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On the Initial Growth of the Interfaces in Nonlinear Diffusion-Convection Processes

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Abstract. We study the qualitative behavior of the fronts or interfaces generated by the solutions of the equation

$$u_t = (u^m)_{xx} + b(u^\lambda)_x,$$

where $m, \lambda > 0$ and b is a real number, $b > 0$. In particular we focus our attention on the waiting time phenomenon and give "necessary" and sufficient conditions on the initial data $u_0(x)$ in order to have such a property. Since the convection term in the equation introduces an asymmetry, a separated study of the left and right fronts is needed. The results depend in a fundamental way on the values of m and λ. In particular, the different answer with respect to the case of pure diffusion ($b = 0$) occurs for $0 < \lambda < 1$, where the left front does not exist and the right one may already be reversing.

1. Introduction.

The nonlinear diffusion-convection processes referred to in the title of this paper are those described by the equation

$$u_t = (u^m)_{xx} + b(u^\lambda)_x, \tag{1}$$

where $m > 0$, $\lambda > 0$ and $b \in \mathbb{R}$, $b > 0$.

This equation arises as a model for a number of different physical phenomena. For instance, when u denotes unsaturated soil-moisture content, the equation describes the infiltration of water in a homogenous porous medium and some natural conditions in this context are $m > 1$ and $\lambda > 0$ (Bear [**3**], Phillip [**15**]). The equation also occurs in the study of the flow of a thin viscous film over an inclined bed for the specific exponents $m = 3$ and $\lambda = 4$ (Buckmaster [**5**]). By analogy with the classical equation from statistical mechanics (Chandrasekhar [**6**]), equation (1) is often referred to as the nonlinear Fokker-Planck equation. Equation (1) is also used in connection with transport of thermal energy in plasma (then $0 < m < 1$ and $\lambda = 1$, Rosenau and Kamin [**16**]). Finally, the equation has additional interest as a generalization of the well-known equation of Burgers approximating the associated hyperbolic conservation law equation.

It is a well-known fact that nonnegative solutions u of (1) may give rise to interfaces (or free boundaries) separating regions where $u > 0$ from ones where $u = 0$. These fronts are relevant in the physical problems modelled and their occurrence is essentially due to slow diffusion ($m > 1$) or to convective phenomena dominating over diffusion ($\lambda < m$) (See, e.g. Gilding [**10,12**] and Diaz-Kersner [**8**]). Another kind of front (the time of extinction of the solution) is intrinsic to fast diffusion ($m < 1$) and will not be considered here (Berryman-Holland [**4**]). Note finally that we cannot expect, in general, to have classical solutions of (1), and that discontinuities of the gradient of solutions take place on the interfaces.

The existence, uniqueness and regularity of weak solutions of the Cauchy problem, the Cauchy-Dirichlet problem and the first boundary-value problem for (1) was given by Diaz-Kersner [**7**] for $m \geq 1$ and $\lambda > 0$ and later extended by Gilding [**11**] to any $m > 0$ (see the references in these articles for earlier works).

The main goal of this work is to study the initial growth of the interfaces

$$\begin{aligned} \varsigma_-(t) &= \inf\{x : u(x,t) > 0\} \\ \varsigma_+(t) &= \sup\{x : u(x,t) > 0\} \end{aligned}$$

in relation to different values of m and λ. For the sake of simplicity, we restrict the discussion to the Cauchy problem for (1). However, we remark that the initial growth of ς_- and ς_+ will depend only on the behavior of the initial data $u_0(x)$ near the boundary of its support $[\varsigma_-(0), \varsigma_+(0)]$. Thus our results may be extended to solutions of other initial boundary-value problems associated with (1) (the extension to the study of "interior" fronts is possible as well).

To be explicit, we shall focus our attention on continuous nonnegative weak solutions of the problem

$$(CP) \qquad \begin{cases} u_t = (u^m)_{xx} + b(u^\lambda)_x & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where u_0 is a given continuous nonnegative function on $\mathbb{R}$ which, for simplicity, is assumed to satisfy

$$(2) \qquad u_0(x) > 0 \text{ on } (a_-, a_+) \quad \text{and} \quad u_0(x) = 0 \text{ on } \mathbb{R} - (a_-, a_+).$$

In analogy with the case of nonconvective flows in porous media ($b \equiv 0, m > 1$) the interfaces may be stationary until a certain finite time, called the "waiting time", of the interface (some discussions of this property for the porous media equation can be found in Aronson [**2**], Knerr [**14**] and Vazquez [**17**]). In the following, we shall place special emphasis on giving "neccesary and sufficient" conditions on u_0 in order to have a waiting time. In contrast with the case $b \equiv 0$, a separate study of ς_- and ς_+ is needed because the convective term introduces an inherent asymmetry into the problem.

As we will show, the initial growth of the interfaces is different in each one of the following regions of (λ, m) parameter space.

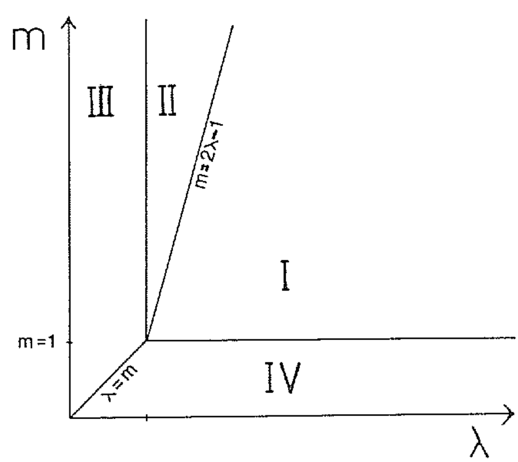

Figure 1

It is a curious fact that the interplay between diffusion and convection may be completely different in other contexts. For instance, the asymptotic behavior of solutions of (CP) was studied by Grundy [**13**], where a different decomposition of the (λ, m) parameter space occurs.

