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OPTIMAL GRADIENT BOUNDS FOR SOME SECOND ORDER QUASILINEAR EQUATIONS.

by

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ABSTRACT: We give a gradient estimate for any solution of a quasilinear second order equation of the form $-\mathrm{div}\,(Q(|\nabla u|)\nabla u)+f(u)=0$ in Ω with $u=k$ on $\partial\Omega$. This includes the p-Laplacian operator $Q(q)=q^{p-2}$ as well as the equation of surfaces of prescribed mean curvature $Q(q)=1/(1+q)^{1/2}$. Our gradient estimates are of the type $|\nabla u|\leq\Phi(u)$ for some suitable function Φ. The inequality becomes an equality in the one-dimensional case. This result was already known for strongly elliptic operators and $f\in C^1$. The generalization to eventual degenerate operators and $f\in C^0$ is motivated for some free boundary problems in continuum mechanics. The associated evolution problem is also considered. Detailed proofs of this preliminary report will appear elsewhere.

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1. ELLIPTIC EQUATIONS. This communication deals with some pointwise gradient estimates for nonnegative solutions of the problem

$$(1)\quad \begin{cases} -\mathrm{div}(Q(|\nabla u|)\nabla u)+f(u)=0 & \text{in } \Omega \\ u=k & \text{on } \partial\Omega , \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a regular open bounded set, k is a positive constant,

$$(2)\quad Q\in C^2(0,\infty)\cap C^0([0,\infty)),\ Q(q)>0 \text{ and } (qQ(q))'>0 \text{ if } q>0 ,$$

$$(3)\quad f\in C^0([0,\infty)) \text{ and } f(t)>0 \text{ if } t>0 .$$

Problems of this type appear in many different contexts:chemical reactions ($Q(q)=1$), non-Newtonian fluids ($Q(q)=q^{p-2}$), surfaces of prescribed mean curvature or the meniscum problem in capillarity ($Q(q)=1/(1+q^2)^{1/2}$): see references in Díaz [2] for the two first problems and Payne-Philippin [6] for the third one.

Our main result is the following

THEOREM 1. Let u be a nonnegative weak solution of (1) such that $u\in W^{2,\infty}(\{x\in\overline{\Omega}: |\nabla u|\neq 0\})\cap C^1(\overline{\Omega})$. Then, for every $x\in\overline{\Omega}$ we have

$$(4)\quad |\nabla u(x)|\leq A^{-1}(F(u(x))-\alpha(u(x)-m)) \quad \text{on } \overline{\Omega} ,$$

where $m\geq 0$ is the minimum of u on $\overline{\Omega}$,

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(5) $A(q)=\int_0^q (Q(s)s)'s\,ds$

(6) $F(t)=\int_m^t f(s)ds$

and

(7) $\alpha=\min\{0, \min_{x\in\partial\Omega} (N-1)H(x)Q(|\frac{\partial u}{\partial n}(x)|)\frac{\partial u}{\partial n}(x)\}$

with $H(x)$ being the mean curvature of $\partial\Omega$.

Proof. Let $J:\overline{\Omega}\to\mathbb{R}$ be defined by

(8) $J(x)=A(|\nabla u(x)|)-F(u(x))+\alpha u(x)$.

In order to prove (4), or equivalently $J(x)\leq\alpha m$ for any $x\in\overline{\Omega}$, we introduce the notation $D_\varepsilon=\{x\in\Omega:|\nabla u(x)|>\varepsilon\}$ for $\varepsilon>0$ and proceed in different steps: First step. We shall prove that if we define $q(x)=|\nabla u(x)|$ and

(9) $T(x)=\Delta J(x)+\frac{Q'(q(x))}{q(x)Q(q(x))}u_k(x)u_l(x)J_{kl}(x)$

then $T\in L^\infty(D_\varepsilon)$ and $T\geq 0$ on D_ε, for any $\varepsilon>0$. (In (9) and in the following we use the Einstein summation convention). Indeed, by differentiating J (in the sense of distributions), we obtain

(10) $\Delta J=(2\frac{Q'}{q}+Q'')u_iu_{ik}u_ju_{jk}+(Q+qQ')(u^2_{jk}+u_j\Delta u_j)-f(u)_ju_j-f(u)\Delta u+\alpha\Delta u$.

