UNIQUENESS OF VERY SINGULAR SELF-SOLUTIONS OF A QUASILINEAR DEGENERATE PARABOLIC EQUATION WITH ABSORPTION[1]

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1. Introduction. The main goal of this communication is to present the results of the work Díaz-Saa [4] concerning the uniqueness of solutions of the following quasilinear elliptic problem

$$-\left(|u'|^{p-2}u'\right)' - \frac{(N-1)}{x}|u'|^{p-2}u' + h(x,u,u') = g(x,u), \quad x>0, \tag{1}$$

$$u'(0) = 0 \quad , \quad \lim_{x\to\infty} u(x) = 0 \tag{2}$$

$$u(x)\geq 0 \quad (\not\equiv 0) \tag{3}$$

in which p>1 and the functions h and g satisfy certain structural conditions which will be made explicit later.

The main motivation for the consideration of such a problem comes from the study of very singular solutions of the quasilinear degenerate parabolic equation with absorption

$$u_t = \Delta_p u^m - u^q \qquad \text{in} \quad Q = \mathbb{R}^N x(0,\infty) \tag{4}$$

where, as usual, $\Delta_p u$ denotes the p-Laplacian operator

$$\Delta_p v = \operatorname{div}\ (|\nabla v|^{p-2}\nabla v) \quad , \ 1<p<\infty,$$

$N\geq 1$ and m and q are nonnegative real numbers. Equation (4) contains, as special cases, the equations

$$u_t = \Delta u^m - u^q \tag{5}$$

and

$$u_t = \Delta_p u - u^q \tag{6}$$

which have been intensively studied in the last years. For many different purposes it is interesting to study singular solutions of (4) i.e. nonnegative functions u satisfying (4) in Q (in the sense of distributions) and such that u(x,0)=0 if $x\in\mathbb{R}^N-\{0\}$. In many cases, the singularity at t=0 of

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such a solution must be as that of the "fundamental solution" i.e.

$$u(x,0) = c\delta(x)$$

for some positive constant c, or, in other words,

$$\lim_{t\to 0} \int_{|x|<r} u(x,t)dx = c \tag{7}$$

for any r>0. Nevertheless, when the absorption is strong enough with respect to the diffusion, there exists another type of singular solution u called "the very singular solution" which has been discovered previously in the following cases:

a) equation (5) with m=1 and $1<q<1+(2/N)$: Brezis, Peletier and Terman [1]

b) equation (5) with m>1 and $m<q<m+(2/N)$: Peletier and Terman [8]

c) equation (6) with p>q and $p-1<q<p-1+(p/N)$: Peletier and Wang [9]

In all those cases this new singular solution satisfies that

$$\lim_{t\to 0} \int_{|x|<r} u(x,t)dx = +\infty \tag{8}$$

for any r>0 and so it is more singular than the fundamental solution. As usual, the existence of a very singular solution is obtained in the class of self-similar solutions

$$W(x,t) = t^{-1/(q-1)}f(|x|/t^{1/\beta}) \tag{9}$$

where β must be suitably chosen. For instance $\beta=2(q-1)/(q-m)$ and $\beta=p(q-1)/(q+1-p)$ in the cases of equations (5) and (6) respectively. More in general, we can consider self-similar solutions W of the equation (4) in which case the natural choice of β is

$$\beta = p(q-1)/(q-m(p-1)) \tag{10}$$

A function W given by (8) is then a very singular solution if f satisfies

$$\left(\left|(f^m)'\right|^{p-2}(f^m)'\right)' + \frac{(N-1)}{x}|(f^m)'|^{p-2}(f^m)' + \frac{1}{\beta}xf' + \frac{1}{(q-1)}f - f^q = 0, \quad \text{in } (0,\infty) \tag{11}$$

$$f \geq 0 \quad \text{in } (0,\infty) \tag{12}$$

$$f'(0)=0 \ , \qquad \lim_{\eta\to\infty} \eta^{p}/(q-m(p-1))f(\eta)=0 \tag{13}$$

The uniqueness of f solution of (11) (12) (13) was only given for the case m=1 and p=2, (see [1]) and was left open in [8] and [9]. The main goal of our work is to give an uniqueness result true for any value of m and p. Simultaneously to the completion of our work [4] (preliminary included in [10]) S. Kamin and L. Veron have find in [7] a new proof of the existence of the very singular solution of the equation (5) as limit of fundamental solutions satisfying (7) when $c\to+\infty$. They also have a proof of the uniqueness of the very singular solution (i.e. a nonnegative function satisfying

