - \Delta\phi(u) = 0 && \text{in } Q = (0,\infty)\times\Omega \\ &\phi(u) = 0 && \text{on } \Sigma = (0,\infty)\times\partial\Omega \\ &u(0,\cdot) = u_0 && \text{on } \Omega \end{aligned}\right\} \qquad (P)$$

and

$$\left.\begin{aligned} &v_t + \phi(-\Delta v) = 0 && \text{in } Q \\ &v = 0 && \text{on } \Sigma \\ &v(0,\cdot) = v_0 && \text{on } \Omega. \end{aligned}\right\} \qquad (P^*)$$

Here Ω represents a regular open bounded set of $\mathbb{R}^N$ and ϕ is a continuous non-decreasing real function such that $\phi(0) = 0$. Problems P and P* arise in many applications leading to functions ϕ of a different behaviour, specially near the origin (see, e.g., [11], [6] and [9]).

An essential tool in our study is the "duality" existing between problems P and P*. To explain this let us note by $\Lambda = -\Delta$ the canonical isomorphism from the Hilbert space $H_0^1(\Omega)$ onto its dual $H^{-1}(\Omega)$. We also recall that problems P and P* are well-posed, in the semigroups sense, on the spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. More concretely, the operators A and B given by

$$D(A) = \{u \in H^{-1}(\Omega) \cap L^1(\Omega) : \phi(u) \in H_0^1(\Omega)\}$$

$$Au = -\Delta\phi(u) \quad \text{if } u \in D(A)$$

and

$$D(B) = \{v \in H_0^1(\Omega) : \Delta v \in L^1(\Omega) \text{ and } \phi(-\Delta v) \in H_0^1(\Omega)\}$$

$$Bv = \phi(-\Delta v) \quad \text{if } v \in D(B)$$

are m-accretives and densely defined in $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ respectively, assumed ϕ satisfying

$$D(\phi) = R(\phi) = \mathbb{R}$$

([3], [4]). Using the usual time-semidiscretization scheme, it is easy to prove (see, e.g., [6]) the following duality result:

Lemma 0. Let $v_0 \in H_0^1(\Omega)$ and $u_0 = \Lambda v_0$. Let $v \in C([0,\infty) : H_0^1(\Omega))$ and $u \in C([0,\infty) : H^{-1}(\Omega))$ such that $u(t) = \Lambda v(t)$ for every $t > 0$. Then, u is the mild solution of P if and only if v is the mild solution of P*.

We shall use this result to prove the extinction in finite time of solutions of P under a suitable condition on ϕ near the origin as well as in the case in which ϕ is multivalued at $r = 0$. This last case is proved by using an abstract result concerning the Cauchy problem

$$\left.\begin{array}{l} \dfrac{du}{dt} + Au \ni f(t) \\ u(0) = u_0 \end{array}\right\} \qquad \text{(CP)}$$

for accretive operators A on a general Banach space X. Finally in Section 3 we apply Lemma 0 to the study of the asymptotic behaviour of solutions of an evolution variational inequality which can be formulated in terms of problem P* for a special ϕ also depending of the x-variable. Again, this study uses an abstract result for the Cauchy problem (CP); in this case a comparison type result between the solutions corresponding to different operators A.

2. Finite extinction time property in fast diffusion problems

A curious and interesting property of solutions of P was exhibited in [12] and [2] when assuming $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and $\phi(u) = u^m$, $0 < m < 1$. They proof that there exists a finite fime T_0 (called the extinction time) such that $u(t,x) \equiv 0$ for all $t \geq T_0$ and $x \in \Omega$. By well-known results, that property does not holds in the linear or slow diffusion $\phi(u) = u^m$, $m = 1$ or $m > 1$, respectively. A natural question arises: For which general functions ϕ the finite extinction time property holds? We note that the equation under consideration may be rewritten by

$$u_t - \operatorname{div}(\phi'(u)\nabla u) = 0.$$

So, if $\phi'(0) < +\infty$ the equation is uniformly parabolic if $\phi'(0) > 0$ or degenerated if $\phi'(0) = 0$. In any case the property can not be satisfied ([11],[9]) and so a first necessary condition is $\phi'(0) = +\infty$ (fast diffusion equation). It turns out that this condition is not enough, being given the wanted characterization by means of the convergence of an improper integral:

Theorem 1. ([6]). Let $u_0 \in H^{-1}(\Omega) \cap L^1(\Omega)$ such that $\Lambda^{-1}u_0 \in L^\infty(\Omega)$ (e.g. $u_0 \in L^p(\Omega)$, $p \geq N/2$ if $N \geq 2$) and let u be the mild solution of P. Then the assumption

$$\int_0 \frac{ds}{\phi(s)} < +\infty$$

is the necessary and sufficient condition for the existence of a finite extinction time T_0 (i.e. such that $u(t,x) = 0$, a.e. $x \in \Omega$, for all $t \geq T_0$).

