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Convexity and Starshapedness of Level Sets for Some Nonlinear Parabolic Problems

A subtitle for this lecture might be: How to treat infinitely many free boundary value problems in ten minutes. It is a report on some essential results from [3].

Consider the following reaction-diffusion problem in ring-shaped domains $\Omega \setminus G$:

$$(P) \quad \begin{cases} u_t - \Delta_p(\phi(u)) + f(u) = 0 & \text{in } (0, \infty) \times (\Omega \setminus G), \\ \phi(u) \equiv 1 & \text{on } (0, \infty) \times G, \\ \phi(u) = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{on } \Omega, \end{cases}$$

where ϕ is positive and nonincreasing (for positive arguments) $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2} \nabla v)$, ∇ denotes the spatial gradient and $p > 1$ is a real number. Notice that Problem (P) is in fact an initial boundary value problem and that $\phi(u) \equiv 1$ is just a convenient way of writing a boundary condition on ∂G . We always assume that $\phi(u_0) \equiv 1$ on \overline{G} . Suppose that Ω , G and the level sets $\{x \in \Omega \mid u_0(x) \geq \text{const.}\}$ of the initial data are starshaped or convex. What can be said about the spatial level sets $\Omega_c(t) := \{x \in \Omega \mid u(t, x) \geq c\}$ of u at fixed time $t > 0$ or about the level sets $\Omega_c := \{(t, x) \in (0, \infty) \times \Omega \mid u(t, x) \geq c\}$ in space and time?

The following answers can be given:

Theorem A. (On Starshapedness). Suppose that Ω , G and the level sets $\{x \in \Omega \mid u_0(x) \geq \text{const.}\}$ of the initial data are starshaped with respect to $x_0 \in G$. Let f be nonnegative and monotone nonincreasing and $f(0) = 0$. If u is a solution of problem (P) and if u_0 satisfies $\Delta\phi(u_0) - f(u_0) \geq 0$, then for any $T > 0$ and for any $0 < t < T$ the spatial level sets of u are starshaped with respect to x_0 and the space-time level sets are starshaped with respect to $(0, x_0)$ and (T, x_0) .

Theorem B. (On Convexity). Let Ω and G be convex and $u_0 = 0$ in $\Omega \setminus \overline{G}$. Let u solve problem (P) and in addition to the assumptions of Theorem A let ϕ be concave. Then the level sets Ω_c of u are convex in space and time.

Remark 1. Notice that Theorems A and B can be applied to diffusion problems with a finite speed of propagation into exterior domains $(\mathbb{R}^n \setminus \overline{G})$, e. g., to $u_t - \Delta u + u^q$ with $0 < q < 1$. In this case one can choose Ω sufficiently large, so that $[0, \infty) \times \Omega$ contains the support of u (see [2] for estimates on its size) and derive geometrical statements on the support of the solution. It should be stressed, however, that Theorems A and B contain statements about the support of $(u - c)^+$ for any $c \in [0, 1]$. Moreover, Theorem A can be used to prove Lipschitz-continuity of the boundary of the support of $(u - c)^+$.

The proofs of Theorems A and B are based on similar ideas for elliptic problems. For the clarity of the exposition we shall outline the proof for the case $p = 2$ and $\phi(u) = u$ only. The extension to the case of nonlinear ϕ is relatively straightforward, but the extension to $p \neq 2$ requires major technical work which has been done in detail in [3]. While (for $p \neq 2$) the proof contains the same steps as for $p = 2$, each individual step requires a more complicated proof. Let us now turn to the major steps in the proof.

Step 1: Regularization

Suppose we approximate the nonlinearity f by a Lipschitzian and monotone increasing nonlinearity f_ε (and — in the general case — regularize ϕ and show that the p -Laplacian operator is nondegenerate for the solution). Then we obtain approximate solutions which are classical in the sense that all the occurring derivatives of those are continuous. These approximate solutions, u_ε , say, converge uniformly on compact sets to the original solution as $\varepsilon \rightarrow 0$.