(5) and (8)) in the class of solutions of the parabolic equation (5). In this way they are giving an indirect proof of the uniqueness of f for p=2 and m>1 arbitrary. It seems that their arguments, jointly with some ideas of Kamin-Vazquez [6], may allow to give the uniqueness of the very singular solution in the class of solutions of (6) or even (4). In any case our arguments are of a different nature to those used in [7] and [6] and can be applied to other elliptic problems not necessarily related with the study of singular solutions of any parabolic equation.

We remark that introducing $v=f^m$ then v satisfies an equation of the type (1) with

$$g(x,u) = u^{q/m} - \frac{1}{(q-1)}u^{1/m}$$

and

$$h(x,u) = -\frac{1}{\beta}\, x\, u^{-\frac{(m-1)}{m}}\, u'$$

So g(x,u) is not monotone in u. Moreover the differential terms in equation (11) may have different homogeneity (m(p-1) and 1 respectively) which leads to some special difficulties (solutions with compact support if m(p-1)>1, etc).

2. <u>The main results.</u> We shall prove the uniqueness of solutions of the problem

$$\left(|(u^m)'|^{p-2}(u^m)'\right)' + \frac{(N-1)}{x}|(u^m)'|^{p-2}(u^m)' + \frac{1}{\beta}x\, u' + G(u) = 0, \quad x>0 \tag{14}$$

$$u(x)\geq 0 \quad (\not\equiv 0) \tag{15}$$

$$(u^m)'(0)=0\ , \quad \lim_{x\to\infty} u(x)=0 \tag{16}$$

where $m>0$, $p>1$, $N\geq 1$, $\beta>0$ and

$$G(u)=\frac{1}{q-1}\, u-u^q\ .$$

In some cases problem (14), (15), (16) does not have any classical solution and it must be solved in a generalized way. This is the case when $m(p-1)>1$ because the solutions have as support a compact interval $[0,x_0]$ and u' may be discontinuous at $x=x_0$ (see part (v) of Lemma 1). To define the notion of weak solution we multiply the equation (14) by a smooth text function $\xi(x)$ with compact support in $[0,\infty]$ but not neccesarily vanishing at x=0. By multiplying by x^{N-1} and integreating (formally) by parts we obtain

$$-\int_0^\infty x^{N-1}|(u^m)'|^{p-2}(u^m)'\xi'\,dx - \frac{1}{\beta}\int_0^\infty x^N u\xi'\,dx + \int x^{N-1}(G(u)-u)\xi dx=0 \tag{17}$$

On the other hand, by standard regularity results, it is clear that $u\in C^0([0,\infty])$ and that in fact $u\in C^2$ on the set where the equation is not degenerate i.e. $\{x\in(0,\infty):u(x)>0$ and $(u^m)'(x)\neq 0\}$. We shall show that this set coincides with the support of u. We can assume that $u^m\in C^1([0,\infty])$, because taking a sequence ξ_n such that lim $\xi_n(x)=1$ if $x\in[x_0-\varepsilon,x_0]$ and lim $\xi_n(x)=0$ otherwise we have that

$$\left[x^{N-1}|(u^m)'|^{p-2}(u^m)'\right]_{x_{0-\varepsilon}}^{x_0}+\left[\frac{x^N}{\beta}u\right]_{x_{0-\varepsilon}}^{x_0}=\int_{x_{0-\varepsilon}}^{x_0}x^{N-1}(G(u)-u)dx.$$

and so $|(u^m)'|^{p-2}(u^m)'(x_0)=0$ (the continuity at x=0 is similarly justified).

In consequence, by a solution of (14),(15),(16) we shall mean a function $u\in C^0([0,\infty))$ such that $u^m\in C^1([0,\infty))$. $u\geq 0$ ($\not\equiv 0$) and satisfies (16) and (17) for any smooth function ξ with compact support in $[0,\infty)$.

Now we are in a condition to state our uniqueness results:

THEOREM 1. *Assume that* $N\geq 1$, $m>0$, $q>0$, $p>1$ *and*

$$m(p-1)>1 \tag{18}$$

and

$$(p-1)m<q<(p-1)m+\frac{p}{N} \tag{19}$$

Then there is at most one solution of problem (14),(15),(16).

