COMPACTNESS OF THE GREEN OPERATOR ASSOCIATED TO THE POROUS MEDIA EQUATION[1]

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Let Ω be an open regular set of $\mathbb{R}^N$, $N\geq 1$, and let $T>0$. Consider the problem

$$u_t-\Delta\varphi(u)=f \quad \text{in} \quad Q=\Omega\times(0,T) \tag{1}$$

$$\varphi(u)=0 \quad \text{on} \quad \partial\Omega\times(0,T) \tag{2}$$

$$u(0,.)=u_0(.) \quad \text{on} \quad \Omega \tag{3}$$

where

$$\varphi \text{ is continuous nondecreasing and } \varphi(0)=0 \tag{4}$$

$$u_0\in L^1(\Omega). \tag{5}$$

Problems as (1),(2),(3) appear in many different contexts as, for instance, unsaturated flows through a porous medium. Existence, uniqueness and regularity results are today well-known in the literature (see e.g.[1]). Here we are interested in showing some compactness properties of the associated Green operator G defined by

$$G: L^1(0,T:L^1(\Omega))\longrightarrow C([0,T]:L^1(\Omega))$$

$$f\mapsto u, \ u \text{ solution of } (1),(2),(3),$$

for a given u_0 satisfying (5). The compactness of G (in some weak sense) has many different applications. For instance, it allows to obtain easily existence results for functional-perturbed equations (see [2] and [8]) or for nonlinear systems

$$u_{i,t}-\Delta\varphi_i(u_i) = f_i(x,t,u_1,u_2) \qquad i=1,2$$

under very weak assumptions on f_i and φ_i (see [7]). Other applications of the compactness of G are concerned with the asymptotic behaviour, as $t\to\infty$, of

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solutions (1), (2), (3) (see [6]).

Our main result is the following

THEOREM 1. *Let* $u_0 \in L^1(\Omega)$ *and assume that*

$$\varphi \text{ is strictly increasing} \tag{6}$$

Let F *a weakly compact subset of* $L^1(Q)$. *Then* G(F) *is a relatively (strongly) compact set of* $C([0,T]:L^1(\Omega))$.

The proof of Theorem 1 is based in the following result of interest by itself

THEOREM 2. *Assume* (6) *and let* S(t) *be the semigroup of contractions on* $L^1(\Omega)$ *associated to the operator* $-\Delta\varphi(.)$. *Then, for any* $t \in (0,T)$, S(t) *transforms any weakly compact set of* $L^1(\Omega)$ *into a relatively compact set of* $L^1(\Omega)$. *In particular, the restriction* $S(t):L^p(\Omega) \to L^q(\Omega)$ *is compact if* $1 \le q < p \le +\infty$.

Remark 1. The study of the compactness of G was previously carried out in the work [3] for the special case of $\varphi(s)=|s|^{m-1}s$ and $m>0$. It is shown there that if $N>2$ and $m>(N-2)/N$ then S(t) is a compact semigroup in $L^1(\Omega)$. Moreover, if S(t) is compact and F is a bounded set of $L^1(Q)$ then G(F) is relatively compact in $C([0,T]:L^1(\Omega))$. Later in [5] it was shown the optimality of that result i.e. if $m \le (N-2)/N$ then S(t) is not compact in $L^1(\Omega)$. Generalizations of the strong compactness of S(t) for more general functions φ were given in [2] and specially in [4] where it was shown that S(t) is compact in $L^1(\Omega)$ if and only if

$$\int^{+\infty} \frac{ds}{\varphi(s)^{N/(N-2)}} < +\infty .$$

Remark 2. The assumption (6) is optimal in order to get the conclusion of Theorem 1. Indeed, take as function φ the one associated to the Stefan problem:

$$\varphi(s)=s+1 \text{ if } s\le -1 , \ \varphi(s)=s-1 \text{ if } s\ge -1 , \ \varphi(s)=0 \text{ if } s\in[-1,1].$$

Take $\Omega=(0,1)$, $u_0=0$ and $F=\{f_n, n\in\mathbb{N}: f_n(t)(x)=\sin nx\}$. Then it is easy to see that although F is weakly compact in $L^1(Q)$, the set of solutions G(F) is given by $G(F)=\{u_n, n\in\mathbb{N}: u_n(t)(x)=t \sin nx\}$ which is not relatively compact in $C([0,T]:L^1(\Omega))$.

Remark 3. The proof of both results was given in [6]. Theorem 2 is shown by reducing the problem to the relative compactness of the set $S(t)(B)$ where B is a bounded set of $L^\infty(\Omega)$. It is shown there that the conclusion comes from the gradient estimate

$$\| \nabla\varphi(S(t)u_0) \|^2_{L^2(\Omega)} \leq \frac{1}{t} \| j(u_0) \|_{L^1(\Omega)}$$

where $j(r)=\int_0^r \varphi(s)ds$. The proof of Theorem 1 uses some approximation arguments. After that the conclusion follows from Theorem 2 by using the formula

$$\|u_f(t+\lambda)-S(\lambda)u_f(t)\|_{L^1(\Omega)} \leq \int_t^{t+\lambda} \|f(s)\|_{L^1(\Omega)} ds,$$

where $\lambda>0$ and u_f denotes the solution of (1),(2),(3) for a given function $f\in L^1(0,T:L^1(\Omega))$.

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