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Recent advances in nonlinear elliptic and parabolic problems

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On space or time localization of solutions of nonlinear elliptic or parabolic equations via energy methods

1. INTRODUCTION

In this article we present some new results on space or time localization of solutions of nonlinear elliptic or parabolic equations with "sources", i.e. with a prescribed right-hand part.

The results will be obtained by suitable energy methods (previously suggested and justified by the authors in [2], [3] and [13]), where homogeneous nonlinear equations were investigated. Here the keystone is the study of several nonhomogeneous nonlinear ordinary differential inequalities satisfied by the corresponding energy functions.

Some of the qualitative properties obtained here seem to be new in the literature. This is the case of the instantaneous extinction time or the nondiffusion of the support properties for nonhomogeneous parabolic equations. Other results of this work generalize to very general formulations. Some qualitative properties only well-known before for some special formulations. This is the case for the waiting time (or metastable localization) of parabolic equations and the nondiffusion of the support for elliptic equations. Both properties were earlier investigated by several authors by other methods (see the review expositions in [12] and [14]).

Energy methods are also applied to systems of combined-type equations. Applications to several nonlinear systems in continuum mechanics will be given in [6].

A first announcement of part of the result of this work was made in [5].

1. Parabolic equations

We consider a general class of nonlinear parabolic equations of the form

$$\frac{\partial\psi(u)}{\partial t} - \operatorname{div} \vec{A}(t,x,u,\nabla u) + B(t,x,u) = f(t,x), \qquad (1)$$

where ψ is a continuous nondecreasing real function such that

$$C_1|r|^{\beta-1}r \leq \psi(r) \leq C_2|r|^{\beta-1}r \qquad (2)$$

for some constants $C_2 \geq C_1 > 0$ and $\beta > 0$ and for every $r \in R$. We will also assume the following structural assumptions on $\vec{A}$ and B:

$$|\vec{A}(t,x,u,\xi)| \leq C_3|\xi|^{p-1}, \qquad (3)$$

$$\vec{A}(t,x,u,\xi)\xi \geq C_4|\xi|^{p-2}, \qquad (4)$$

$$B(t,x,u)u \geq C_5|u|^{q+1}, \qquad (5)$$

for some positive constants C_3, C_4, $p > 1$ and $C_5 \geq 0$, $q \geq 0$. Here $u \in R$ and $\xi \in R^N$.

Notice that (1) contains, as a particular case, the following generalization of the porous media equation:

$$\frac{\partial u^{1/m}}{\partial t} - \Delta_p u + \lambda u^q = 0, \qquad (6)$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $\lambda \geq 0$, $m > 0$ and the expressions $u^{1/m}$ and u^q must be substituted by $|u|^{1/m-1}u$ and $|u|^{q-1}u$ if u changes sign.

1.1 Localization in time: Instantaneous extinction time

Let us begin by studying the vanishing, in a finite time, of global solutions of (1) satisfying the following initial and boundary conditions:

$$u(0,x) = u_0(x) \text{ in } \Omega, \qquad (7)$$

$$u(t,x) = 0 \quad \text{on } \Sigma = (0,T) \times \partial\Omega, \qquad (8)$$

where Ω is a bounded regular open set of R^N and $T > 0$. The existence and uniqueness of weak solutions of problems (1), (7), (8) have been considered by many authors (see, e.g. [1], [8] and [9]). In particular, it is well known that if $u_0 \in L^{\beta+k}(\Omega)$ and $f \in L^{p'}(0, T: L^{(\beta+k)/\beta}(\Omega))$ then there exists a unique weak solution $u \in L^p(0, T: W_0^{1,p}(\Omega)) \cap L^1(0, T: L^{\beta+k}(\Omega))$ for any $k \geq 0$.

The following result is peculiar to two alternative phenomena: a "fast diffusion", which corresponds to the range of parameters $\beta > (p-1)$ and $C_3 \geq 0$; or a "strong absorption with respect to the accumulation term" which corresponds to the assumption $C_3 > 0$ and $q < \beta$.

<u>THEOREM 1</u>: Assume that one of the following conditions holds:

$$\beta > (p-1) \text{ and } C_3 \geq 0, \qquad (9)$$

or

$$C_3 > 0 \text{ and } \beta > q. \qquad (10)$$

Let u_0 and f as mentioned with K large enough. We also assume that f vanishes after a finite time $T_f < T$ and that

$$\int_\Omega |f(t,x)|^{(\beta+k)/\beta}dx \leq C_6(T_f-t)_-^{\ldots} \quad \text{for a.e. } t \in (t_1,T), \qquad (11)$$

where $h_+ = \max(h,0)$, $0 \leq t_1 < T_f$, α and C_6 are suitable positive constants $\alpha \in (0,1)$ and C_6 small. Then there exists a constant C_7 (depending on C_6, $\|u_0\|_{\beta+k}$, and α) such that if

$$\int_\Omega |u(t_1,x)|^{\beta+k}dx \leq C_7(T_f-t_1)^\alpha, \qquad (12)$$

then $u(t,\cdot)$ vanishes on Ω for any $t \geq T_f$. More precisely

$$\int_\Omega |u(t,x)|^{\beta+k}dx \leq C_7(T_f-t)_+^\alpha \qquad (13)$$

for any $t \in (t_1,T)$.