Using the equation in (1) and differentiating there with respect to x we get (after, at least, five minutes of computations) that

$$T=(Q+qQ')u^2_{jk}+u_ju_{jk}u_{ik}u_i(\frac{2Q'}{q}+Q''-\frac{Q'}{q^2}-\frac{(Q')^2}{qQ})+$$

$$+(u_iu_{il}u_l)^2(-\frac{Q'}{q^3}-\frac{Q''}{q^2}-\frac{Q}{q^4}+\frac{3(Q')^2}{q^2Q}+\frac{(Q')^3}{qQ^2}+\frac{Q}{q}+\frac{(Q')^2}{Q})-$$

$$-u_iu_{il}u_lf(u)(\frac{Q'}{qQ}+\frac{(Q')^2}{q^2Q^2})-\frac{f^2(u)}{Q}.$$

since the right side of the equality is a bounded function in D_ε we have $T\in L^\infty(D_\varepsilon)$. On the other hand, using Cauchy-Schwarz inequality $u_{ik}u_{ik}u_ju_j\geq u_{ik}u_iu_{jk}u_j$, as well as the identities

$$u_iu_{il}u_l=-\frac{fq^2}{Q+qQ'}+\text{terms containing } J_l$$

$$(u_iu_{il}u_l)^2=\frac{f^2q^4}{(Q+qQ')^2}+\text{terms containing } J_l$$

$$u_{ik}u_iu_{jk}u_j=\frac{f^2q^4}{(Q+qQ')^2}+\text{terms containing } J_l,$$

we conclude that $T(x)\geq 0$ for $x\in D_\varepsilon$. Second step. We claim that J cannot take its maximum value on $\partial\Omega$ unless $q\equiv 0$ on $\partial\Omega$. Indeed, since $u=K$ on $\partial\Omega$ and $u\leq k$ in Ω, it follows that $Q(|\nabla u|)\partial u/\partial n\geq 0$ on $\partial\Omega$. By the divergence theorem

$$\int_{\partial\Omega}Q(u)\frac{\partial u}{\partial n}=\int_\Omega f(u)>0.$$

Then, there is $p^*\in\partial\Omega$ such that $\frac{\partial u}{\partial n}(p^*)>0$ and, in consequence $\frac{\partial u}{\partial n}(p)>0$. But

(11) $\frac{\partial J}{\partial n}=[\frac{\partial u}{\partial n}Q'(q)+Q(q)]\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2}+(\alpha-f(u))\frac{\partial u}{\partial n}\leq 0$

Hence, from equation (1)

(12) $Q(q)(\frac{\partial^2 u}{\partial n^2}+(N-1)H\frac{\partial u}{\partial n})+Q'(q)\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2}=f(u)$ on $\partial\Omega$

(remember that $\Delta u=\frac{\partial^2 u}{\partial n^2}+(N-1)H\frac{\partial u}{\partial n}$ on $\partial\Omega$; Sperb |7|). Combining (11) and (12) we conclude that $\frac{\partial J}{\partial n}(p)\leq 0$, which is a contradiction with the Hopf's maximum principle (which can be applied because $T\geq 0$ and $p\in\partial D_\varepsilon$ for ε small enough). Third step. $J(x)$ takes its maximum value in every $p^*\in\Omega$ such that $u(p^*)=m$: To prove that, let $p\in\overline{\Omega}$ such that $J(p)=\max J(x)$. By the above step $p\in\Omega$ and $\nabla J(p)=0$. If $\nabla u(p)=0$, from the definition of J and p we conclude that $u(p)=m$. If $|\nabla u(p)|=\delta>0$ we take $\varepsilon<\delta$ and as $T\geq 0$ on D_ε, by the strong maximum principle we conclude that $J(x)\equiv J(p)$ $\forall x\in D_\varepsilon$. Since this is true for all $\varepsilon<\delta$ and J is continuous we get $J(x)=J(p)$ in $S\equiv\{x\in\Omega:|\nabla u(x)|>0\}$. Now, let $p^*\in\Omega$ such that $u(p^*)=m$. If $p^*\in S$, then $J(p)=J(p^*)$ and the statement follows. If $p\in S$, there is a largest ball B centered at p^* in which $\nabla u=0$. Then $u(x)=m$ and $\nabla u(x)=0$ in $\overline{B}$ and as $\overline{B}$ intersects S we get, from the definition of J, that $J(p^*)=J(p)$. ∎

REMARKS 1. It is not difficult to show that the estimate (4) is optimal in the sense that, in fact, the equality is true if N=1.

2. Theorem 1 extends previous results due to Payne-Phillipin |6| for the case of strongly elliptic quasilinear equations and $f\in C^1$. Our proof is also inspired on the adaptation made by Mossino |5| of Payne's method, for semilinear equations.

3. The regularity assumed on u is not restrictive. This is well-known in many important particular cases including the p-Laplacian and the minimal surfaces operators (see Di Benedetto |4|).

4. Optimal pointwise gradient estimates are of a great interest in the study of the free boundary given by the boundary of the support of u. In particular, estimate (4) is used in Díaz-Saa-Thiel |3| in order to obtain a necessary condition for the existence of the free boundary for the equation (1) (which generalizes results collected in Díaz |2|).