In order to describe our results we remark that the behavior of the interface $\varsigma_\mp(t)$ depends only on the values of m and λ as well as on the local behavior of the initial data $u_0(x)$ near $a_\mp$. For the sake of simplicity we shall use the notation $u_0(x) \sim |x - a_\mp|^\alpha$ to indicate that $u_0(x) \leq C\,|x - a_\mp|^\alpha$ and $u_0(x) \geq C\,|x - a_\mp|^{\tilde{\alpha}}$ for x near $a_\mp$, where C, $\tilde{C} > 0$ and $\tilde{\alpha}$ is some number $< \alpha$.

As a final general remark we point out that the conclusions of this paper also hold upon replacing pointwise comparison conditions on u_0 by more general assumptions indicating how the mass $M_\pm(x)$ of u_0 grows near $a_\mp$ $\left(\text{here } M_\pm(x) = \pm \int_x^{a_\mp} u_0(s)ds\right)$. For instance, conditions assuring the existence of a waiting time can be formulated in terms of the relation $\limsup_{x\to a_\mp} M_\mp(x)\,|x-a_\mp|^{-\alpha-1} = 0$ (see Alvarez-Diaz [**1**]; a pioneer work in this direction is Vazquez [**17**], where the case of nonconvection was examined). We also point out that our proof of the nonexistence of a waiting time always leads to growth estimates on $\varsigma_\pm(t)$.

In Section 2 we study the interfaces for (λ, m) in the region I, defined by $\{(\lambda,m) : \lambda \geq \frac{1}{2}(m+1), m > 1\}$. This case corresponds to a slow diffusion dominating convection, in the sense that the results are of the same nature as in the equation without convection; the interface $\varsigma_\pm(t)$ has a waiting time if and only if $u_0(x) \sim |x-a_\mp|^{2/(m-1)}$. The presence of the convection term only leads to a natural displacement of the interfaces compared to the case without convection (such a property occurs for any value of λ when there is some interface).

Section 3 is devoted to the case in which (λ, m) belongs to the region II defined by $\{(\lambda,m) : 1 < \lambda < \frac{1}{2}(m+1)\}$. Some differences compared to the case of pure diffusion appear, namely the criterion for the existence of a waiting time for $\varsigma_-(t)$ is weaker $(u_0(x) \sim |x-a_-|^{1/(\lambda-1)})$ while for the front $\varsigma_+(t)$ a stronger criterion is needed: $u_0(x) \leq C_0\,|x-a_+|^{1/(m-\lambda)}$ for x near a_+ and $C_0 = [b(m-\lambda)/m]^{1/(m-\lambda)}$. We also show that this last condition is "necessary" for the existence of a waiting time. We would describe this by saying that convection already dominates over diffusion but in a weak way, because many other properties of $\varsigma_\mp(t)$, for small t, remain unchanged: $\varsigma_-(t)$ is finite and nonincreasing, $\varsigma_+(t)$ is finite and nondecreasing, etc.

The region III, defined by $\{(\lambda,m) : \lambda \leq 1 \text{ and } \lambda < m\}$, is examined in Section 4 and reflects the case in which there is a great contrast with pure diffusion phenomena (especially when $0 < \lambda < 1$) because then the interface $\varsigma_-(t)$ does not exist (Diaz-Kersner [**8,9**]). Here we show that convection dominates strongly over diffusion; namely, it is enough to know that $u_0(x) \leq C\,|x-a_+|^{1/(m-\lambda)}$ for some $C < C_0$ to conclude that $\varsigma_+(t)$ is initially a reversing front and that $\varsigma_+(t) \leq a_+ - \bar{C}t^{(m-\lambda)/(m+1-2\lambda)}$ for some suitable $\bar{C} > 0$ and any t small. Moreover, if $u_0(x) \geq C\,|x-a_+|^{1/(m-\lambda)}$ for some $C > C_0$ then $\varsigma_+(t)$ is initially a progressing front and $\varsigma_+(t) \geq a_+ + \underline{C}t^{(m-\lambda)/(m+1-2\lambda)}$ for some suitable $\underline{C} > 0$ and any t small. Moreover, if $u_0(x) \geq C\,|x-a_+|^{1/(m-\lambda)}$ for some suitable $C > C_0$ then $\varsigma_+(t)$ is initially a progressing front and $\varsigma_+(t) \geq a_+ + \underline{C}t^{(m-\lambda)/(m+1-2\lambda)}$ for some suitable

$\underline{C} > 0$ and any t small.

We finish the introduction with two remarks. The first is that the region IV, defined by $\{(\lambda, m) : m \leq 1 \text{ and } \lambda \geq m\}$, corresponds to a fast or linear diffusion with a weak convection. In this case none of the fronts exist (Gilding [**12**]), hence this region is not of interest to us. Second, we can consider a more general formulation of the equation,

$$u_t = (u^m)_{xx} + f(u)_x, \tag{3}$$

where $m > 0$ and f is a continuous real function. This program will be developed elsewhere. As an illustration, consider the function $f(s) = \mu s + bs^\lambda$ with $\mu, b, \lambda > 0$. Making the change of variables ($\bar{x} = x$, $\bar{t} = t - \mu x$) it is easy to see that the function $v(\bar{x}, \bar{t}) = u(t, x)$ will satisfy the equation (1). Thus a waiting time phenomenon for v means that u follows the characteristics of the hyperbolic conservation law $u_t = f(u)_x$ during some finite time (see figure 2). Obviously, a systematic study of the different possibilities can be carried out in terms of the values of λ, m and the assumptions on the behavior of u_0 near the boundary of its support.

