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Nonlinear parabolic equations: qualitative properties of solutions

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Qualitative properties of free boundaries for some nonlinear degenerate parabolic equations

1. Introduction

This paper is a short survey of some recent work by the authors, concerning qualitative properties of nonlinear degenerate parabolic equations. The associated stationary problem was considered by the authors in [7] by using a local comparison technique involving some kind of local radial super-solutions, which was previously introduced by the first author in [5]. There the main interest was the study of the dead core, namely the subset where the (positive) solutions vanish identically; some necessary and/or sufficient conditions for the existence of a (non-empty) dead core, together with additional information about its size and location, were obtained (see [1] and [11] for related work as well as the monograph [6]).

Here we apply the same kind of arguments to a rather large class of nonlinear (possibly) degenerate parabolic equations complemented with non-zero Dirichlet boundary conditions (see Problem (P) below). Some results for the case of pure powers, i.e., $\varphi(u) = u^m$ and $f(u) = u^p$ were obtained in [8]. Here we extend this investigation to nonlinearities φ and f which are not necessarily powers but have only a similar qualitative behaviour (see assumptions (H_1) and (H_2) below) near the origin. We refer the reader to [2] - [4] and [13] - [15] for other related work.

Very roughly speaking, a large part of our results seem to be new in this more general situation, and some of them extend to the case $0 < p < 1$ theorems known for $p \geq 1$. More detailed information can be found below (see also [8][9]). An extended version of this survey, including also work in [8], with full proofs and many complementary results and applications will appear in [9]: in particular, we will give there applications to some reaction-diffusion systems arising in combustion theory (see [2][8]) and population dynamics with nonlinear diffusion ([12]).

2. Main theorems

In this section we consider the following degenerate parabolic problem:

$$\begin{cases} u_t - \Delta\varphi(u) + f(u) = 0 & \text{in } Q = \Omega\times(0,\infty) \\ u(x,t) = h(x,t) & \text{on } \Sigma = \partial\Omega\times(0,\infty) \\ u(x,0) = u_o(x) & \text{in } \Omega, \end{cases} \qquad (P)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, under the following assumptions:

φ is a continuous increasing function, $\varphi(0) = 0$ and $\varphi' > 0$, $\varphi'' > 0$ in $(0,\infty)$; (2.1)

f is continuous, $f(0) = 0$; there exists a continuous increasing function f_o such that $0 \le f_o(s) \le f(s)$ for every $s \ge 0$; (2.2)

$h \in L^\infty(\Sigma)$, $h \ge 0$ in Σ; $u_o \in L^\infty(\Omega)$, $u_o \ge 0$ on Ω. (2.3)

Our main result in this section is the following theorem.

THEOREM 2.1. *Suppose that $u \in C(\overline{Q})$, $u \ge 0$, is a solution of problem (P) with (2.1) - (2.3). Moreover assume that*

$$\int_0^1 \frac{ds}{\left[\int_0^s f_o(\varphi^{-1}(t))dt\right]^{1/2}} < +\infty \qquad (H_1)$$

and

$$\int_0^1 \frac{ds}{f_o(s)} < +\infty \qquad (H_2)$$

are satisfied. Then there exists $T_o > 0$ such that for every $t \ge T_o$ we have

$$N(u(.,t)) \equiv \{x \in \Omega | u(x,t) = 0\} \supset \{x \in \Omega | d(x, \bigcup_{\tau>0} S(h(.,\tau)) \ge L\}$$

where S denotes the support of the corresponding function, and L is a constant depending on φ, f_o, h, u_o, Ω and N.

The main tool for the proof of Theorem 2.1 is the following Lemma, which generalizes Lemma 2.1 in [7]. Its proof can be found in [6].

LEMMA 2.1. *If we define $\eta(s) = \psi^{-1}1/N(s)$, where*

$$\psi_\mu(r) = \int_0^r \frac{ds}{\left[\mu\int_0^s \frac{1}{2} f_o(\varphi^{-1}(t))dt\right]^{1/2}},$$

then for any $x_o \in \Omega$ we have

$$-\Delta\eta(|x-x_o|) + \frac{1}{2} f_o(\varphi^{-1}(\eta(|x-x_o|))) \ge 0 \quad in \quad \Omega. \qquad (2.4)$$

Moreover $\eta(0) = \eta'(0) = 0$ and $\eta(s) > 0$ if $s \ne 0$.

Sketch of the proof of Theorem 2.1. We define (this is an idea adapted from [10])

$$\overline{u}(x,t) = \varphi^{-1}(\eta(|x-x_o|) + \varphi(U(t)))$$

where $\eta(s)$ and $\psi_\mu(r)$ are as in Lemma 2.1 (we remark that by (H_1) we have $\psi_\mu(r) < +\infty$) and U is a positive solution of the ordinary differential equation

$$\frac{dV}{dt} + \frac{1}{2} f_o(V) = 0 \qquad (2.5)$$

$$V(0) = \|u_o\|_{L^\infty}.$$

It is not difficult to see that, as a consequence of (H_2), we have $U(t) = 0$ for any $t \ge T_o = \int_0^{\|u_o\|_{L^\infty}} \frac{ds}{2f_o(s)}$.