THEOREM 2. *The conclusion of Theorem* 1 *holds replacing the assumption* (18) *by*

$$m(p-1)=1. \tag{20}$$

Before giving the proofs we shall make some remarks on the assumptions of both results. First of all we indicate that the reasonable assumption on the parameters m and p is $m(p-1)\geq 1$, because otherwise the parabolic equation (4) corresponds to a "fast diffusion" and solutions vanish after a finite time. On the other hand, it is naturalto expecta different behaviour of solutions of (14),(15),(16) according to whether m(p-1) is greater or equal to one. Indeed, the first case corresponds to slow diffusion, and the solutions of (4) have compact support for any value of t, although when $m(p-1)=1$ the solutions of (4) are strictly positive in $\mathbb{R}^N\times(0,\infty)$. Finally the assumption (19) include the assumptions made in [1],[8] and [9] for the existence of very singular solutions, as it has been indicated in the Introduction. This also explains how the boundary condition (16) implies the one given in (13).

The following Lemma collects several properties of solutions of (14),(15),(16).

LEMMA 1. *Assume* $m(p-1)\ge 1$ *and condition* (17). *Let* u *be any solution of* (14), (15), (16). *Then* $u\in C^0$ *and* $u^m\in C^1$. *Moreover*

(i) $$\lim_{x\downarrow 0} \frac{|(u^m)'(x)|^{p-2}(u^m)'(x)}{x} = -\frac{1}{N}G(u(0)) \tag{21}$$

(ii) $u(x)\le M$ *for any* $x\ge 0$ *with* $M=\left(\frac{1}{q-1}\right)^{1/(q-1)}$

(iii) $\forall x_0\in[0,\infty)$ *such that* $u(x_0)=0$ *then* $u(x)=0 \quad \forall x\ge x_0$

(iv) $u(x)$ *is non-increasing in* $[0,\infty)$ *and* $u'(x)<0$ *for any* $x>0$ *such that* $u(x)>0$.

(v) *If* $m(p-1)>1$ *then* u *has compact support* $[0,x_0]$ *and*

$$\lim_{x\uparrow x_0} \frac{|(u^m)'(x)|^{p-1}}{u(x)} = \frac{x_0}{\beta}. \tag{22}$$

(vi) *If* $m(p-1)=1$ *then* $u(x)>0$ *for any* $x\in[0,\infty)$.

The proof of Lemma 1 can be found in [4]. Condition (22) is equivalent to the differential equation of the interface of the solution of the parabolic equation (4) which also comes from the Darcy law.

Proof of Theorem 1. . The first step is to introduce a change of variables in such a way that the absorption term of the new equation be monotonically non-increasing. Let v(x) defined by

$$u(x)=v(x)^{\mu} \tag{23}$$

If we take

$$\mu=(p-1)/(m(p-1)-1) \tag{24}$$

it is easy to see that v satisfies

$$\left(|v'|^{p-2}v'\right)' + \frac{N-1}{\kappa}|v'|^{p-2}v' + \mu\frac{|v'|^p}{v} + \frac{\mu}{\beta a}\kappa\frac{v}{v} + \frac{1}{(q-1)a}\,\frac{v^{\mu(q-1)}}{a} = 0 \tag{25}$$

$$v\ge 0 \ (\not\equiv 0) \quad \text{in } [0,\infty) \tag{26}$$

$$v'(0)=0\ , \quad \lim_{x\,\infty} v(x)=0 \tag{27}$$

where

$$a=(m\mu)^{(p-1)}. \tag{28}$$

Now let v_1 and v_2 be two solutions of (25), (26) and (27). Let $x_0\in[0,\infty)$ be such that

$$0<(v_1-v_2)(x_0) = \sup_{[0,\infty)}(v_1-v_2) \equiv h\ .$$

By comparing the value of

$$\lim_{x\downarrow 0} \frac{|v_i'(x)|^{p-2}v_i'(x)}{x}$$

for i=1,2, it is not difficult to see that $x_0>0$. On the other hand, by using part (v) of Lemma 1 one can show that $v_2(x_0)>0$ (see details in [4]). Then there exists a constant $L>1$ such that

$$\max\{v_1(o)-v_2(o)\ ,\ v_1(y_0)-v_2(y_0)\} < \frac{v_1(x_0)-v_2(x_0)}{L} \tag{29}$$

