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ON THE BEHAVIOUR NEAR THE FREE BOUNDARY OF SOLUTIONS OF SOME NONHOMOGENEOUS ELLIPTIC PROBLEMS

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ABSTRACT: We study the behaviour, near the free boundary, of nonnegative solutions of nonhomogeneous elliptic problems of the type $-\Delta u+u^q=f(x)$ in Ω, $u=0$ on $\partial\Omega$, with $0<q<1$. We prove a pointwise "nondegeneracy" property (u grows faster than some function of the distance to the free boundary), and we give an application to the numerical approach of the free boundary.

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1. INTRODUCTION AND THE MAIN RESULT: Consider the Dirichlet problem

$$(1)\quad \begin{cases} -\Delta_p u+\lambda u^q=f(x) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a regular open set of $\mathbb{R}^N$, $\lambda>0$, $q>0$ and

$$\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u).\quad 1<p<\infty$$

Problem (1) have been largely studied in the literature and appears in many different contexts (see Díaz |4| and its references). It is also well known that under the assumption

$$(2)\qquad q<p-1$$

there exists a free boundary F defined by the boundary of the support of u. For the sake of simplicity in the exposition we shall always assume

$$(3)\qquad f\in L^\infty(\Omega),\ f\geq 0$$

which allows us to assume that $u\in C^{1,\alpha}(\Omega)$ and $u\geq 0$ on $\overline{\Omega}$. So the free boundary F is defined by

$$F=\partial\{x\in\Omega:u(x)>0\}\cap\Omega\ .$$

Our main goal is the study of the behaviour of u near the free boundary. This was firstly carried out in Phillips |7| and Alt-Phillips |1| for the case of $p=2$, $f\equiv 0$ and nonhomogeneous boundary conditions ($u=h$ on $\partial\Omega$). The analysis of the nonhomogeneous case ($f\not\equiv 0$) cannot be treated by their methods. (We remark that in fact the assumption $u=0$ on $\partial\Omega$ is not an important restriction because of the local nature of our results).

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It turns out that the behaviour of f near the bounda[ry] of its support has an important role for our purposes. Indeed, as it was first noticed in Díaz |4|, if f "grows slowly" near the boundary of its support then there is "nondiffusion" of the support of u and $F=\partial\, supp(f)$. This is a global version of the following local property: "Let $x_0\in\partial S(f)$ ($S(f)\equiv$ support of f) be such that

(4) $$0\leq f(x)\leq C|x-x_0|^{pq/(p-1-q)} \quad \text{a.e. } x\in B_R(x_0)\cap\Omega$$

for some R>0 and C>0. Then, there exists C*>0 such that

(5) $$0\leq u(x)\leq C^*|x-x_0|^{p/p-1-q)} \quad \text{a.e. } x\in B_R(x_0)\cap\Omega".$$

(see Díaz |4| Th. 1.15). We point out that the main difficulties in the study of the behaviour of u near F are located at the points of F where there is nondiffusion of the support (i.e. on the region $F\cap\partial S(f)$). Indeed: if $x_0\in F-\partial S(f)$ and F and $\partial S(f)$ are regular, u satisfies the homogeneous equation on $B_\varepsilon(x_0)\subset S(u)-S(f)$ for some $\varepsilon>0$ and so the behaviour of u near x_0 is a consequence of the results of Phillips |7| (when p=2). Thus, we shall concentrate our attention on points $x_0\in F\cap\partial S(f)$. (We remark that assumption (4) is, in some sense, optimal in order to conclude that a point $x_0\in\partial S(f)$ is such that $x_0\in F$: see Alvarez-Díaz |2|).

Inspired in the homogeneous case, and for further applications, we want to prove some "nondegeneracy properties" ensuring that u grows faster than some parabola $|x-x_0|^\alpha$ near $x_0\in F$. The following one-dimensional example shows that we need extra-assumptions to (4) in order to have such a kind of properties:

EXAMPLE: Consider the boundary value problem

(6) $$\begin{cases} -u''(x)+u(x)^q=f(x) & \text{in }]-2\pi,2\pi[\\ u(-2\pi)=u(2\pi)=0 \end{cases}$$

where $0<q<1$ and $f(x)=C\sin x\, e^{-1/x}$ if $0\leq x\leq\pi$ for some C>0 and f(x)=0 if $x\notin[0,\pi]$. It is not difficult to show that $\partial S(f)=F$ if C is small enough and that the function $\bar{u}(x)=Ke^{-1/qx}$ is a supersolution on the set $[0,R)$ for K and 1/R large enough.