Suppose furthermore that we can prove Theorems A and B for the approximate solutions. Then we are done, because starshapedness and convexity of level sets are preserved under uniform convergence. To see this for instance for the case of starsphapeness with respect to $(0, 0) \in \mathbb{R}^{1+n}$ one has to observe that each u_ε is starshaped w. r. t. $(0, 0)$ if and only if

$$S_\varepsilon(\lambda, t, x) = u_\varepsilon(t, x) - u_\varepsilon(\lambda t, \lambda x) \geq 0 \quad \text{in } (0, 1) \times (0, \infty) \times \Omega$$

for each ε . But because of the uniform convergence of u_ε this inequality is preserved as $\varepsilon \rightarrow 0$. In a similar fashion, using the function Q in Step 3 below, one can show that convexity of level sets is a property which is closed under uniform convergence.

Therefore, in order to prove Theorems A and B, we may assume without loss of generality that the solution u of Problem (P) is classical and that f is Lipschitzian and monotone increasing.

Step 2: Sign of first order derivatives

If one differentiates the differential equation with respect to t , it is easily seen from the maximum principle that

$$u_t(t, x) > 0 \quad \text{in } (0, \infty) \times (\Omega \setminus G). \quad (1)$$

Similarly one can show that

$$(x - x_0) \cdot \nabla u(t, x) < 0 \quad \text{and} \quad tu_t + (x - x_0) \cdot \nabla u \leq 0 \quad \text{in } (0, \infty) \times (\Omega \setminus G) \quad (2)$$

for every $x_0 \in G$. But (1) and (2) prove Theorem A.

Step 3: Gabriel-Lewis method

To prove the convexity of level sets we use the well-known fact that a continuous function $u : D \rightarrow \mathbb{R}$ has convex level sets if and only if

$$Q(z_1, z_2) = u((z_1 + z_2)/2) - \min\{u(z_1), u(z_2)\} \geq 0 \quad \text{in } D \times D.$$

For the stationary elliptic problem one can set $D = \Omega$ and prove that Q is nonnegative. This was done for functions which are harmonic or p -harmonic in [4] and [7]. For nonlinear elliptic problems one can use the same approach; we refer to [5], [6]. In the parabolic situation we interpret the time-space cylinder $(0, \infty) \times \Omega$ as a convex domain in \mathbb{R}^{1+n} and thus have to show that

$$Q(t_1, t_2, x_1, x_2) = u((t_1 + t_2)/2, (x_1 + x_2)/2) - \min\{u(t_1, x_1), u(t_2, x_2)\} \quad (3)$$

is nonnegative for any pair $(t_1, x_1), (t_2, x_2)$ of points in $(0, \infty) \times D$. To prove (3) we can argue by contradiction. Suppose that Q attains a negative minimum at $(t_1, x_1) = z_1, (t_2, x_2) = z_2$ and set $z_{12} = (z_1 + z_2)/2$. We want to apply a maximum principle argument to Q . Therefore, we have to show first that Q can only attain a negative minimum in points z_1, z_2 and z_{12} in which the differential equation is valid. In this case the structure of the differential equation will lead to a contradiction. Let us postpone the discussion of this case to Step 4 and investigate the other possible cases.

Case 1: $t_1 = 0$. This case is ruled out by (1) and (2).

Case 2: $t_1 \rightarrow \infty$. In this case $(t_1 + t_2)/2 \rightarrow \infty$. Now (1) and the convexity of level sets for the stationary problem lead to a contradiction. Notice that u tends asymptotically to a stationary solution. Therefore both t_1 and t_2 are finite and positive.

Case 3: $x_1 \in \bar{G}$. If $(x_1 + x_2)/2 \in \bar{G}$, then Q must be nonnegative. But if $(x_1 + x_2)/2 \in \Omega \setminus \bar{G}$, we obtain a contradiction to (2). So we are left with the remaining possibility that t_1 and t_2 are finite and positive and x_1, x_2 and $x_{12} = (x_1 + x_2)/2$ are in $\Omega \setminus \bar{G}$.