where $y_0>x_0$ is such that $\mathrm{supp}\ v_2=[o,y_0]$. We also chose $k>0$ such that

$$\max\ \{\frac{h}{L}\ ,\ \frac{h}{2}\} < k < v_1(x_0)-v_2(x_0) = h. \tag{30}$$

Now, we shall first pay attention to the case $p\geq 2$. We multiply the equations of $v_1(i=1,2)$ by $x^{N-1}\xi$ with ξ given by

$$\xi = e^{pw}-1, \quad w = (v_1-v_2-k)^{+}. \tag{31}$$

Integreating on $(0,+\infty)$ we have

$$\int_{[\xi'\neq 0]} x^{N-1}(|v_1'|^{p-2}v_1'-|v_2'|^{p-2}v_2')\xi'$$

$$\int \mu x^{N-1}\left(\frac{|v_1'|^{p}}{v_1} - \frac{|v_2'|^{p}}{v_2} - \frac{v_1^{\mu(q-1)}}{a} + \frac{v_2^{\mu(q-1)}}{a}\right)\xi$$

$$+ \int \mu\ \frac{x^N}{\beta a}\left(\frac{v_2'}{v_1} - \frac{v_2'}{v_2}\right)\xi$$

Using that $\xi'=pw'e^{pw}$(where $w'=v_1'-v_2'$ on $w>0$ and $w'=0$ otherwise), the inequality (see e.g. Díaz [2] p 264)

$$(|\alpha_1|^{p-2}\alpha_1-|\alpha_2|^{p-2}\alpha_2)(\alpha_1-\alpha_2)\geq p|\alpha_1-\alpha_2|^p \quad \forall\alpha_1,\alpha_2\in\mathbb{R}^+ \tag{32}$$

and the fact that $(v_2-v_1)\xi\leq 0$ we deduce that

$$p\int x^{N-1}|w'|^pe^{pw}dx \leq \mu\int\frac{x^{N-1}}{v_1}(|v_1'|^p-|v_2'|\xi dx + \frac{\mu}{\beta a}\int x^N(\mathrm{Ln}v_1-\mathrm{Ln}v_2)'\xi dx$$

Moreover, applying the inequality (true for any $p>1$)

$$|\alpha_1|^p-|\alpha_2|^p \leq C|\alpha_1-\alpha_2| \qquad \forall\alpha_1,\alpha_2\in\mathbb{R},$$

using that $v_1(x)>h/2$ for any x such that $\xi(x)>0$, and integreating by parts in the last integral we deduce that

$$p\int_{[w'\neq o]} x^{N-1}|w'|^pe^{pw}dx \leq \frac{2\mu}{h}\ C\int_{[w'\neq o]} x^{N-1}|w'|(e^{pw}-1)$$

$$+ \frac{\mu}{\beta a}\ \int_{[w'\neq o]} x^N\ (\mathrm{Ln}v_1-\mathrm{Ln}v_2)|\xi'|dx$$

As the logarithm function is concave we have

$$p\int_{[w'\neq o]} x^{N-1}|w'|^pe^{pw}dx \leq C\int_{[w'\neq o]} x^{N-1}|w'|e^{pw} + \frac{\mu}{\beta a}\int_{[w'\neq o]}\frac{v_1-v_2}{v_2}\ x^N|\xi'|dx.$$

where C denotes again a generic constant and so it will denotes in the following. But $[w'\neq o]\subset[w\neq o]$ and from the choice of w we deduce that there exists three positive constants $\delta_1,\delta_2,\delta_3$ such that if $x\in[0,\infty)$ satisfies that

$w(x)>0$ then $x<\delta_1$ (because supp v_1 and supp v_2 are bounded), $x>\delta_2$ because from (29) and (30) $w(0)=0$) and $v_2(x)>\delta_3$ (because $w(y_0)=0$ and v_1 is non-increasing). In consequence, we deduce

$$C\int_{[w'\neq 0]} |w'|^p e^{pw} dx \le \int_{[w'\neq 0]} |w'| e^{pw}\, dx.$$

or equivalently

$$C\int_{[w'\neq 0]} |(e^w)'|^p \le \int_{[w'\neq 0]} |(e^w)'| e^{(p-1)w}$$

Using Hölder inequality we have

$$C\left(\int_{[w'\neq 0]} |(e^w)'|^p\right)^{\frac{p-1}{p}} \le \left(\int_{[w'\neq 0]} e^{pw}\right)^{\frac{p-1}{p}}$$