Our main result is the following

THEOREM 1: Let $x_0\in\partial S(f)$ and $x_1\in S(f)$ such that

(7) $$\begin{cases} f(x)\geq C(R-|x-x_1|)^\gamma \text{ a.e. } x\in B_R(x_1) \text{ with } R=d(x_0,x_1), \gamma\geq pq/(p-1-q) \text{ and} \\ C>0. \end{cases}$$

Then

(8) $$u(x)\geq K|x-x_0|^{\gamma/q} \quad \text{for any } x\in[x_0,x_1] \text{ and for some } K>0,$$

where $[x,y]$ denotes the segment between x and y ($[x,y]=\{z=tx+(1-t)y,\ t\in[0,1]\}$).

Proof: Define

(9) $$\underline{u}(x)=\begin{cases} K(R-|x-x_1|)^{\gamma/q} & \text{if } R/2\leq|x-x_1|\leq R \\ 2K(\frac{R}{2})^{\gamma/q}-K|x-x_1|^{\gamma/q} & \text{if } 0\leq|x-x_1|\leq R/2 . \end{cases}$$

We claim tha[t K]>0 can be chosen such that $\underline{u}$ is a subsolution on $B_R(x_1)$. Indeed, $\underline{u}\in C^1$ and on $R/2\leq|x-x_1|\leq R$ we have that if $r=|x-x_1|$ then

$$-\Delta_p\underline{u}+\lambda\underline{u}^q=-(K\frac{\gamma}{q})^{p-1}\Big[(\frac{\gamma}{q}-1)(p-1)(R-r)^{(\frac{\gamma}{q}-1)(p-1)}+$$

$$-\frac{(N-1)}{r}(R-r)^{(\frac{\gamma}{q}-1)(p-1)}\Big]+\lambda K^q(R-r)^\gamma\leq\Big[(\frac{K\gamma}{q})^{p-1}(N-1)+\lambda K^q\Big](R-r)^\gamma$$

Moreover on $0\leq|x-x_1|\leq R/2$

$$-\Delta_p\underline{u}+\lambda\underline{u}^q\leq\Big[(\frac{K\gamma}{q})^{p-1}(\frac{\gamma}{q}-1)(p-1)+(N-1)+(2K)^q\lambda\Big](\frac{R}{2})^\gamma .$$

Then, if we take K such that

$$\max\{(\frac{K\gamma}{q})^{p-1}(N-1)+\gamma K^q,\ (K\frac{\gamma}{q})^{p-1}(\frac{\gamma}{q}-1)(p-1)+(N-1)+(2K)^q\lambda\}<C$$

we have that

$$-\Delta_p\underline{u}+\lambda\underline{u}^q\leq C(R-|x-x_1|)^\gamma\leq f(x) \quad \text{a.e. } x\in B_R(x_1).$$

On the other hand, as $\underline{u}=0$ on $\partial B_R(x_1)$ we deduce, from the comparison principle that $\underline{u}\leq u$ in $B_R(x_1)$, which implies (8). ∎

COROLLARY 1: Let f such that $\partial S(f)$ satisfies the uniform interior sphere condition (i.e. $\exists\, R>0$ such that $\forall\, x_0\in\partial S(f), \exists\, B_R(x_1): x_0\in\partial B_R(x_1)$). Assume that

(10) $$\begin{cases} f(x)\geq Cd(x,\partial S(f))^\gamma \text{ a.e. } x\in S(f) \text{ with } d(x,\partial S(f))\leq R,\ \gamma\geq pq/p-1-q) \text{ and} \\ C>0 . \end{cases}$$

Then

(11) $$u(x)\geq Kd(x,\partial S(f))^{\gamma/q} \text{ for any } x\in S(f) \text{ with } d(x,\partial S(f))\leq R \text{ and for some } K>0 .$$

Proof. Let $x\in S(f)$ with $d(x,\partial S(f))\leq R$. Let $x_0\in\partial S(f)$ such that $d(x,\partial S(f))=|x-x_0|$. Due to the assumption on $\partial S(f)$ we have that $x\in B_R(x_1)$ for some $x_1\in S(f)$ with $R=d(x_0,x_1)$. Moreover, without loss of generality we can assume that $x\in[x_0,x_1]$. Then

$$f(x)\geq Cd(x,\partial S(f))^\gamma\geq C(R-|x-x_1|)^\gamma$$

and by Theorem 1 we conclude that

$$u(x)\geq K|x-x_0|^{\gamma/q}=Kd(x,\partial S(f))^{\gamma/q} . ∎$$

The above pointwise non-degeneracy properties imply another non-degeneracy property (now in measure) that is very useful for many purposes:

COROLLARY 2. Under the assumptions of Corollary 1 ther exists $\varepsilon_0>0$ such that for any compact D [⊂] Ω we have

(12) $$|\{x\in D\cap S(f): 0<u(x)<\varepsilon^{\gamma/q}\}|\leq K_D\varepsilon$$

for any $\varepsilon<\varepsilon_0$ and for some positive constant K_D .