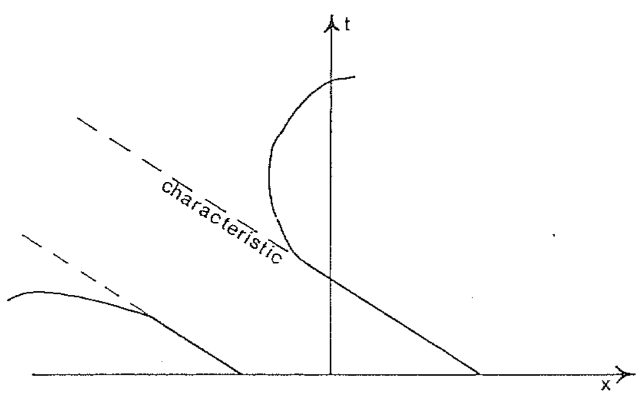

Figure 2

2. Region I: Slow diffusion dominating convection.

We shall assume in this section that $\lambda \geq \frac{1}{2}(m+1)$ and $m > 1$. Under these assumptions it turns out that the initial behavior of the interfaces is the same as for the porous medium equation ($b \equiv 0$). We start by recalling a well-known result

THEOREM 1 Gilding [**10**]. *Let u be a weak solution of the Cauchy problem (CP) and suppose that*

$$u_0(x) \le C\,|x - x_0|^{2/(m-1)} \tag{4}$$

for some constants $C > 0$, $\delta > 0$ and for x such that $|x - x_0| \le \delta$. Then there exists a finite time $t^* > 0$ such that $u(x_0, t) = 0$ for any $t \in [0, t^*)$.

IDEA OF THE PROOF: Without loss of generality we may choose $x_0 = 0$. Then, it is enough to show that the separable function

$$\bar{u}(x,t) = (Ax^2)^{1/(m-1)}\{\tau_0/(\tau_0 - t)\}^{\alpha}$$

is a supersolution on the set $(x,t) \in [-\delta, \delta] \times [0, \tau_0)$, for some positive constants A, α and τ_0.

□

Concerning the nonexistence of a waiting time, we have

THEOREM 2. *Let u be a weak solution of (CP) and suppose that*

$$u_0(x) \ge C\,|x - a_{\mp}|^{\gamma}, \tag{5}$$

for some $\gamma \in (0, 2/(m-1))$, $C > 0$, $\delta > 0$ and for x respectively such that $0 \le x - a_- \le \delta$ or $0 \le a_+ - x \le \delta$. Then there exist positive constants τ and $\bar{c}$ such that

$$\varsigma_-(t) \le a_- - \bar{c}t^{1/(2-\gamma(m-1))} \tag{6}$$

for any $t \in [0, \tau]$, and (if in addition $\gamma(m-1) \ge 1$)

$$\varsigma_+(t) \ge a_+ - \bar{c}t^{1/(2-\gamma(m-1))} \tag{7}$$

for any $t \in [0, \tau]$. In particular, $u(a_{\mp}, t) > 0$ for any $t > 0$.

In order to prove Theorem 2 we will define a family of auxiliary functions depending on two parameters K and $\bar{x}$ in the following way:

$$\underline{u}(x,t;K,\bar{x}) = \left[K^2 t - KM_1(1 - M_2 t)(x - \bar{x})\right]_+^{1/(m-1)}, \tag{8}$$

where $[a]_+ = \max\{a, 0\}$ and M_1 and M_2 will be chosen later. The following lemma shows that $\underline{u}$ is a subsolution of the equation on sets of the form $[-\delta, \infty) \times [0, \tau]$ for some suitable $\tau > 0$.

LEMMA 1. *Let M and δ be given positive constants. Then there exist M_1, M_2 and $\tau > 0$ such that $\underline{u}$ is a subsolution of (1) in the region $(x,t) \in [-\delta, +\infty) \times [0,\tau]$ for every $K \in (0, M)$ and $\bar{x} \in [-\delta, 0]$.*

PROOF: It is easy to see that if u satisfies (1) then the function $v = m/(m-1)u^{m-1}$ satisfies the equation

$$\mathcal{L}(v) = -v_t + (m-1)vv_{xx} + (v_x)^2 + q(v^p)_x = 0,$$

where

$$q = \frac{bm\lambda}{m+\lambda-2}\left(\frac{m-1}{\lambda}\right)^{\frac{m+\lambda-2}{m-1}}, \quad p = 1 + \frac{\lambda-1}{m-1}.$$

Now, take $\underline{v}(x,t;K,\bar{x}) = \frac{m}{m-1}\left[K^2 t - KM_1(1-M_2 t)(x-\bar{x})\right]_+$. A direct computation gives

$$\mathcal{L}(\underline{v}) = \frac{m}{m-1}K\left[K\left\{\frac{mM_1^2}{m-1}(1-M_2 t)^2 - 1\right\} - M_1 M_2 (x-\bar{x}) + \right.$$
$$\left. + pM_1\left(\frac{m}{m-1}\right)^{p-1} q(M_2 t - 1)\left\{K^2 t - KM_1(1-M_2 t)(x-\bar{x})\right\}^{p-1}\right]_+ .$$

In order to prove that $\mathcal{L}(\underline{v}) \geq 0$ in $[-\delta, +\infty) \times [0,\tau]$ for some $\tau > 0$ we study the following two cases separately.

Case 1: $x \leq -K$. Since $K \in (0, M)$, $\bar{x} \in [-\delta, 0]$ and $\lambda \geq \frac{1}{2}(m+1)$ then $\underline{v}$ is uniformly bounded in $[-\delta, +\infty) \times [0,\tau]$ and moreover $p - 1 \geq \frac{1}{2}$. Now, let

$$\tau = \min\{1/2M_1, 1\} \quad \text{and} \quad M_1 \geq 2\left(\frac{m-1}{m}\right)^{1/2}.$$

Then, there exists $N = N(M, \delta) > 0$ such that

$$\mathcal{L}(\underline{v}) \geq \frac{m}{m-1}KM_1|x|\left(M_2 - N(1+M_1)^{1/2}\right),$$

and so, if we take $M_2 = N(1+M_1)^{1/2}$ we have $\mathcal{L}(\underline{v}) \geq 0$.

