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Free Boundary Problems: Theory and Applications

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On the Initial Behaviour of Interfaces in Nonlinear Diffusion–Convection Processes

We study the initial behaviour of the fronts, or interfaces, generated by the solutions of the equation

$$u_t = (u^m)_{xx} + b(u^\lambda)_x,$$

where $m, \lambda > 0$ and b is a real number, $b > 0$. In particular we will focus our attention on the waiting time phenomenon. We give necessary and sufficient conditions on the initial value $u_0(x)$ assuring that the fronts do not move until a finite (or infinite) time. Since the convection term at the equation introduces an asymetry, a separated study of the left and right fronts is needed. The results depend in a fundamental way on the values of m and λ.

We consider the Cauchy problem

$$(CP) \begin{cases} u_t = (u^m)_{xx} + b(u^\lambda)_x & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}, \end{cases}$$

where $m, \lambda > 0$ and b is a real number, $b > 0$. For simplicity, we assume that u_0 is a given continuous nonnegative function such that

$$u_0(x) > 0 \quad \text{on} \quad (a_-, a_+) \quad \text{and} \quad u_0(x) = 0 \quad \text{on} \quad \mathbb{R} \setminus (a_-, a_+).$$

Problems of this nature arise as models for a number of different physical phenomena: infiltration of water in a homogeneous porous medium ([3]); transport of thermal energy in plasma([10]) etc. The existence, uniqueness and regularity of weak solutions have been studied by several authors in the last ten years (see [5],[7] and their references).

The main goal of this communication is to present the results of the article of the authors [2] where the initial behaviour of the interfaces

$$\xi_-(t) = \inf\{x : u(x,t) > 0\} \quad \text{and} \quad \xi_+(t) = \sup\{x : u(x,t) > 0\}$$

is studied. In analogy with the case of nonconvective flows in porous media ($b \equiv 0$, $m > 1$) the interfaces may be stationary until a certain time, called the waiting time of this interface (see e. g. [11] and its references). We will give necessary and sufficient conditions on u_0 in order to have a waiting time for $\xi_-(t)$ or $\xi_+(t)$. In contrast with the nonconvective case ($b \equiv 0$) a seperate study of $\xi_-(t)$ and $\xi_+(t)$ is needed due to the asymmetry introduced by the convection.

It turns out that the initial behaviour of the interfaces is different in each one of the following regions of (λ, m) parameter space.

Notice that this decomposition of the (λ, m) parameter space is different to the one shown in [9] when studying the asymptotic behaviour of the interfaces $\xi_\pm(t)$ depends only on the values of m and λ as well as on the local behaviour of the initial data u_0 near the points $a_\pm$. For the sake of the explanation we shall use the notation $u_0(x) \sim |x - a_\pm|^\alpha$, $\alpha > 0$, to indicate that $u_0(x) \leq C|x - a_\pm|^\alpha$ for x near $a_\pm$ and for some positive constant C when proving the existence of the waiting time, or that $u_0(x) \geq C|x - a_\pm|^{\tilde\alpha}$ for x near $a_\pm$ and for some $C > 0$ and $\tilde\alpha < \alpha$.

Region I is defined by $\{(\lambda, m) \; \lambda \geq 1/2(m + 1) \quad m > 1\}$ and corresponds to the case in which the (slow diffusion) dominates over convection. Indeed, the results are of the same nature as in the equation without convection: the interface $\xi_\pm$ has a waiting time (finite in this case) if and only if

$$u_0(x) \sim |x - a_\pm|^{2/(m-1)}. \tag{1}$$

The presence of the convection term only leads to a natural displacement of the interfaces compared to the case without convection. In fact such a property occurs for any value of λ (see a proof in [1]).

In region II, $\{(\lambda, m) \; 1 < \lambda < 1/2(m + 1)\}$ some differences compared to the case of pure diffusion appears, namely the criterion for the existence of a waiting time becomes different for each interface. The convection helps in the case of the front $\xi_-(t)$ and the waiting time holds if and only if

$$u_0(x) \sim |x - a_-|^{1/(\lambda-1)}. \tag{2}$$

In the contrast to that, the formation of the waiting time for $\xi_+(t)$ is more difficult and so this property holds if and only if

$$u_0(x) \leq C_0 |x - a_+|^{1/(m-\lambda)} \tag{3}$$

with

$$C_0 = [b(m - \lambda)/m]^{1/(m-\lambda)} \tag{4}$$

(notice that the right hand side function in (3) is a stationary solution of the differential equation and that when (λ, m) is in region II the following inequality holds $1/(\lambda - 1) \geq 2/(m - 1) \geq 1/(m - \lambda)$. We would say that when (λ, m) is in region II convection already dominates over diffusion but in a weak way because many other proporties of the fronts $\xi_\pm(t)$ remain unchanged with respect to the case of pure diffusion: $\xi_-(t)$ is finite and nonincreasing in t, $\xi_+(t)$ is finite nondecreasing etc. (see [8] for the proof of these and other qualitative properties of $\xi_\pm(t)$).