Proof. Let $\varepsilon_0>0$ given by

$$\varepsilon_0^{\gamma/q}<\min\{u(x): x\in D\cap S(f) \text{ and } d(x,\partial S(f))\geq R\} .$$

Let $x \in D \cap S(f)$ such that $d(x, \partial S(f)) < R$. By Corollary 1 $u(x) \geq K d(x, \partial S(f))^{\gamma/q}$. So if $R > d(x, \partial S(f)) \geq \frac{\varepsilon}{K^{q/\gamma}}$ we conclude that $u(x) \geq \varepsilon^{\gamma/q}$. Then

$$|\{x \in D \cap S(f) : 0 < u(x) < \varepsilon^{\gamma/q}\}| \leq |\{x \in D \cap S(f) : d(x, \partial S(f)) < \frac{\varepsilon}{K^{q/\gamma}}\}| \leq K_D \varepsilon$$

for some $K_D > 0$. ■

REMARK 1: Non-degenerancy properties of the type (11) or (12) were obtained by other different methods in Phillips |7| and Alt-Phillips |1| for the case of p=2 and u=h on $\partial\Omega$ (but $f \equiv 0$). Their results are concerned with the critical exponent $\gamma = pq/(p-1-q)$ and give us information on the growing of u on the part of the free boundary $F - \partial S(f)$ (i.e. on the part of F where the support diffuses). Indeed, it is dear that $S(f) \subseteq S(u)$. Then in the region $\Omega - S(f)$ u satisfies the homogeneous equation and $u > 0$ on $(\partial S(f)) \cap S(u)$.

2. APPLICATION TO THE NUMERICAL APPROXIMATION OF THE FREE BOUNDARY.

As it has been shown in Nochetto |6|, non degeneracy properties are one of the main ingredients in order to approximate the free boundary F. To apply this general philosophy to problem (1) we consider a descomposition of Ω in finite elements, and let $h \in (\mathbb{R}^+)^I$ be a discretization parameter whose components tends to zero. Let u_h be the discrete solution of (1) (i.e. of a sequence of approximate problems $(P_1)h$). In contrast with the obstacle problem it is possible that $\partial S(u_h) \cap \Omega$ be the empty set. So we define the discrete free boundary by means of

$$F_h = \partial\{x \in \Omega : u_h(x) > \delta_h\} \cap \Omega$$

where $\delta_h > 0$ is a constant to be determined later. We make the assumption that "there exists a function $\sigma(h) : (\mathbb{R}^+)^I \to \mathbb{R}^+$, $\sigma(h) \downarrow 0$ as $h \downarrow 0$ such that for some s, $1 \leq s \leq \infty$

(13) $\quad ||u - u_h||_{L^s(\Omega)} < \sigma(h)$.

Then we have

COROLLARY 3. Assume f as in Corollary 1 and suppose that (13) holds. Then taking $\delta_h = \sigma(h)^{\alpha s/(1+\alpha s)}$, $\alpha = p/(p-1-q)$. for any compact set D we have

$$|(\Omega^+ \Delta \Omega_h^+) \cap D| \leq C_D \sigma(h)^{s/(1+\alpha s)} \quad \text{for any compact set } D,$$

where $\Omega^+ = \{x \in \Omega : u(x) > 0\}$, $\Omega_h^+ = \{x \in \Omega : u_h(x) > \delta_h\}$ and $A \Delta B = (A-B) \cup (B-A)$ for any sets A and B. Moreover, if (13) holds for $p = \infty$, then

$$(F_h \cap D) \subset \mathcal{O}_{(2\sigma(h))^{1/\alpha}}(F) \cap \Omega^+ \cap D \ ,$$

where $\mathcal{O}_t(E) = \{x \in \Omega : d(x, E) < t\}$, for any $E \subset \Omega$. ■

The proof is a direct consequence of Theorems 5.7 and 5.8 of Nochetto |6| once that the assumptions of those theorems are implied by (11) and (13).

REMARK 2. The assumption (13) is well-known in some particular cases (see Nochetto |6| and its references for p=2 and Cortey-Dumond |3| for the study of the case $p \neq 2$).

REMARK 3. Corollary 1 can be also applied to show the continuous dependence of the free boundary with respecto to f in the class of functions f satisfying (10). See Díaz- Nochetto |5| for a related result.

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