Case 2: $x \geq -K$. Since $\underline{v} \equiv 0$ if $x \geq 2K/M_1$ it suffices to assume $x < 2K/M_1$. Then

$$\mathcal{L}(\underline{v}) \geq \frac{m}{m-1}K^2\left[\frac{m}{m-1}\frac{M_1^2}{4} - 1 - 2N(1+M_1)^{1/2} + M_1 N(1+M_1)^{1/2}\right].$$

Therefore, if M_1 is big enough we conclude that $\mathcal{L}(\underline{v}) \geq 0$.

□

PROOF OF THEOREM 2: We shall first prove (7). Without loss of generality, we may assume $a_+ = 0$, $1 < \gamma(m-1) < 2$ and $u_0(x) = C|x|^\gamma$ for $x \in [-\delta, 0]$. We shall compare the function $\underline{u}$, defined by (8), and u in a region $[-\delta, \infty) \times [0, \tau]$ when the parameter $\bar{x}$ belongs to $(-\delta, 0)$ and τ is given in Lemma 1. First of all, we remark that by the continuity of u, there exists a value $\theta > 0$ such that $u(-\delta, t) \geq \theta$ for every $t \in [0, \tau]$ (recall (2)). The initial inequality $\underline{u}(x, 0; K, \bar{x}) \leq C|x|^\gamma$ is verified if and only if

$$[-M_1 K(x - \bar{x})]_+ \leq C^{m-1}|x|^{\gamma(m-1)}. \tag{9}$$

Since $\gamma(m-1) > 1$, by a convexity argument, (9) is satisfied if we choose

$$K = K(\bar{x}) = \frac{\gamma(m-1)C^{m-1}}{M_1}|\bar{x}|^{\gamma(m-1)-1}.$$

Next, choosing $\bar{x}$ small enough we have

$$\underline{u}(-\delta, t; K(\bar{x}), \bar{x}) \leq \theta \leq u(-\delta, t) \quad \text{for any } t \in [0, \tau]$$

and

$$\underline{u}(x, 0; K(\bar{x}), \bar{x}) \leq u_0(x) \qquad\qquad \text{if } x \in [-\delta, +\infty).$$

Then, by the comparison principle, we conclude that

$$\underline{u}(x, t; K(\bar{x}), \bar{x}) \leq u(x, t) \ \text{ in } [-\delta, +\infty) \times [0, \tau].$$

A direct computation shows that $\underline{u}(0, t; K(\bar{x}), \bar{x}) > 0$ if

$$t > t(\bar{x}) = \frac{M_1^2|\bar{x}|^{2-\gamma(m-1)}}{\gamma(m-1)C^{m-1} + M_1^2 M_2|\bar{x}|^{2-\gamma(m-1)}}. \tag{10}$$

Then for $t_0 \in (0, \tau)$ we choose $\bar{x} = -\beta t_0^{1/(2-\gamma(m-1))}$ with $\beta > 0$ large enough, and derive from (10) that

$$\varsigma_+(t_0) \geq \left(\frac{\gamma(m-1)C^{m-1}\beta^{\gamma(m-1)-1}}{M_1^2(1 - M_2 t_0)} - \beta\right) t_0^{1/(2-\gamma(m-1))}; \tag{11}$$

this proves the inequality (7) because t_0 is arbitrary. By a well-known result (Gilding [**10**, Theorem 3]) we have, in fact, $u(a_+, t) > 0$ for any $t > 0$.

The proof of the inequality (6) is similar. As a matter of fact, the study of the left interface $\varsigma_-(t)$ is easier due to the sign ($b > 0$) of the convection term (see Remark 1). In particular the proof that $u(t, a_-) > 0$ given in Knerr [**14**] remains true without any significant change.

□

REMARK 1: The assumptions (4) and (5) are the same as in the the proof of the corresponding results for the porous media equation, $b \equiv 0$ (see, for instance, Knerr [**14**]). (We point out that the proof of Theorem 2 is different from the one given in Knerr [**14**] and that the estimates (6) and (7) seem to be new in the literature.) As already mentioned, in region I the diffusion dominates the convection. Nevertheless, it is clear that the presence of the convection terms modifies the behavior of the solution (and so of the fronts) compared to the solution of the pure diffusion equation, *independent of the value of* λ. In particular, it is shown in Alvarez-Diaz [**1**] that if we denote by w the solution of the Cauchy problem (CP) without convection ($b \equiv 0$) and by $\xi_-(t)$ and $\xi_+(t)$ the left and right fronts generated by w, then

$$\varsigma_-(t) \leq \xi_-(t) \quad \text{and} \quad \xi_+(t) \leq \varsigma_+(T). \tag{12}$$

Notice that (12) makes sense only when $m > 1$ because otherwise $\xi_-(t) = -\infty$ and $\xi_+(t) = +\infty$.

3. Region II: Convection weakly dominating diffusion.

We shall now assume that $1 < \lambda < \frac{1}{2}(m+1)$. As we shall show, in this case convection dominates diffusion weakly and therefore the influence of convection is different for each front, making it necessary to study $\varsigma_-(t)$ and $\varsigma_+(t)$ separately.

We start by studying the left interface $\varsigma_-(t)$. A sufficient condition for the existence of a waiting time is given by the following result.

THEOREM 3. *Let u be a weak solution of (CP) and assume that*

$$u_0(x) \leq C\,|x - a_-|^{1/(\lambda-1)} \quad \text{if } 0 \leq x - a_- \leq \delta \tag{13}$$

for some positive constants C and δ. Then there exists a finite time $t^* > 0$ such that $u(a_-, t) = 0$ for any $t \in [0, t^*]$.