<u>PROOF</u>: We first consider the case of assumption (9). Multiplying the equation by $|u|^{k-1}u$, integrating by parts and using Sobolev-Poincaré, Hölder and Young inequalities it is not difficult to show that the function

$$y(t) = \int_\Omega |u(t,x)|^{\beta+k}dx \qquad (14)$$

satisfies the ordinary differential inequality

$$\frac{dy}{dt}(t) + C_8\, y^{\alpha}(t) \le C_9(T_f-t)^{\alpha/(1-\alpha)} \text{ for } t \in (t_1, \cdot), \tag{15}$$

where $\alpha = (p-1+k)/(\beta+k)$ and C_8 and C_9 are given in the following way: First, we fix $\varepsilon > 0$ and then

$$C_9 = \frac{\beta+k}{\beta p'} C_9 \varepsilon^{-p'}, \quad C_8 = \frac{\beta+k}{\beta}\left(C_4 C_{10} - \frac{\varepsilon^p}{p}(1 + C_4^2 C_{10})\right)$$

and C_{10} is the constant of the Sobolev-Poincaré inequality in $W_0^{1,p}(\Omega)$ (when assuming (9), K must be any positive real number such that $K \ge (N(\beta-p+1)/p)-\beta$ if $p < N$ and $K \ge 1$ arbitrary if $p \ge N$).

In the case of assumption (10) the inequality (15) is also obtained for another exponent $\alpha \in (0,1)$ and positive constants C_8 and C_9. To do that an interpolation inequality of the form

$$\|V\|^a_{\beta+k(\Omega)} \le \varepsilon^{-\mu} \|V\|^{q+k}_{q+k(\Omega)} + C\varepsilon^{\mu'} \int_\Omega |V|^{k-1}|\nabla v|^p dx$$

must be used (see [14] or [15] for the case of homogeneous equations), where now $k \ge 1$, $q > 0$, $p > 1$ and $\varepsilon > 0$ are arbitrary and the constants a, μ and C are suitably chosen.

The conclusion of Theorem 1 comes from the investigation of inequality (15) which is made in the following lemma.

LEMMA 1: Let $y(t) \ge 0$ be such that $y'(t) + \phi(y(t)) \le F((T_f-t)_+)$ a.e. on (t_1,T), where ϕ is a nondecreasing function such that $\phi(0) = 0$ and $(1/\phi(\cdot)) \in L^1(0,1)$. For any $\mu > 0$ and $\tau > 0$ we define

$$\theta_\mu(\tau) = \int_0^\tau \frac{ds}{\mu\phi(s)} \tag{16}$$

and $\eta_\mu(s) = \theta_\mu^{-1}(s)$. Assume

$$\exists \bar\mu < 1 \text{ such that } F(s) \le (1-\bar\mu)\phi(\eta_{\bar\mu}(s))$$

and

$$(T_f-t_1) \ge \theta_{\bar\mu}(y(t_1)).$$

Then $y(t) \equiv 0$ for any $t \in [T_f,T]$.

PROOF OF LEMMA 1: The function $\bar y(t) = \eta_{\bar\mu}((T_f-t)_+)$ satisfies $\bar y' + \phi(\bar y) = (1-\bar\mu)\phi(\eta_{\bar\mu}((T_f-t)_+)$, and so it is a supersolution for the ordinary differential inequality. □

REMARK 1: The conclusion of Theorem 1 can be interpreted as an instantaneous extinction time (the solution vanishes from the time in which f vanishes). When $t_1 = 0$, condition (12) only affects the initial datum u_0. If $t_1 > 0$ (12) can be obtained throughout u_0 and f by using the well-known a priori estimate

$$\|u(t_1)\|_{\beta+k(\Omega)} \le \|u_0\|_{\beta+k(\Omega)} + \int_0^{t_1} \|f(s,\cdot)\|_{L^{(\beta+k)/\beta}(\Omega)}.$$

REMARK 2: The instantaneous extinction time also holds for other boundary conditions. The case of the Cauchy problem, $\Omega = R^N$ can also be considered under assumptions stronger than (9) or (10). So, if, for instance, we consider the case of "fast diffusion", (9) must be replaced by

$$\beta > \max\ p-1 < \frac{N(p-1)}{N-p}\ . \tag{17}$$

REMARK 3: Inequality (15) is also useful to show the "nondegeneracy" of the energy y(t) near its first zero T_0. Applications to the continuous dependence of T_0 with respect to u_0 and f can be obtained from this property. Those results will be published elsewhere.

