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Nonlinear parabolic equations: qualitative properties of solutions

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The one dimensional porous media equation with convection

1. Introduction

This work* deals with the one dimensional porous media equation with convection. We shall concentrate our attention on nonnegative solutions of the Cauchy problem associated with the simple equation

$$u_t - (u^m)_{xx} - b.(u^\lambda)_x = 0 \qquad (t > 0,\ x \in R), \tag{1}$$

where $m \geq 1$ and $b, \lambda > 0$. Equation (1) (sometimes called the nonlinear Fokker-Planck equation) arises for example in the study of the flow of a fluid through a porous medium. Very roughly speaking, equation (1) describes a fluid moving in a vertical column (in the case of a horizontal column the gravity action is negligible and there is no convection term : $b \equiv 0$). It turns out that the value of the parameter λ is of a great relevance : $\lambda \geq 1$ occurs in downward infiltration problems, $0 < \lambda < 1$ in evaporation type problems (concerning the physical derivation of the equation, we refer the reader to [10], [24]).

From a mathematical point of view, we note that (1) is a quasilinear equation which is nonuniformly parabolic (it is degenerate near the set where $u = 0$) if $m > 1$; moreover, the convection term becomes singular (again where $u = 0$) if $0 < \lambda < 1$. As we shall indicate later, there is an extensive literature concerning the filtration problem ($\lambda > 1$), in contrast with the limited treatment given to the general case ($\lambda > 0$). Our treatment will be general, including the case $0 < \lambda < 1$ (if $\lambda = 1$, equation (1) reduces to the standard porous media equation by an easy change of variables).

In Section 2 we review some results on the existence, regularity and uniqueness of solutions of (1). Section 3 deals with the study of the free boundaries. Finally, in Section 4 we explain how the previous results can be suitably applied to certain (first order) conservation laws equations, which are hyperbolic, yet have an unbounded dependence domain.

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In Section 2 we review some results on the existence, regularity and uniqueness of solutions of (1). Section 3 deals with the study of the existence and qualitative behaviour of the free boundaries. Finally, in Section 4 we explain how the previous results can be suitably applied to certain (first order) conservation laws equations, which are hyperbolic, yet have an unbounded dependence domain.

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2. Existence, regularity and uniqueness

Since the equation (1) becomes degenerate or singular near the region $\{u = 0\}$, we cannot expect it to have classical solutions. Several weaker notions of solution may be introduced. We recall the one given in [18] for the Cauchy problem:

(CP) "to find u satisfying (1), as well as $u(0,.) = u_0(.)$",

where $u_0 \geq 0$ is a given bounded continuous function on $(-\infty,+\infty)$.

DEFINITION. *A function u is a generalized solution of* (CP) *if*

(i) *u is continuous, bounded and nonnegative in* $\overline{Q}$, *where* Q *denotes the strip* $(-\infty,+\infty) \times (0,T]$ *for some fixed* $T > 0$;

(ii) $u(0,x) = u_0(x)$ for any $x \in R$;

(iii) *for every rectangle* $P = [x_0,x_1] \times [t_0,t_1] \subset \overline{Q}$ *and* $\Psi \in C^{2,1}_{x,t}(P)$ *such that* $\Psi(x_1,t) = \Psi(x_2,t) = 0$ for any $t \in [t_0,t_1]$ *we have*

$$0 = I(u,\Psi,P) = \iint_P \{u\Psi_t + u^m\Psi_{xx} - bu^\lambda\Psi_x dxdt - \int_{x_0}^{x_1} u\Psi\Big|_{t_0}^{t_1} dx -$$

$$- \int_{t_0}^{t_1} u^m\Psi_x\Big|_{x_0}^{x_1} dt.$$

The existence of generalized solutions of (CP) may be established by following the constructive method in [21]. Thus we shall obtain a generalized solution as the pointwise limit of a decreasing sequence of classical solutions of (1). This is made in two steps: first, we construct the required sequence; secondly, we study the continuity of the corresponding limit function.