The keystone part of the proof is to obtain primilary the similar result for solutions v of P* corresponding to initial data v_0 smooth enough. To do that we use the comparison principle and the construction of suitable super and subsolutions. Here the maximum or comparison principle may be proved by using the fact that problem P* is "well-posed" in semigroups sense in the space $L^\infty(\Omega)$. The proof of Theorem 1 ends by using Lemma 0, some regularizing effects as well as the maximum principle for solutions of P which now is obtained through the fact that P is also "well-posed" in semigroups sense in the space $L^1(\Omega)$. For details we refer the reader to [6].

The conclusion of Theorem 1 may be easily extended to solutions of the nonhomogeneous equation

$$u_t - \Delta\phi(u) = f(t)$$

where $f \in L^1((0,\infty) : H^{-1}(\Omega))$ is assumed such that $f(t) \equiv 0$ for a.e. $t \geq t_0$, for some $t_0 \geq 0$. A general reference containing many other variants of Theorem 1 to other equations, other methods of proof and many references is [9].

The case in which in the equation of P ϕ represents a general maximal monotone graph of $\mathbb{R}^2$ have some interest in applications ([7]) and may be also treated by using Lemma 0. The only new difference appear when ϕ is multivalued at $r = 0$, i.e. $\phi(0) = [\phi^-(0), \phi^+(0)]$ for some numbers $\phi^-(0)$, $\phi^+(0)$, $-\infty \leq \phi^-(0) \leq 0 \leq \phi^+(0) \leq +\infty$. In that case a necessary condition for the finite extinction time property is that $\Lambda^{-1}(f(t)) \in \phi(0)$ for t large enough and a.e. $x \in \Omega$, (take $u \equiv 0$ at the equation). That condition turns out to be "almost sufficient", and we have

Theorem 2. Let $u_0 \in H^{-1}(\Omega) \cap L^1(\Omega)$, such that $\Lambda^{-1}u_0 \in L^\infty(\Omega)$ and $f \in L^1_{loc}((0,\infty) : L^\infty(\Omega))$ such that, there exists $t_0 \geq 0$ and $\varepsilon > 0$ satisfying that

$$\phi^-(0) + \varepsilon \leq \Lambda^{-1}(f(t)) \leq \phi^+(0) - \varepsilon \quad \text{a.e.} \quad t \in (t_0,+\infty).$$

Then there exists a finite extinction time $T_0 \geq t_0$ for the corresponding solution u. ∘

Due to Lemma 0, the proof of Theorem 2 reduces to show the finite extinction time property for the solution of

$$v_t + \phi(-\Delta v) \ni F(t)$$

with homogeneous Dirichlet conditions and the corresponding initial data, being $F(t) = \Lambda^{-1}(f(t))$. We first recall an abstract result

Theorem 3. ([7]). Let X be a Banach space, A be an accretive operator on X, $f \in L^1_{loc}((0,\infty) : X)$, $u_0 \in \overline{D(A)}$ and $u \in C([0,\infty) : X)$ be the (integral) solution of the Cauchy problem (CP). Assume that there exist $\varepsilon > 0$ such that

$$B(f(t),\varepsilon) \subset A(0) \qquad \text{a.e.} \quad t \geq t_0$$

for some $t_0 \geq 0$ (here $B(h,\varepsilon) = \{w \in X : |h - w| < \varepsilon\}$). Then there exists a finite extinction time T_0 for u (i.e. $u(t) \equiv 0$, for all $t \geq T_0$).

Returning to the proof of Theorem 2, it follows from the application of Theorem 3 to the realization of the operator $\phi(-\Delta v)$ on the space $X = L^\infty(\Omega)$. Details, references and other applications of Theorem 3 can be found in [7] and [9].

3. On a fully nonlinear parabolic equation

Another context in which Lemma 0 turns to be very useful is the study of the problem

$$\left.\begin{array}{lll} v_t = \min\{\Psi,\Delta v\} & \text{in} & Q \\ v = 0 & \text{on} & \Sigma \\ v(0,\cdot) = v_0 & \text{on} & \Omega \end{array}\right\} \qquad (P^*_\Psi)$$

where $\Psi \in L^2(\Omega)$, $\Psi \geq 0$ is a given function. Such problem arises in heat control theory and can be equivalently formulated in terms of the following evolution variational inequality

$$v_t \in K = \{w \in H^1_0(\Omega) : w \leq \Psi \quad \text{a.e. on} \quad \Omega\}$$

$$\int_\Omega v_t(w-v_t)dx + \int_\Omega \nabla v.\nabla(w-v_t)dx \geq 0 \quad \text{for all} \quad w \in k.$$

The existence and uniqueness of solutions of the above variational inequality was proved in [5]. There it is also shown that $v(t,x)$ converges weakly in $H^1_0(\Omega)$, when $t \to \infty$, to a function $v_\infty(x) \in H^1_0(\Omega)$ satisfying

$$\min\{\Delta v_\infty,\Psi\} = 0 \quad \text{on} \quad \Omega.$$

Two natural questions arises: a) identify v_∞ in terms of v_0; and b) try to show a strong converge criterium. Some results to the first question are given in [8]. In particular, it is easy to show that if $\Psi(x) > 0$ then necessarily $v_\infty \equiv 0$. The study of the asymptotic behaviour in this last case may be carried out by using Lemma 0.