Step 4: Use a quasiconcavity maximum principle

In this step one needs only modifications of the corresponding proof for the stationary problem. In fact, if Q attains a local extremum, we expect certain equalities for first order derivatives of u and certain inequalities for second order derivatives of u . This is how the structure of the differential equation will come into play. However, Q is not differentiable in the extremal pair of points, since one can show (see [6])

$$u(z_1) = u(z_2) > u(z_{12}), \quad (4)$$

where $z_{12} = (z_1 + z_2)/2$. So one has to study derivatives of Q under the side constraint that $u(t_1, x_1) = u(t_2, x_2)$. Using the notation $Du = (u_t, \nabla u)$ for the gradient in \mathbb{R}^{1+n} and $A = |Du((z_1 + z_2)/2)|, B = |Du(z_1)|, C = |Du(z_2)|$ one can derive the following equality

$$\frac{1}{A} = \frac{1}{2} \left(\frac{1}{B} + \frac{1}{C} \right), \quad (5)$$

and the inequality

$$\frac{1}{A^2} \Delta(u(z_{12})) \geq \frac{\mu}{B^2} \Delta(u(z_1)) + \frac{1-\mu}{C^2} \Delta(u(z_2)), \quad (6)$$

where $\mu = C/(B+C) \in (0, 1)$. If we use the three relations (4), (5), (6) as ingredients and stir them well, they can be shown to contradict each other. In fact, due to the differential equation and the monotonicity of f , (4) and (6) lead to

$$\frac{1}{A^2} u_t((z_1+z_2)/2) > \frac{\mu}{B^2} u_t(z_1) + \frac{1-\mu}{C^2} u_t(z_2). \quad (7)$$

Because of (1) and (5) we have

$$\frac{1}{u_t(z_{12})} = \frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)}. \quad (8)$$

Now (7), (8) and the definition of μ give

$$0 > \frac{\mu}{B^2} u_t(z_1) + \frac{1-\mu}{C^2} u_t(z_2) - \frac{1}{A^2} \left(\frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)} \right)^{-1} = A \left(\frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)} \right)^{-1} \left[\left(\frac{1}{2B^3} u_t(z_1) + \frac{1}{2C^3} u_t(z_2) \right) \left(\frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)} \right) - \frac{1}{A^3} \right].$$

To complete the proof of Theorem B we have to show that $[\dots] \geq 0$, but

$$\begin{aligned} & \left(\frac{1}{2B^3} u_t(z_1) + \frac{1}{2C^3} u_t(z_2) \right) \left(\frac{1}{2u_t(z_1)} + \frac{1}{2u_t(z_2)} \right) \\ & \geq \left(\frac{1}{2} B^{-3/2} + \frac{1}{2} C^{-3/2} \right)^2 \geq (A^{-3/2})^2 = A^{-3}, \end{aligned}$$

which is the desired contradiction.

Remark 2. Unfortunately Theorem B does not apply to the porous medium equation. The assumptions of Theorem A can be considerably modified. We refer to [3] for details.

An open problem

The following is a simple looking interior boundary value problem, whose solution could contribute a lot to the study of more complicated problems. Again let $\Omega \subset \mathbb{R}^n$ be convex and let u solve the problem

$$u_t - \Delta u = 0 \text{ in } (0, \infty) \times \Omega, \quad (3.1)$$

$$u = 0 \text{ on } (0, \infty) \times \partial\Omega, \quad (3.2)$$

$$u(0, x) = u_0(x) \text{ in } \Omega. \quad (3.3)$$

Suppose that the function $u_0(x)$ has convex level sets $\{x \in \Omega \mid u_0(x) > \text{const.}\}$. Is it true that for every fixed but positive t the function $u(t, x)$ has level sets $\{x \in \Omega \mid u(t, x) > \text{const.}\}$, which are convex in space?

If the space dimension is $n = 1$, the answer to this problem is positive and follows from any of the papers [8], [9], [10], [11]. Unfortunately the proof that works in one dimension fails in higher dimensions.

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