From (2.1), (2.2) we obtain:

$$\overline{u}_t - \Delta\varphi(\overline{u}) + f_o(\overline{u})$$

$$= \frac{d}{dt}(\varphi^{-1}(\eta(|x-x_o|) + \varphi(U(t))) - \Delta\eta(|x-x_o|) +$$

$$+ f_o(\varphi^{-1}(\eta(|x-x_o|) + \varphi(U(t))) \ge$$

$$\ge \frac{\varphi'(U)}{\varphi'(\varphi^{-1}(\eta+\varphi(U)))}\frac{dU}{dt} - \Delta\eta + \frac{1}{2} f_o(\varphi^{-1}(\eta)) + \frac{1}{2} f_o(U) \ge$$

$$\ge \frac{dU}{dt} - \Delta\eta + \frac{1}{2} f_o(\varphi^{-1}(\eta)) + \frac{1}{2} f_o(U) \ge 0$$

by (2.4) and (2.5), taking into account that

$$\eta+\varphi(U) \geq \varphi(U)$$

implies the inequality

$$\varphi^{-1}(\eta+\varphi(U)) \geq U,$$

hence

$$\varphi'(\varphi^{-1}(\eta+\varphi(U))) \geq \varphi'(U),$$

once again by (2.1).

Concerning the boundary condition, it is easy to show that if we have

$$0 \leq h(x,t) \leq \|h\|_{L^\infty} \leq \varphi^{-1}(\eta(|x-x_0^\circ|)) \leq \bar{u}(x,t),$$

then the inequality $h(x,t) \leq \bar{u}(x,t)$ holds at the boundary. Indeed, if $x \notin S(h(.,\tau))$, $h(x,\tau) = 0$ and the inequality is automatically satisfied. If not, it is sufficient that

$$\varphi(\|h\|_{L^\infty}) \leq \eta(|x-x_0|) \text{ for any } x \in \partial\Omega;$$

this is equivalent to

$$\psi_{1/N}[\varphi(\|h\|_{L^\infty})] \leq |x-x_0|$$

or, otherwise stated, be

$$d(x_0, \bigcup_{\tau\geq 0} S(h(.,\tau)) \geq L,$$

where $L = \psi_{1/N}(\varphi(\|h\|_{L^\infty}))$.

As for the initial condition, it is easily seen that

$$0 \leq u_0(x) \leq \|u_0\|_{L^\infty} \leq \varphi^{-1}(\eta(|x-x_0|)) + \varphi(\|a_0\|_{L^\infty})).$$

Thus we obtain (recall (2.2)):

$$\begin{cases} u_t - \Delta\varphi(u) + f_0(u) \leq 0 \leq \bar{u}_t - \Delta\varphi(\bar{u}) + f_0(\bar{u}) & \text{in } Q \\ u(x,t) \leq \bar{u}(x,t) & \text{on } \Sigma \\ u_0(x) \leq \bar{u}(x,0) & \text{in } \Omega ; \end{cases}$$

it follows from comparison results for problem (P) with f_0 that $0 \leq u(x,t) \leq \bar{u}(x,t)$. The proof ends by recalling that $u(x_0,t) = 0$ if $t \geq T_0$ and x_0 satisfies the above inequality.

REMARK 2.1. It is also possible to prove similar results when replacing $f(u)$ by $c(x,t).f(u)$, with $c(x,t) \geq 0$ (see [8][9]). This seems to be particularly interesting for applications to reaction-diffusion systems.

REMARK 2.2. If $\varphi(s) = s^m$, $f_0(s) = s^p$, then (H_1) is equivalent to $p < m$ and (H_2) is equivalent to $p < 1$. Now, for $m = 1$, (H_1) and (H_2) coincide. But if $\varphi(s) = s$ and f_0 is not a power, then (H_1) implies (H_2) but the converse is not true (see [10]).

REMARK 2.3. Our theorem extends some work by Kersner [14] for the case $N = 1$, and also, for $h \equiv 0$ and $\Omega = \mathbb{R}$, results by Kalashnikov [13] and Véron [15] concerning extinction of solutions in finite time. On the other hand, for $m = 1$, $h \equiv 1$, $u_0 \equiv 1$, estimates for the dead core as $N(u(.,t)) \supset \{x \in \Omega | d(x,\partial\Omega) \geq L\}$ can be found in [2] (see also [8]).

REMARK 2.4. If (H_2) is satisfied but (H_1) does not hold, it is still possible to get estimates of the kind

$$0 \leq u(x,t) \leq U(t)$$

extending in this way some work by Berstch, Nanbu and Peletier [4], respectively Véron [15]. Similar arguments also allow us to prove the estimate

$$N(u(.,t)) \supset \{x \in \Omega - S(u_0) | d(x, S(u_0) \cup (\bigcup_{\tau\geq 0} S(h(.,\tau))) \geq L'\}$$

for some constant L'.