The region III defined by $\{(\lambda, m) : \lambda \leq 1 \quad \text{and} \quad \lambda < m\}$, corresponds to the case in which the convection leads to greatest contrast with respect to the case of the pure diffusion equation. Indeed, it was already well-known that in this case the interface $\xi_-(t)$ does not exist ([6],[8]). Concerning $\xi_+(t)$ we prove that the criterion for the existence of

a waiting time is the same than (3) and beside that this waiting time, when existing, is infinite. To be explicit, we show that if

$$u_0(x) \leq C|x - a_+|^{1/(m-\lambda)} \quad \text{for some} \quad C < C_0, \tag{5}$$

then $\xi_+(t)$ is initially a reversing front and

$$\xi_+(t) \geq a_+ - kt^{(m-\lambda)/(m+1-2\lambda)} \quad \text{for } t \text{ small and some} \quad k > 0. \tag{6}$$

If by contrary, u_0 satisfies

$$u_0(x) \geq C|x - a_+|^{1/(m-\lambda)}, \quad \text{for some} \quad C > C_0, \tag{7}$$

then $\xi_+(t)$ is initially a progressing front and

$$\xi_+(t) \geq a_+ + kt^{(m-\lambda)/(m+1-2\lambda)} \quad \text{for } t \text{ small and some} \quad k > 0. \tag{8}$$

(Estimates on the growth of $\xi_\pm$ similar to the ones given in (6) and (7) are also shown in [2] when (λ, m) is in the region I or II, with suitable changes in the time exponent).

The last region, region IV $\{\lambda, m\} : m \leq 1 \quad \text{and} \quad \lambda \geq m\}$ corresponds to a fast or linear diffusion with a weak convection and then do not exist any of the fronts ([8]).

The conclusions of our paper can be also obtained under more general assumptions on the initial datum u_0. Namely, it is possible to replace the pointwise behaviour assumptions on u_0 of the type $u_0(x) \sim |x - a_\pm|^\alpha$ by more general assumptions indicating how the mass of u_0 grows near the points $a_\pm$ (here that mass is defined by $M(x) = \pm \int_x^{a_\pm} u_0(s)ds$). A proof of that can be found in [1] (a pioneer result in this direction but for the nonconvective equation, is [11]).

Under some technical assumptions, the above results on the initial behaviour of the fronts can be generalized to the case of more general formulation such as, for instance,

$$u_t = (u^m)_{xx} + f(u)_x, \tag{9}$$

where $m > 0$ and f is a continuous real function. In particular when $f'(0) \neq 0$, as, for instance, $f(s) = \mu s + bs^\lambda$ with $\mu, b, \lambda > 0$ our results explain some interesting qualitative behaviour of the fronts. Indeed, making the change of variables $\overline{x} = x$ and $\overline{t} = t - \mu x$, it is easy to see that the function $v(\overline{x}, \overline{t}) = u(t, x)$ satisfies the equation (1) and so the waiting time phenomenon for v means that the support $u(t, \cdot)$ follows the characteristics of the hyperbolic conservation laws equation

$$u_t = f(u)_x$$

during some finite time (seeFigure 2). Obviously, a systematic study of the different possibilities is now obtained for different values of λ and m.

The proofs are based on the construction of suitable super or subsolutions depending on several parameters which later are taken such that they converge to some suitable real numbers (see [2] for details). As a final remark we notice that the results only need information on the behaviour of the initial datum $u_0(x)$ near the boundary of its support $[a_-, a_+]$. So, our conclusions may be extended to solutions of other initial–boundary value problems associated to the same equation.
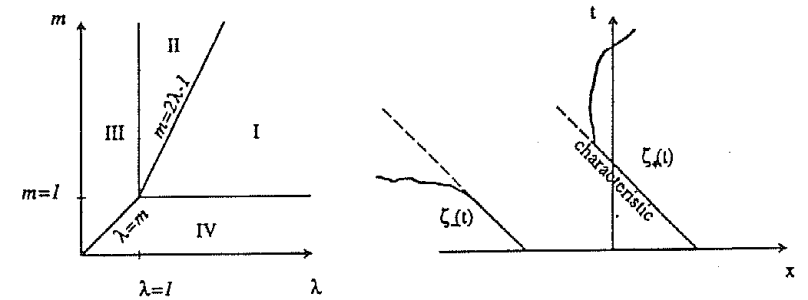
References

[1] ALVAREZ, L. AND DIAZ, J. I. Sobre un criterio de comparación en masa para le ecuatión de los medios porosos con convecccción, *Comm. to the X CEDYA*, Univ. of Valencia, 1987.

[2] ALVAREZ, L., DIAZ, J. I. AND KERSNER. R. On the initial growth of the interfaces in nonlinear diffusion-convection processes. To appear in the proceedings of the Mikroprogram *Nonlinear diffusion equations and their equilibrium states*, J. Serrin, W. Ni and L. A. Peletier editors Sept. 1986. Math. Science Research Institute serie, Springer.

[3] BEAR, J. *Dynamics of fluid in porous media*, American Elsevier Publishing Company, New York, 1972.

[4] BUCKMASTER, J. Viscous sheets advancing over dry beds, *J. Fluid. Mech.* 81 (1977) 735–756.

[5] DIAZ, J. I. AND KERSNER, R. On a nonlinear degenerate parabolic equation in infiltration or evaporation through a porous medium, *J. Differential Equations* 69 (1987) 368–403.

[6] DIAZ, J. I. AND KERSNER, R. Non existence d'une de frontieres libres dans une equation dégénérée en theorie de la filtration, *C. R. Acad. Sci. Paris* 296 (1983) 505–508.

[7] GILDING, B. H. Improved theory for a nonlinear degeneate parabolic equation, Twente University of Technology, Dept. of Applied Math. Memorandum 587 (1986) 42pp.

[8] GILDING, B. H. The occurrence of interfaces in nonlinear diffusion–advection processes. *Arch. Rational mech. Anal.* (to appear).

[9] GRUNDY, R. E. Asymptotic solution of a model nonlinear convective diffusion equation, *IMA Journal of Appl. Math.* 31 (1983) 121–137.

[10] ROSENAU, P. AND KAMIN, S. Thermal waves in an absorbing and convective medium, *Physica* 8D (1983) 273–283.

[11] VASQUEZ, J. L. The interface of one–dimensional flows in porous media, *Trans. Amer. Math. Soc.* 285 (1984) 717–737.

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