PROOF: Without loss of generality we may choose $a_- = 0$ and $\delta \leq 1$. As in Theorem 1, we shall derive our conclusion from the construction of a suitable supersolution. We define

$$\bar{u}(x,t) = \left(\frac{1}{K_1 - K_2 t}\right)^{1/(m-1)} |x|^{1/(\lambda-1)} \tag{14}$$

with

$$K_1 = \min\left\{1, \quad C^{1-m}, \quad M^{1-m}\delta^{(m-1)/(\lambda-1)}\right\}, \quad M = \|u_0\|_{L^\infty(\mathbb{R})},$$
$$K_2 = (m-1)\left[\frac{m(m-\lambda+1)}{(\lambda-1)^2} + b\frac{\lambda}{\lambda-1}\right].$$

From the choice of K_2, $\lambda < m$ and $m + 2 - 2\lambda > 1$, it is not difficult to verify that $\underline{u}$ satisfies the inequality

$$\bar{u}_t - (\bar{u}^m)_{xx} - b(\bar{u}^\lambda)_x \geq 0.$$

Moreover, from the definition of K_1 we have

$$\begin{aligned} &\bar{u}(x,0) \geq C|x|^{1/(\lambda-1)} && \text{if } x \in [-\delta,\delta] \\ &\bar{u}(\delta,t) \geq M && \text{if } t \in [0,t^*], \end{aligned}$$

with $t^* = K_1/K_2$. Then, applying the comparison principle to the region $(-\delta,\delta) \times [0,t^*)$ we conclude that $0 \leq u \leq \bar{u}$ and the result follows.

□

The nonexistence of any waiting time for the front ς_- can also be proved when the opposite inequality in (13) holds.

THEOREM 4. *Let u be a weak solution of (CP) and suppose that*

$$u_0(x) \geq C\,|x - a_-|^\gamma \ \text{ if } 0 \leq x - a_- \leq \delta \tag{15}$$

for some $\gamma \in (0, 1/(\lambda-1))$, $C > 0$ and $\delta > 0$. Then there exist positive constants τ and $\bar{c}$ such that

$$\xi_-(t) \leq a_- - \bar{c}t^{1/1-\gamma(\lambda-1)}$$

for every $t \in [0,\tau]$. In particular, $u(a_-,t) > 0$ for any $t > 0$.

PROOF: As before, it suffices to take $a_- = 0$ and to construct a suitable subsolution, which we shall choose as the following "traveling wave solution" depending on two parameters $K > 0$ and $\bar{x} \in (0,\delta)$:

$$v(x,t;K,\bar{x}) = \mu_K\left([x + Kt - \bar{x}]_+\right) \tag{17}$$

where μ_K is defined by

$$y = m\int_0^{\mu_K(y)} \frac{s^{m-2}}{K - bs^{\lambda-1}}\,ds. \tag{18}$$

Given τ small enough, we shall compare u and v on the region $(-\infty,\delta)\times[0,\tau)$. From the continuity of u and (2) there exists $\tau > 0$ such that $u(\delta,t) \geq \theta$ for any $t \in [0,\tau]$. Then we shall choose $\bar{x}$ and K such that

$$v(\delta,t;K,\bar{x}) \leq \theta \leq u(\delta,t) \quad \text{for } t\in[0,\tau] \tag{19}$$

and

$$v(x,0;K,\bar{x}) \leq u_0(x) \qquad \text{for } x \in (-\infty,\delta). \tag{20}$$

In order to have (20) we take $K = K(\bar{x})$ and since

$$\sup_{x\in\mathbb{R}} v(x,0;K,\bar{x}) = bK^{1/(\lambda-1)},$$

using (15) it is enough to choose

$$K = K(\bar{x}) = (C/b)^{(\lambda-1)}\bar{x}^{\gamma(\lambda-1)}. \tag{21}$$

On the other hand, by taking $\bar{x}$ small enough it is easy to see that (19) holds. Then, by the comparison principle we conclude that $v \leq u$ in $(-\infty,\delta)\times[0,\tau)$. Now a direct computation shows that $v(0,t;K(\bar{x}),\bar{x}) > 0$ if $t > t(\bar{x}) = (b/c)^{(\lambda-1)}\bar{x}^{1-\gamma(\lambda-1)}$. Since $\gamma < 1/(\lambda-1)$, it is clear that $t(\bar{x}) \to 0$ when $\bar{x} \to 0^+$. Now let $t_0 \in [0,\tau)$ and choose $\bar{x} = \beta t_0^{1/(1-\gamma(\lambda-1))}$ with $\beta > 0$ small. Then we have

$$\varsigma_-(t_0) \leq a_- - \left[\beta - (b/c)^{(\lambda-1)}\beta^{1-\gamma(\lambda-1)}\right] t_0^{1/(1-\gamma(\lambda-1))}.$$

Then, if β is large enough, we have $\beta > (b/c)^{(\lambda-1)}\beta^{1-\gamma(\lambda-1)}$. This proves the estimate (16). The fact that $u(a_-,t) > 0$ for any $t > 0$ again follows from the initial positivity of $u(a_-,t)$ and Theorem 2 of Gilding [**12**].

□

We shall now study the initial behavior of the right front $\varsigma_+(t)$. We start by giving a sufficient condition for the waiting time property.

THEOREM 5. *Let u be a weak solution of (CP) and suppose that*

$$u_0(x) \le C\,|x - a_+|^{1/(m-\lambda)} \quad \text{if } 0 \le a_+ - x \le \delta \tag{22}$$

for some positive constants δ and $C > 0$, $C < C_0$, where

$$C_0 = \left(\frac{b(m-\lambda)}{m}\right)^{1/(m-\lambda)}. \tag{23}$$

Then there exists a finite time $\tau^* > 0$ such that $u(a_+, t) = 0$ for every $t \in [0, \tau^*]$.