Concerning the first step, a possible choice is the following: $u_k(t,x)$ are classical solutions of (1) in $Q_k = (-k-1,\ k+1) \times (0,T]$, which satisfy the boundary and initial conditions

$$u_k(\pm(k+1),t) = M = \|u_0\|_\infty \quad \text{and} \quad u_k(x,0) = u_{0,k}(x);$$

here the sequence $u_{0,k}$ tends uniformly to u_0 on compact subsets of $(-\infty,\infty)$ and satisfies

$$1/k \leq u_{0,k} \leq M, \quad u_{0,k+1} \leq u_{0,k} \text{ for all } k \geq 1.$$

It follows from a straightforward application of the maximum principle that

$$1/k < u_k \leq M \text{ and } u_{k+1} \leq u_k \quad \text{in } \overline{Q}_k.$$

Then there exists a function u defined on $\overline{Q}$ by $u(x,t) = \lim_{k\to\infty} u_k(x,t)$ for every $(x,t) \in \overline{Q}$. It is easy to see that this function u satisfies condition (iii).

Continuity and other regularity properties of u are proven by obtaining estimates on the modulus of continuity of u_k. One method to get that is to use the Bernstein's method (see the adaptation made in [1]). A crucial point here is choosing some auxiliary functions in a convenient way. In particular, the following result is proven in [10].

THEOREM 1. *Let $u_o \in C_b(R)$, $u_o \geq 0$, such that u_o^β is Lipschitz continuous for some $\beta > 0$ with $\max\{(m-1),(m-\lambda)^+\} \leq \beta$. Then there exists at least one generalized solution u of (CP); u satisfies $(u^\alpha)_x \in L^\infty(\overline{Q})$, where $\alpha=\max\{1,\beta\}$.*

REMARK 1. The above result was previously established in [18] and [17] for $\lambda > 1$. We point out that the modulus of continuity of u given in Theorem 1 is optimal ([10]) and that, due to Nash's theorem, $u \in C^\infty$ where $u > 0$. We also note that $(u^m)_x \in L^\infty(Q)$ and u satisfies (CP) in a stronger sense, namely:

$$\iint [\{(u^m)_x - bu^\lambda\}\vartheta_x - u\vartheta_t]dxdt = \int_{-\infty}^{\infty} \vartheta(x,0)u_o(x)dx \tag{2}$$

for any $\vartheta \in C^1(\overline{Q})$ such that $\vartheta(.,T) = 0$, $\vartheta(x,t) = 0$ for $t > 0$ and $|x|$ large enough. Finally, we point out that the regularity assumptions on u_o may be considerably weakened (see [4]).

The uniqueness of generalized solutions, as well as their continuous dependence on the initial data, is a consequence of the following comparison result.

THEOREM 2. *Let u be the limit solution constructed in the proof of* Theorem 1. *Let $\overline{u}$ (respectively $\underline{u}$) be a generalized supersolution (respectively subsolution) of (CP) [i.e. satisfying (i) and $I(\overline{u},\zeta,P) \leq 0$ (respectively $I(\underline{u},\zeta,P) \geq 0$ when $\Psi \geq 0$ in (iii)]. Then for every $0 < t \leq T$ we have*

$$\int_{-\infty}^{\infty} (u(x,t) - \overline{u}(x,t))^+dx \leq \int_{-\infty}^{\infty} (u(x,0) - \overline{u}(x,0))^+dx \tag{3}$$

or, respectively,

$$\int_{-\infty}^{\infty} (\underline{u}(x,t) - u(x,t))^+dx \leq \int_{-\infty}^{\infty} (\underline{u}(x,0) - u(x,0))^+dx. \tag{4}$$

In particular, under the assumptions of Theorem 1 there exists a unique generalized solution of (CP).

Inequality (3) (or (4)) is proven previously for each classical solution u_k instead of u; the result follows easily by an already classical duality argument. By approaching suitably the solutions, some simplifications are made without loss of generality. The crucial point of this method is to obtain sharp "a priori" estimates on the test functions $\Psi(x,t)$, which solve a retrograde linear parabolic boundary value problem (see details in [10]).

REMARK 2. Theorem 2 improves on previous uniqueness results, where different restrictions on λ were made (see [18], [17] and [22]). The uniqueness of generalized solutions has been recently investigated in [4] by a different method. The main content of Theorem 2 is a comparison principle: indeed, from (3) it is obvious that $u_o \leq \overline{u}_o$ implies $u \leq \overline{u}$ on $\overline{Q}$. Finally, we point out that (3) shows that the semigroup associated with the equation (1) is a semigroup of contractions on the space $L^1(\mathbb{R})$ (see [3], [4], [23] and [24] for a different approach).

REMARK 3. The above results on existence, regularity and uniqueness of generalized solutions are, in fact, particular statements of some more general results dealing with the equation

$$u_t = \varphi(u)_{xx} + b(x,u)_x - c(x,u) \tag{5}$$

under suitable assumptions on φ,b and c. Similar results are also available for other initial-boundary value problems (see [10]). They also can be established for the case of higher space dimension (for an adaptation of the uniqueness argument see [5], [6]).