and hence

$$C\left[\left(\int_{[w'\neq 0]} e^{pw}\right)^{1/p} + \left(\int_{[w'\neq 0]} |(e^w)'|^p\right)\right] \le (1+C)\left(\int_{[w'\neq 0]} e^{pw}\right)^{1/p}$$

so

$$C\|e^w\|_{W^{1,p}([w'\neq 0])} \le \|e^w\|_{L^p([w'\neq 0)]} \tag{33}$$

Assume now that $p<N$. Applying Sobolev and Hölder inequalities we obtain

$$C\|e^w\|_{L^{p^*}([w'\neq 0])} \le \|e^w\|_{L^p([w'\neq 0])} \le \|e^w\|_{L^{p^*}([w'\neq 0])} |\mathrm{supp} w'|^{\frac{(p^*-p)}{p^*}}$$

where $p^* = pN/(N-p)$' In particular

$$|\mathrm{supp}\ w'| \ge C>0 \tag{34}$$

In the case $p\ge N$ conclusion (34) is obtained from the Sobolev inequality by replacing p^* by any number greater than p^*. Since these inequalities are independent of K they must hold as k tends to h. That is, the function v_1-v_2 attain its supremum on a set of positive measure, where at the same time $(v_1-v_2)'=0$, which is a contradiction with the inequality (34).

In the case $1<p<2$ inequality (32) must be replaced by

$$(|\alpha_1|^{p-2}\alpha_1-|\alpha_2|^{p-2}\alpha_2)(\alpha_1-\alpha_2) \ge C\, \frac{|\alpha_1-\alpha_2|^2}{(|\alpha_1|+|\alpha_2|)^{2-p}}$$

(see e.g. Díaz [2] p. 264). This justifies a change in the text function ξ which now is taken as

$$\xi = w = (v_1-v_2-k)^+ \tag{35}$$

Multiplying the equations of v_i by $x^{N-1}\xi$ and integrating on $(0,\infty)$ we have

$$C\int_{[w'\neq 0]} x^{N-1}\frac{|w'|^2}{|v_1'|+|v_2'|}\, dx \le \mu\int \frac{x^{N-1}}{v_1'}(|v_1|^p-|v_2|^p)w + \frac{\mu}{\beta a}\int x^N(Lnv_1-Lnv_2)'\xi dx$$

But there exists $\delta_4>0$ such that $|v_1'|+|v_2'|>\delta_4$ on $[w'\neq 0]$ (recall part (iv) of Lemma 1). Then it is easy to see that all the above arguments allow to obtain the inequality

$$C\|w\|_{W^{1,2}([w'\neq o])} \leq \|e^{w}\|_{L^2([w'\neq o])} \tag{36}$$

(instead of (33)) and so the conclusion follows. ∎

REMARK. The idea of obtaining a contradiction via Sobolev inequalities was already used in Trudinger [11] (see also [5] Theorem 10.7) to compare solutions of non-degenerate quasilinear elliptic problems. In that work the test function is defined as in (35). Finally we point out that our arguments can be also applied in order to obtain comparison results for solutionsof more general equations, as for instance

$$-\Delta_p u - \lambda\frac{|\nabla u|^p}{u} + B(x,u,|\nabla u|) + f(x,u) = 0$$

where $u\mapsto f(x,u)$ and $u\to B(x,u,\eta)$ are non-decreasing and $\eta\to B(x,u,\eta)$ is Lipschitz continuous (see [4]). In particular, this allows to generalize the uniqueness result of [3].

Proof of Theorem 2. As in the previous theorem, we introduce a change of unknown in order to arrive to a new equation with a monotone perturbation term. More precisely, let v(x) defined by

$$u(x)=e^{v(x)} \qquad x>0$$

(recall that u(x)>0; see part of Lemma 1). It is easy to see that v satisfies

$$(|v'|^{p-2}v')' + |v'|^{p-2}v' + \frac{x}{\beta} v' + \frac{1}{q-1} - e^{(q-1)v} = 0$$

Now the proof reduces to repeat the same arguments as before (even in a easier way because v(x)>0 on $(0,\infty)$).

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