PROOF: We first remark that the function $z(x) = C_0[a_+ - x]_+^{1/(m-\lambda)}$ is a stationary solution of the equation. From (22) and the continuity of u we deduce that there exists $\tau^* > 0$ such that $u(a_+ - \delta, t) \le z(-\delta)$ for any $t \in [0, \tau^*]$. The conclusion follows from the comparison principle applied to the solutions u and z in the region $(a_+ - \delta, +\infty) \times [0, \tau^*]$.

□

The optimality of the assumption (22) is given by the following result, showing the expanding nature of $\varsigma_+(t)$ under an opposite hypothesis on u_0.

THEOREM 6. *Let u be a weak solution of (CP) and suppose that*

$$u_0(x) \ge C\,|x - a_+|^{1/(m-\lambda)} \quad \text{if } 0 \le a_+ - x \le \delta \tag{24}$$

for some $\delta > 0$ and $C > C_0$, C_0 given in (23). Then there exist positive constants τ and $\bar{c}$ such that

$$\varsigma_+(t) \ge a_+ + \bar{c}t^{(m-\lambda)/(m-2\lambda+1)}$$

for every $t \in [0, \tau]$. In particular, $u(a_+, t) > 0$ for every $t > 0$.

PROOF: Without loss of generality we may assume $a_+ = 0$. We define a traveling wave solution, depending on two parameters, in the following way:

$$w(x, t : K, \bar{x}) = \nu_K\left([Kt - x + \bar{x}]_+\right) \tag{26}$$

where ν_K is defined by

$$y = m \int_0^{\nu_K(y)} \frac{s^{m-2}}{K + bs^{\lambda-1}}\, ds. \tag{27}$$

Given τ small enough, we shall compare u and w on the region $(-\delta, \infty) \times [0, \tau)$. By the comparison principle it suffices to have

$$w(x, 0; K, \bar{x}) \leq u_0(x) \text{ for } x \in (-\delta, +\infty) \tag{28}$$

and

$$w(-\delta, t; K, \bar{x}) \leq u(-\delta, t) \text{ for } t \in [0, \tau]. \tag{29}$$

As for (28) we remark that, by (24), it is equivalent to the following condition:

$$[\bar{x} - x]_+ \leq m \int_0^{C|x|^{1/(m-\lambda)}} \frac{s^{(m-2)}}{K + bs^{(\lambda-1)}}\, ds \tag{30}$$

for $x \in (-\delta, \infty)$. But if $\alpha > 0$ we have

$$\begin{aligned} m \int_0^{C|x|^{1/(m-\lambda)}} & \frac{s^{(m-2)}}{K + bs^{(\lambda-1)}}\, ds \\ & \geq m \int_\alpha^{C|x|^{1/(m-\lambda)}} \frac{s^{(m-2)}}{\frac{K}{\alpha^{(\lambda-1)}} s^{(\lambda-1)} + bs^{(\lambda-1)}}\, ds \\ & = \frac{m}{(m-\lambda)\left(\frac{K}{\alpha^{(\lambda-1)}} + b\right)} \left(C^{m-\lambda}|x| - \alpha^{m-\lambda}\right). \end{aligned}$$

Since $C > C_0$, there exists $\varepsilon > 0$ such that

$$\frac{mC^{m-\lambda}}{(m-\lambda)(\varepsilon + b)} = 1.$$

Then, choosing

$$K = K(\bar{x}) = \varepsilon \left(\frac{(m-\lambda)(\varepsilon + b)}{m}\right)^{\frac{(\lambda-1)}{(m-\lambda)}} |\bar{x}|^{\frac{(\lambda-1)}{(m-\lambda)}}$$

and $\alpha = (K/\varepsilon)^{1/(\lambda-1)}$ we deduce (30). Finally, condition (29) is satisfied for $\bar{x}$ small enough (use the same argument as in the proof of Theorem 4). Then $w \leq u$ in $(-\delta, \infty) \times [0, \tau)$. A direct computation shows that $w(0, t; K(\bar{x}), \bar{x}) > 0$ if

$$t > t(\bar{x}) = \frac{1}{\varepsilon} \left(\frac{m}{(m-\lambda)(\varepsilon + b)}\right)^{\frac{(\lambda-1)}{(m-\lambda)}} |\bar{x}|^{1-(\lambda-1)/(m-\lambda)}.$$

Since $1 > (\lambda-1)/(m-\lambda)$, then $t(\bar{x}) \to 0$ as $\bar{x} \to 0^-$. On the other hand, from the inequality $w \le u$ in $(-\delta,\infty)\times[0,\tau)$ we deduce that

$$\varsigma_+(t_0) \ge -|\bar{x}| + K(\bar{x})t_0 \text{ for any } t_0 \in [0,\tau).$$

Then, choosing $\bar{x} = -\beta t_0^{(m-\lambda)/(m-2\lambda+1)}$ with β small, we have that

$$\varsigma_+(t_0) \ge \left[-\beta + \varepsilon\left(\frac{(m-1)(\varepsilon+b)\beta}{m}\right)^{(\lambda-1)/(m-1)}\right] t_0^{(m-\lambda)/(m-2\lambda+1)}$$

Since $(\lambda-1)/(m-\lambda) < 1$ we conclude that the coefficient of $t_0^{(m-\lambda)/(m-2\lambda+1)}$ is positive if we choose β big enough and, since t_0 is arbitrary in $[0,\tau)$, (25) is proved. As before, this also proves that $u(a_+,t) > 0$ for any $t > 0$.

□

REMARK 2: When (λ, m) is in the region II the asymmetry caused by the convection is quite clear. So, in that case the previous results and the inequalities

$$\frac{1}{\lambda-1} \ge \frac{2}{m-1} \ge \frac{1}{m-\lambda}$$

show that, since the convection plays an important role, the condition on u_0 to have a waiting time in the interface $\varsigma_-(t)$ is stronger than for $\varsigma_+(t)$ and also stronger than $\varsigma_-(t)$ with (λ,m) belonging to the region I.