Theorem 4. ([8]). Let $v_0 \in H^1_0(\Omega)$ and Ψ such that $\Psi \in H^2(\Omega)$, with $\Delta\Psi \geq 0$. Then, if $\Psi > 0$ on Ω then $v(t) \to 0$ strongly in $H^1_0(\Omega)$ when $t \to \infty$. Moreover, if $\Psi(x) \geq \delta$ for some $\delta > 0$ then there exist a finite time T_0 such that $v_t - \Delta v = 0$ on $(T_0,\infty) \times \Omega$.

The proof of Theorem 4 is made in several steps. First we show (Lemma 0) that if v_0 is smooth enough, and $v(t)$ is the solution of (P^*_Ψ) then the function $u(t) = \Lambda v(t)$ satisfies

$$\left.\begin{array}{lll} u_t - \Delta\phi(x,u) = 0 & \text{in} & Q \\ \phi(x,u) = 0 & \text{on} & \Sigma \\ u(0,\cdot) = \Lambda v_0 & \text{on} & \Omega \end{array}\right\} \qquad (P_\Psi)$$

where $\phi(x,r) = -\min\{\Psi(x),-r\}$. Note that the range of ϕ is not the whole $\mathbb{R}$ which leads to some extra difficulties in the study of the realization of the operator in the space $L^1(\Omega)$. Nevertheless using some regularity results for stationary variational inequalities it can be shown ([8]) that (P_Ψ) is well-posed (in the semigroups sense) on the space $L^1(\Omega)$, assumed $\Psi \in H^1(\Omega)$ and $(-\Delta\Psi)^- \in L^2(\Omega)$. As a second step we shall apply the following abstract comparison result to the L^1-realization of the operator $-\Delta\phi(x,u)$.

<u>Theorem 5.</u> ([1]). Let X be a Banach lattice. For $i = 1,2$, let A_i be m-T-accretive operator in X and let $u_{0,i} \in D(A_i)$. Assume that there exists $\Theta : X \to X$ continuous and such that: i) $(I-\Theta)$ is order to preserving; ii) $A_2 u \subset A_1\Theta(u)$ for every $u \in D(A_2)$, and iii) $u_{0,2} \in D^+(A_2)$, $D^+(A_2) = \{u \in D(A_2) : A_2 u \geq 0\}$. Then

$$\|[u_1(t) - \Theta(u_2(t))]^+\|_X \leq \|[u_{0,1} - \Theta(u_{0,2})]^+\|_X$$

for every $t > 0$.

Returning to the proof of Theorem 4 we note that by comparison arguments we can always assume, without loss of generality, $v_0 < 0$ a.e. on Ω. Taking $u_0 = \Lambda^{-1} v_0$, we apply Theorem 5 to the case of $X = L^1(\Omega)$, $A_1 u$ the realization in $L^1(\Omega)$ of the linear operator $-\Delta u$ and $A_2 u$ that of the operator $-\Delta\phi(x,u)$. A function Θ satisfying the requirements of the theorem is given by the Nemitsky operator in $L^1(\Omega)$ of the Lipschitz function $\phi(x,r)$. Finally, it is not difficult to show the existence of a $\hat{u}_0 \in L^\infty(\Omega)$ such that $-\Delta\phi(x,\hat{u}_0(x)) \geq 0$ and $u_0(x) \leq \phi(x,\hat{u}_0(x))$ a.e. $x \in \Omega$. In consequence, if $h(t,x,z)$ denotes the solution of the linear heat equation

$$\begin{array}{lll} h_t = \Delta h & \text{in} & Q \\ h = 0 & \text{on} & \Sigma \\ h(0,\cdot) = z & \text{on} & \Omega, \end{array}$$

we deduce from Theorem 5 that $h(t,x,\hat{u}_0) \leq \phi(x,u(t,x)) \leq 0$. By Lemma 0 applied to P_Ψ as well as to the linear heat equation we obtain that

$$h(t,x,\Lambda^{-1}(\hat{u}_0)) \leq -\min\{\Psi(x),\Delta v(t,x)\} \leq 0 \quad \text{a.e.} \quad (t,x) \in Q,$$

where now $v(t,x)$ is the solution of the original problem (P^*_Ψ). From the well-known results on the asymptotic behaviour of solutions of the linear heat equation we obtain the wanted conclusions. (See details in [8].)

<u>Remark.</u> Duality arguments similar to Lemma 0 are also useful in the the regularity of the semigroup solution as well as for some other qualitative properties (such as, for instance, the finite speed of propagation) of solutions of nonlinear problems P and P* (see, e.g. [10] and [9]).

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J.I. DIAZ (*)

Facultad de Matemáticas
Universidad Complutense de Madrid

28040 Madrid
Spain

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