REMARK 2.5. The same technique of proof allows us also to obtain local (namely depending on the point $x_o \in \Omega$ and on the norm $\|u_o\|_{L^\infty(B(x_o,\varepsilon))}$) estimates for the extinction time $T_o(\varepsilon > 0$; see [2],[8],[9]).

THEOREM 2.2. *Assume that* $u \in C(\overline{Q})$, $u \geq 0$, *is a solution of problem* (P) *with* (2.1) - (2.3) *and* (H_1). *If* $x_o \in \Omega$ *satisfies*

$$0 \leq u_o(x) \leq \varphi^{-1}(\eta(|x-x_o|), 1/N) \tag{2.5}$$

for any $x \in B(x_o,\varepsilon)$, *where* $\varepsilon = \psi_{1/N}(\varphi(M))$, $M = \|u\|_{L^\infty(Q)}$, $\eta(r,\mu) = \psi_\mu^{-1}(r)$, ($\psi_\mu(r)$ *as above*), *then* $u(x_o,t) = 0$ *for any* $t > 0$.

Sketch of the proof. On the set $B(x_o,\varepsilon) \times (0,\infty)$ define the function

$$\overline{u}(x) = \varphi^{-1}(\eta(|x-x_o|), 1/N).$$

Now, reasoning as in [6] we obtain

$$\begin{cases} u_t - \Delta\varphi(u) + f_o(u) \leq 0 \leq \overline{u}_t - \Delta\varphi(\overline{u}) + f_o(\overline{u}) & \text{in } B(x_o,\varepsilon) \times (0,\infty) \\ u(x,0) = u_o(x) \leq \overline{u}(x) = \varphi^{-1}(\eta(|x-x_o|)) & \text{in } B(x_o,\varepsilon) \\ u(x,t) \leq M \leq \overline{u}(x) & \text{on } \partial B(x_o,\varepsilon) \times (0,\infty), \end{cases}$$

where $\|u\|_{L^\infty(Q)} \leq M$. Then a comparison argument gives $0 \leq u(x,t) \leq \overline{u}(x)$.

REMARK 2.6. Theorem 2.2 improves on some results in [4] for $h \equiv 0$; indeed, we only need the local estimate (2.5). If $\varphi(s) = s^m$, $f_o(s) = \lambda s^p$, then

$$u(x) = K_\lambda |x-x_o| \frac{2}{1-f_m}$$

for some $K_\lambda > 0$.

THEOREM 2.3. *Assume* $u \in C(\overline{Q})$, $u \geq 0$, *is a solution of the problem*

$$\begin{aligned} u_t - \Delta u + f_o(u) &= 0 && \text{in } Q \\ u &= 0 && \text{on } \Sigma \\ u(x,0) &= u_o(X) && \text{on } \Omega, \end{aligned}$$

where (2.2),(2.3) *and* (H_1) *are satisfied. If, moreover,* $u_t \in L^\infty(Q)$, *then we have the estimate*

$$S(u(.,t)) \subset S(u_o) + B(0, \psi_{1/N}(Ct))$$

for any $t > 0$ *and some* $C > 0$, *where* C *depends on* $\|u_t\|_{L^\infty(Q)}$.

Sketch of the proof. Let $t_o > 0$ and $x_o \in S(u(.,t_o)) - S(u_o)$. We consider the region

$$R(t_o) = \{(x,t) | 0 < t < t_o,\ u(x,t) > 0,\ x \notin S(u_o)\}$$

and the function

$$\overline{u}(x) = \eta(|x-x_o|, 1/N).$$

The function $z(x,t) = u(x,t) - \overline{u}(x)$ satisfies

$$z_t - \Delta z + B(x,t)z \leq 0 \quad \text{on} \quad Q$$

for a suitable $B(x,t)$; then the Strong Maximum Principle implies that z takes its maximum on the parabolic boundary of $R(t_o)$. But, on the other hand, $0 = u(x,t) \leq \overline{u}(x)$ for $(x,t) \in \partial_p R(t_o) - S(u_o)$, and $z(x_o,t_o) > 0$. Hence there exists a point $(\overline{x},\overline{t})$ in $\partial S(u_o) \times (0,t_o)$ satisfying $\overline{u}(\overline{x}) < \overline{u}(\overline{x},\overline{t})$. This in turn implies

$$d(x_o,S(u(.,t)) \leq |x-x_o| \leq \psi_{1/N}(u(x,t)) \leq \psi_{1/N}(u(\overline{x},\overline{t}) - u(\overline{x},0)) \leq$$

$$\leq \psi_{1/N}(Ct) \leq \psi_{1/N}(Ct_o),$$

which gives the result.

REMARK 2.7. The proof follows an idea of Evans and Knerr [10]. If $\Delta u_o \in L^\infty(\Omega)$, $u_o \in H_o^1(\Omega)$ and $h \in L^\infty(\Sigma) \cap H^1(\Sigma)$, then, following a theorem by Bénilan-Ha, $u_t \in L^\infty(Q)$.

REMARK 2.8. If $f_o(s) = s^p$, $0 < p < 1$, then $\psi_{1/N}(Ct) = Ct^{1-p/2}$.

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