REMARK 3: Some other qualitative properties on ς_- and ς_+ are well-known when that $\lambda > 1$ and $m > 1$. Thus in Gilding [10,12] it is shown that $\varsigma_-(t)$ is monotone and nonincreasing and $\varsigma_-(t) \searrow -\infty$ as $t \nearrow +\infty$, $\varsigma_+(t)$ is monotone nondecreasing and $\varsigma_+(t) \nearrow +\infty$ if $\lambda \ge m$ or $\varsigma_+(t) \nearrow A_+$ if $\lambda < m$ for some real number A_+, as $t \nearrow +\infty$. He also proves that both interfaces satisfy the equation

$$\varsigma_\pm'(t) = \{-[(u^m)_x + bu^\lambda]/u\}\,(\varsigma_\pm(t),t)$$

in a certain sense.

4. Region III: Convection strongly dominating over diffusion.

In this last section we shall assume that $\lambda \le 1$ and $\lambda < m$. The first important difference compared to the cases in which (λ,m) belongs to regions I or II appears already for $\lambda = 1$. Indeed, in that case it is well-known that $m > 1$ implies the existence of the interfaces $\varsigma_-(t)$ and $\varsigma_+(t)$ (see Diaz-Kersner [8]), nevertheless, the following result shows that in this case $\varsigma_-(t)$ can never exhibit a waiting time.

THEOREM 7. *Let $m > 1$, $\lambda = 1$, and u be a weak solution of (CP). Then $\varsigma_-(t) \leq a_- - bt$ for all $t \geq 0$.*

PROOF: We introduce the transformation $v(x,t) = u(x - bt, t)$. Then it is easy to see that v satisfies the equation $v_t = (v^m)_\alpha (v^m)_{xx}$ and $v(x,0) = u_0(x)$. Moreover if $\xi_-(t)$ is the left face generated by v, i.e. $\xi_-(t) = \inf\{x : v(x,t) > 0\}$, then we have $\varsigma_-(t) = \xi_-(t) - bt$. Since the fronts for the porous media equation are nonincreasing the result follows at once.

□

When $\lambda < 1$ it turns out that the convection dominates diffusion in such a strong way that, in fact, the left interface $\varsigma_-(t)$ does not exist, i.e. $\inf\{x : u(x,t) > 0\} = -\infty$ for any $t > 0$. This result was first proved in Diaz-Kersner [8] for $m \geq 1$, and later, for any $m > 0$, in Gilding [**12**].

The behavior of $\varsigma_+(t)$ is completely different than that of $\varsigma_-(t)$ when (λ, m) is in the region III. Thus, this front does exist for any value of (λ, m) in that region (see Diaz-Kersner [8] for $\lambda < 1 \leq m$ and Gilding [**12**] for the general case $\lambda < m$). Besides, the stationary solution $z(x) = C_0[a_+ - x]_+^{1/(m-\lambda)}$, C_0 given by (23), shows that (CP) admits solutions with an infinite waiting time. The following result gives a stronger result ensuring that, under a suitable assumption on u_0, $\varsigma_+(t)$ is in fact a "reversing front" near $t = 0$ (the property $\varsigma_+(t) \searrow -\infty$ as $t \nearrow +\infty$ without any additional assumption on the initial value u_0, was first proved in Diaz-Kersner [8], see also Gilding [**12**,Theorem 5]).

THEOREM 8. *Let u be a weak solution of (CP) and suppose that*

$$u_0(x) \leq C\,|x - a_+|^{1/(m-\lambda)} \quad \text{if } 0 \leq a_+ - x \leq \delta, \tag{31}$$

for some positive constants δ and C, $C < C_0$, where C_0 is given by (23). Then there exist positive constants $\bar{c}$ and τ such that

$$\varsigma_+(t) \leq a_+ - \bar{c}t^{(m-\lambda)/(m-2\lambda+1)} \tag{32}$$

for all $t \in [0, \tau]$.

PROOF: We assume $a_+ = 0$ and introduce the following traveling wave solution

$$v(x,t;K) = \alpha_K\left([-x - Kt]_+\right) \tag{33}$$

where α_K is defined by

$$y = m \int_0^{\alpha_K(y)} \frac{s^{m-2}}{bs^{\lambda-1} - K}\, ds, \tag{34}$$

and K is a positive constant to be chosen such that $K < b$ if $\lambda = 1$.

In order to compare v with u in the region $(-\delta, +\infty) \times [0, \tau)$, with τ suitably chosen, we notice that the condition

$$v(x, 0; K) \geq u_0(x) \text{ for } x \in (-\delta, +\infty) \tag{35}$$

is verified if

$$|x| \geq m \int_0^{C_*|x|^{1/(m-\lambda)}} \frac{s^{m-2}}{bs^{\lambda-1} - K}\, ds$$

for $-\delta < x \leq 0$ for any $C_* \in [C, C_0)$. But if $x \in (-\delta, 0]$ we have

$$\begin{aligned} m \int_0^{C_*|x|^{1/(m-\lambda)}} & \frac{s^{m-2}}{bs^{\lambda-1} - K}\, ds \\ & \leq m \int_0^{C_*|x|^{1/(m-\lambda)}} \frac{s^{m-2}}{bs^{\lambda-1} - K\left(\frac{s}{C_*|x|^{1/(m-\lambda)}}\right)^{(\lambda-1)}}\, ds \\ & = \frac{mC_*^{m-\lambda}}{(m-\lambda)\left(b - \frac{K}{C_*^{(\lambda-1)}}\delta^{(\lambda-1)/(m-\lambda)}\right)} |x|. \end{aligned}$$