2. PARABOLIC EQUATIONS. Pointwise spacial gradient estimates can also be obtained for nonnegative solutions of

$$
(13)\quad \begin{cases} u_t - \operatorname{div}(Q(|\nabla u|)\nabla u) + f(u) = 0 & \text{in } (0,T)\times\Omega \\ u = k & \text{on } (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{on } \Omega, \end{cases}
$$

where Q and f satisfies (2) and (3) and $T>0$. We have

THEOREM 2. Assume that $||u_0||_\infty \leq k$ as well as

(14) f is nondecreasing or f is locally Lipschitz continuous,

(15) $Q'(s)\leq 0$ if $s>0$,

(16) $-\operatorname{div}(Q(|\nabla u_0|\nabla u_0)+f(u_0)\leq 0$ on Ω .

Let $u \in C^0([0,T]\times\overline{\Omega})$ be a nonnegative weak solution of (13) such that $u(t,\cdot)\in W^{2,\infty}(\{x\in\overline{\Omega}: |\nabla u(t,x)| \neq 0\})\cap C^3(\overline{\Omega})$. Define $m = \min_{\overline{\Omega}} u$, A and F given by (5) and (6) and let

(17) $\alpha = \min\{0, \min (N-1)H(x)Q(|\frac{\partial u}{\partial n}(t,x)|)\frac{\partial u}{\partial n}(t,x)\}$

Then if

(18) $|\nabla u_0(x)| \leq A^{-1}(F(u_0(x))) - \alpha(u_0(x)-m)$ on Ω ,

we have

(19) $|\nabla u(t,x)| \leq A^{-1}(F(u(t,x))) - \alpha(u(t,x)-m)$ on $[0,T]\times\Omega$.

Proof. Due to assumptions (14) and (16) is not difficult to show that $u_t \geq 0$ in $D'((0,T)\times\Omega)$ (this can be obtained by comparison of $u(t,\cdot)$ with $u(t+h,\cdot)$ for any $h>0$). Now, define

$$J(t,x) = A(|\nabla u(t,x)|) - F(u(t,x)) + \alpha u(t,x)$$

and

$$D_\varepsilon(t) = \{x\in\Omega:\ |\nabla u(t,x)|>\varepsilon\},\quad D_\varepsilon^T = \bigcup_{t\in(0,T)} D_\varepsilon(t)\times\{t\}.$$

In order to prove (19) (or, equivalently, $J(t,x)\leq \alpha m$ for any $(t,x)\in[0,T]\times\overline{\Omega}$) we see that

$$\Delta J = \left(2\frac{Q'}{q} + Q''\right)u_j u_{jk} u_i u_{il} + (Q+qQ')(u_{jk}u_{jk} + u_j \Delta u_j) - f(u)_j u_j - (f-\alpha)\Delta u$$

and so

$$J_t - Q\Delta J - \frac{Q'}{q}u_k u_l J_{kl} = \left[\frac{QQ'}{q} + (Q')^2\right](u_i u_j u_{il} u_{jl}) - (Q^2 + qQ'Q)(u_{jk}u_{jk}) +$$

$$+ \left[\frac{qQQ'' - QQ' - 3q(Q')^2}{q^3}\right](u_i u_k u_{ik})^2 + \left(\frac{QQ'}{q} + (Q')^2\right)u_i u_l u_{li}\Delta u + (f(u)-\alpha)f(u)$$

$$\leq \alpha(f(u)-\alpha)\left(1+\frac{qQ'}{Q}\right) \leq 0 \text{ in } D_\varepsilon^T$$

where we have used similar arguments to the elliptic part as well as (15) and $u \geq 0$. The maximum of J in $[0,T]\times\Omega$ must be attained in the parabolic boundary. But this maximum is not attained in the spacial boundary $(0,T)\times\partial\Omega$ (use that $u(t,x)\leq k$ on $(0,T)\times\overline{\Omega}$, $u = 0$ on $(0,T)\times\Omega$ and argue as in the elliptic case). On $t=0$ we have $J(0,x)\leq\alpha m$ by (18). Finally, if the maximum of J is not at $t=0$ it must be at some $(t_0,x_0)\in(0,T]\times\Omega$ and, as in the elliptic case, $\nabla J(t_0,x_0)=0$ and $u(t_0,x_0)=m$. ∎

REMARKS 6. Theorem 2 extends previous results due to Sperb |7| and Friedman-McLeod |1| for the semilinear case $Q(s)\equiv 1$. We point out that in this case assumption (16) is not needed.

7. When $\alpha\equiv 0$ assumption (18) can be removed. Indeed in this case we can prove that $|\nabla u(t,x)| \leq A(F(u(t,x))/t)$.

8. If (15) is not assumed, the problem becomes degenerate and it seems that conclusion (19) only holds for very special initial data u_0 or $N=1$.

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