3. On the free boundaries

Comparison of solutions and conservation of mass, namely

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u_0(x)dx \qquad (t > 0)$$

([15],[11]) are useful tools in order to study the existence or nonexistence of the free boundaries $\zeta_i(t)$ $(i = 1,2)$, defined by

$$\zeta_1(t) = \inf\{x\in(-\infty,\infty) : u(x,t)>0\},\ \zeta_2(t) = \sup\{x\in(-\infty,\infty) : u(x,t)>0\}.$$

Here we assume that u_0 satisfy the conditions of Theorem 1 and moreover

$$\text{supp } u_0 = [a_1,a_2] \text{ with } -\infty < a_1 < a_2 < +\infty.$$

The behaviour of the free boundaries $\zeta_i(t)$ is quite different as to whether $\lambda \geq 1$ or $0 < \lambda < 1$.

The case $\lambda > 1$ corresponds to filtration problems and has been widely treated in the literature (see [18], [15] and [16]). In that case both curves $\zeta_i(t)$ are continuous functions on t, whose behaviour may be illustrated by the following properties:

(a) $\zeta_1(t)\downarrow-\infty$ when $t\to\infty$ if $\lambda \geq m$ and $\zeta_1(t)\downarrow a_1-K$ when $t\to\infty$ if $1 < \lambda < m$, for some $K > 0$;

(b) $\zeta_2(t)\uparrow+\infty$ when $t\to\infty$;

(c) if $u(x_0,t_0) > 0$ for some $(x_0,t_0) \in \overline{Q}$, then $u(x_0,t) > 0$ for every $t \geq t_0$.

When $\lambda = 1$ an adaptation of the Barenblatt-Pattle solution shows that now the properties (a) and (b) are not satisfied.

The case $0 < \lambda < 1$ was considered in [11]. Now there is a strong singularity in the convection term preventing the formation of the free boundary $\zeta_2(t)$, as the following theorem shows.

<u>THEOREM 3.</u> *Let $0 < \lambda < 1$ and u_0 as in Theorem 1. Then the function $v = u^{m-\lambda}$ satisfies*

$$v_x > -\frac{Cv}{t} \quad \text{in } Q, \tag{6}$$

where C is a positive constant only depending on $m-\lambda, \|u_0\|_\infty$ and T. In particular, $\zeta_2(t) = +\infty$ for every $t > 0$.

The proof of (6) consists in the study of the parabolic equation satisfied by v_x and the application of the comparison principle to a suitable subsolution. (An additional argument must be added to the proof of (6) in [11], as it has been kindly communicated to the authors by P. Bénilan. See also the proof of (6) in [14]). With respect to the other free boundary $\zeta_1(t)$, its behaviour is given in the following theorem.

<u>THEOREM 4.</u> *There exists $K > 0$ and $C > 0$ such that*

$$Ct-K < \zeta_1(t) < +\infty \quad \textit{for every } t > 0. \tag{7}$$

Moreover $\zeta_1(t)\uparrow+\infty$ when $t \to \infty$.

The second assertion of the above theorem is consequence of the principle of conservation of mass. Estimate (7) is obtained by comparing the solution u with a supersolution $\overline{u}$ of the form

$$\overline{u}(x,t) = \begin{cases} [f(x-Ct+K)]^{1/(m-\lambda)} & \text{if } x \geq Ct-K \\ 0 & \text{if } x < Ct-K \end{cases}$$

for a suitably chosen function f [11].

<u>REMARK 4.</u> Many of the above results can be established for some more general equations like (5), other initial-boundary value problems or in higher space dimension (see, for instance, [16], [12] and [9]).

4. An application to certain conservation laws equations

As a byproduct of Theorem 3, we can study the domain of dependence of the conservation law equation

$$u_t + f(u)_x = 0 \quad \text{in} \quad Q \tag{8}$$

when f is a continuous but not locally Lipschitz continuous function with $f(0) = 0$.