Since $C_* < C_0$, there exist positive constants ε and ε' such that

$$1 - \frac{mC_*^{(m-\lambda)}}{(m-\lambda)(b-\varepsilon)} = \varepsilon'.$$

Thus (35) holds if we take

$$K = \varepsilon C_*^{(\lambda-1)} \delta^{(\lambda-1)/(m-\lambda)}. \tag{36}$$

On the other hand, by continuity, we deduce from (31) that for any $C_* \in (C, C_0)$ there exists a $\tau^* > 0$ such that

$$u(-\delta, t) \leq C_* \delta^{1/(m-\lambda)} \quad \text{if} \quad t \in [0, \tau_*].$$

Then, choosing $\tau \leq \tau^*$ the condition

$$v(-\delta, t; K) \geq u(-\delta, t) \text{ for } t \in [0, \tau) \tag{37}$$

holds if

$$\alpha_K([\delta - Kt]_+) \geq C_* \delta^{1/(m-\lambda)}. \tag{38}$$

From the definition of α_K we deduce that (38) holds if $\delta - K\tau \geq (1-\varepsilon')\delta$, that is,

$$\tau \leq \varepsilon' \frac{\delta}{K} = \frac{\varepsilon'}{\varepsilon C_*^{\lambda-1}} \delta^{(m-2\lambda+1)/(m-\lambda)}. \tag{39}$$

Then, for τ small enough we deduce from the comparison principle that $v \geq u$ in $(-\delta, +\infty) \times [0, \tau)$. This shows that

$$\varsigma_+(t_0) \leq -\varepsilon C_*^{\lambda-1} \delta^{(\lambda-1)/(m-\lambda)} t_0 \quad \text{if } t_0 \in [0, \tau).$$

Obviously, the same arguments may be applied to any $\delta_0 \in (0, \delta)$. Thus, in particular, by choosing

$$\delta_0 = \left(\frac{\varepsilon C_*^{\lambda-1}}{\varepsilon'} t_0 \right)^{(m-\lambda)/(m-2\lambda+1)}$$

we obtain the estimate (32) and the proof is finished.

□

Our last result shows that even when $\varsigma_+(t) \searrow -\infty$ as $t \nearrow +\infty$, the front $\varsigma_+(t)$ may expand initially under an assumption on u_0 which is the opposite to that in the above theorem.

THEOREM 9. *Let u be a weak solution of (CP) and suppose that*

$$u_0(x) \geq C\,|x - a_+|^{1/(m-\lambda)} \quad \text{if } 0 \leq a_+ - x \leq \delta, \tag{40}$$

for some positive constants δ and C, $C > C_0$, where C_0 is given by (23). Then there exist positive constant $\bar{c}$ and τ such that

$$\varsigma_+(t) \geq a_+ + \bar{c} t^{(m-\lambda)/(m-2\lambda+1)} \tag{41}$$

for all $t \in [0, \tau]$.

PROOF: As before, we can assume without loss of generality that $a_+ = 0$. We introduce the traveling wave solution defined in Theorem 6 but now for $\bar{x} = 0$, i.e.,

$$w(x, t; K) = \nu_K([Kt - x]_+)$$

where ν_K is defined by (27). To compare u with w we first remark that the condition

(42) $$w(x,0;K) \leq u_0(x) \quad \text{for } x \in (-\delta, +\infty)$$

holds if

$$|x| \leq m \int_0^{C_*|x|^{1/(m-\lambda)}} \frac{s^{m-2}}{bs^{\lambda-1} - K}\, ds$$

for $-\delta < x \leq 0$ and any $C_* \in (C_0, C)$. But for $x \in (-\delta, 0]$, arguing as in Theorem 8 we have

$$m \int_0^{C_*|x|^{1/(m-\lambda)}} \frac{s^{m-2}}{bs^{\lambda-1} - K}\, ds \geq \frac{mC_*^{m-\lambda}}{(m-\lambda)\left(b + \frac{K}{C_*^{(\lambda-1)}}\delta^{(\lambda-1)/(m-\lambda)}\right)}|x|.$$

Since $C_* > C_0$, there exist positive constants ε and ε' such that

$$\frac{mC_*^{(m-\lambda)}}{(m-\lambda)(b+\varepsilon)} - 1 = \varepsilon.$$

Thus, taking

$$K = \varepsilon C_*^{(\lambda-1)}\delta^{(\lambda-1)/(m-\lambda)}.$$

condition (42) holds. On the other hand, from (40) and the continuity of u we deduce that for any $C_* \in (C, C_0)$ there exists a $\tau^* > 0$ such that

$$u(-\delta, t) \geq C_*\delta^{1/(m-\lambda)}.$$

Then, choosing $\tau \leq \tau^*$, the condition

(43) $$w(-\delta, t; K) \leq u(-\delta, t) \quad \text{for } \; t \in [0, \tau)$$

holds if

(44) $$\nu_K(K\tau + \delta) \leq C_*\delta^{1/(m-\lambda)}.$$

As before, (44) is verified when

(45) $$\tau \leq \frac{\varepsilon'}{\varepsilon C_*^{(\lambda-1)}}\delta^{(m-2\lambda+1)/(m-\lambda)}.$$

It now follows from the comparison principle that $w \leq u$ in the region $(-\delta, \infty) \times [0, \tau]$ for τ small enough. This shows that

$$\varsigma_+(t_0) \geq \varepsilon C_*^{(\lambda-1)}\delta^{(\lambda-1)/(m-\lambda)}t_0 \quad \text{if } t_0 \in [0, \tau).$$

and the same conclusion holds if δ is replaced by any $\delta_0 \in [0, \delta]$. Then, choosing

$$\delta_0 = \left(\frac{\varepsilon}{\varepsilon'} C_*^{(\lambda-1)} t_0\right)^{(m-\lambda)/(m-2\lambda+1)}$$

we obtain the estimate (41) and this establishes the result.

□

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