As is well-known ([20],[19],[7] and [2]), if f is locally Lipschitz continuous and $u_0 \in L^1(R) \cap L^\infty(R)$, then

$$\operatorname{supp} u_0 = [a_1,a_2] \text{ implies } \operatorname{supp} u(t,.) \subset [a_1-Kt,\ a_2+Kt],$$

where

$$K = \operatorname{Sup} \{|f'(r)|, -\|u_0\|_\infty \leq r \leq \|u_0\|_\infty\}.$$

The local Lipschitz continuity assumed for f may be replaced by an integral condition near the origin. In particular, in [13] it is shown that supp u(t,.) is a compact set of $(-\infty,\infty)$ for every fixed $t \geq 0$, provided that f satisfies the condition

$$\int_0 \frac{ds}{|f^{-1}(s)|} < +\infty.$$

Nevertheless, both assumptions fail when f near the origin behaves like the function $f(s) = s^\lambda$, with $0 < \lambda < 1$. In fact, an explicit example due to Kruskov and Hildebrant shows that in that case the support of u(t,.) may be unbounded for every $t > 0$. The following result shows that this property is peculiar to a class of nonlinear functions f.

THEOREM 5. *Let* $u_0 \in L^1(R) \cap L^\infty(R)$, $u_0 \geq 0$, *with* $\operatorname{supp} u_0 = [a_1,a_2]$. *Let* f *be a continuous function such that*

$$\int_0 \frac{ds}{|f(s)|} < +\infty. \tag{9}$$

Then if u *is the unique (entropy) solution of* (CP), *there exists a function* $\zeta_1(t)$ *such that* $u(t,x) > 0$ *for* $x > \zeta_1(t)$.

The idea of the proof consists in applying Theorem 3 - or, more precisely, a generalization of it - to the solutions of the equation

$$u_t - \varepsilon u_{xx} + f(u)_x = 0, \quad \epsilon > 0. \tag{10}$$

Indeed, as noted in Remark 4, the conclusion of Theorem 3 is true in a more general context, which includes, in particular, the case of equation (10)

under the assumption (9) on the convection term [12]. Finally, by well-known results (see, e.g., [4]), $\lim u_\varepsilon = u$ when $\varepsilon \downarrow 0$, and the conclusion is obtained by means of some uniform estimates on $\zeta_{1,\varepsilon}(t)$ [4].

REMARK 5. A different proof of Theorem 5 for the special case of $f(s) = |s|^{\lambda-1}s$, with $0 < \lambda < 1$, may be obtained via the estimate $u_t \geq -u/t$ proven in [8]. On the other hand, we point out that a result similar to Theorem 5 is also available for suitable N-dimensional conservation laws equations.

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A FASANO

A free boundary problem in combustion

1. Introduction

Most of the material presented here (namely the existence, uniqueness and continuous dependence theorems for Problem (P) below) is a condensed form of a joint paper with J.R. Cannon and J.C. Cavendish [1]. A section on special solutions is added. We consider an idealized model for the combustion of a half-space full of solid fuel by a half-space full of gaseous oxidizer.

Neglecting heat conduction in the solid, evaporation of the solid and compressibility and convection in the gas, we are led to the following problem.

PROBLEM (P). For any given $T > 0$, find a curve $x = s(t)$ and two functions $u(x,t)$, $v(x,t)$ such that

(i) $s \in C^1(0,t] \cap C[0,t]$,

(ii) u, v are continuous and bounded in $-\infty < x \leq s(t)$, $0 \leq t \leq T$,
u_x, v_x are continuous in $-\infty < x \leq s(t)$, $0 < t \leq T$,
u_t, v_t, u_{xx}, v_{xx} are continuous in $-\infty < x < s(t)$, $0 < t \leq T$,

(iii) the following system is satisfied:

$$u_t = \alpha u_{xx}, \quad v_t = \beta v_{xx} \text{ in } -\infty < x < s(t),\ 0 < t \leq T, \tag{1.1}$$

$$s(0) = 0, \tag{1.2}$$

$$u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), \ -\infty < x < 0, \tag{1.3}$$

$$\alpha u_x(s(t),t) = -(\gamma + u(s(t),t))\,\dot{s}(t), \quad 0 < t \leq T, \tag{1.4}$$

$$\beta v_x(s(t),t) = -(-\mu + v(s(t),t))\,\dot{s}(t), \ 0 < t \leq T, \tag{1.5}$$

$$\dot{s}(t) = \nu f(u(s(t),t)) \exp\{-\delta/(v(s(t),t) + v_0)\}, \quad 0 < t \leq T, \tag{1.6}$$

where:

(a) α, β, γ, δ, μ, ν, v_0 are positive given constants,

(b) φ, ψ are continuous on $-\infty < x \leq 0$, and

$$0 < \varphi(x) \leq \varphi^*, \qquad 0 < \psi_* \leq \psi(x) \leq \psi^*, \tag{1.7}$$