REMARK 4: Lemma 1 is inspired in Theorem 1.15 of [12]. The particular case of $\phi(s) = Cs^\alpha$ with $\alpha \in (0,1)$ can be investigated by other methods.

1.2. Space localization: Waiting time and nondiffusion of the support

In this section we will study the local vanishing of solutions of (1). The local nature of this property will allow us to work merely with local solutions of (1), i.e. functions satisfying (1) on sets of the form $\sigma \times (0,T)$ with $\sigma \subset \Omega$ but without any information on the values of u or of $\nabla u \cdot \vec n$ on $\partial\Omega$.

The main conclusion of this section is, again, peculiar to two alternative phenomena: a "slow diffusion", which corresponds to the assumption $\beta < (p-1)$ and $C_3 \geq 0$; or a "strong absorption with respect to the diffusion", which corresponds to conditions $C_3 > 0$ and $q < (p-1)$. Concerning the first case, we have:

THEOREM 2: Assume $\beta < (p-1)$ and $C_3 \geq 0$. Let $B_\rho(x_0) = \{x \in \Omega: |x-x_0| < \rho\}$ and assume that there exists $\rho_0 > 0$, $t_f > 0$, $\delta > 0$ and $\varepsilon > 0$ (ε small enough) such that

$$\int_{B_\rho(x_0)} |u(0,x)|^{\beta+1} dx + \left(\int_0^{t_f} \|f(s,\cdot)\|^{p/(p-\theta)}_{L^{r(r-1)}(B_\rho(x_0))} ds\right)^{(p-\theta)/p(1-\lambda)}$$

$$\leq \varepsilon(\rho-\rho_0)_+^{1/(1-\sigma)}$$

for a.e. $\rho \in (0,\rho_0+\delta)$, where

$$\sigma = \frac{p}{p-1}\left(1 - \left(\frac{\theta}{p} + \frac{1-\theta}{\beta+1}\right)\right), \quad \theta = \left(\frac{1}{\beta+1} - \frac{1}{r}\right)/\left(\frac{1}{\beta+1} - \frac{N-p}{Np}\right),$$

$$\lambda = \left(\frac{1-\theta}{\beta+1} + \frac{\theta}{p}\right), \qquad \beta + 1 \leq r \leq \frac{Np}{N-p}$$

Then there exists $t^* > 0$, $t^* \leq t_f$, such that $u(t,x) = 0$ a.e. $x \in B_{\rho_0}(x_0)$ and $t \in [0,t^*]$.

PROOF: We introduce the energy functions ([13])

$$E(t,\rho) \equiv \int_0^t \int_{B_\rho} \vec{A}(s,x,u,\nabla u)\cdot\nabla u \, dx\, dt, \quad b(t,\rho) = \operatorname*{ess\,sup}_{\tau\in[0,t]} \int_{B_\rho} |u(\tau,x)|^{\beta+1} dx$$

Integrating by parts, using the interpolation inequality

$$\|u(t,\cdot)\|_{L^r} \leq C \|u(t,\cdot)\|^{1-\theta}_{L^{\beta+1}} \left(\|\nabla u(t,\cdot)\|_{L^p} + \|u(t,\cdot)\|_{L^{\beta+1}}\right)^\theta$$

with $r \in [\beta+1, N_p/(N-p)]$, $\theta = (1/(\beta+1) - 1/r)/(1/(\beta+1) - (N-p)/N_p)$, and using the interpolation-trace lemma (see [13]), we conclude that

$$(E + b) \leq Ct^{\frac{(1-\theta)H}{p}} \left(\frac{\partial E}{\partial \rho}\right)^{\frac{(p-1)H}{p}} + \varepsilon(\rho-\rho_0)_+^{\frac{1}{1-\sigma}} \tag{19}$$

for any $\rho > 0$ and $t \leq t_f$, where

$$H = \frac{1}{1 - \left(\frac{\theta}{p} + \frac{(1-\theta)}{\beta+1}\right)} . \tag{20}$$

The proof of Theorem 2 ends with the following lemma:

LEMMA 2: Let $y \in C^0([0,t_1] \times [0,\rho_0+\rho])$, $y \geq 0$, be such that for any $t \leq t_1$ and for some $\omega > 0$ and $\delta > 0$

$$\phi(y(t,\rho)) \leq Ct^\omega \frac{\partial y}{\partial \rho}(t,\rho) + G((\rho-\rho_0)_+), \quad \text{a.e. } \rho \in (0,\rho_0+\delta), \tag{21}$$

where ϕ is a nondecreasing continuous function such that $\phi(0) = 0$ and $1/\phi(\cdot) \in L^1(0,1)$. As in Lemma 1, given $\mu > 0$, we define $\theta_\mu(\tau)$ and $\eta_\mu(s)$. We suppose that

$$\exists\bar{\mu} > 0 \text{ and } \varepsilon < 1 \text{ such that } G(s) < \varepsilon\phi(\eta_{\bar{\mu}}(s)) \text{ a.e. } s \in (0,\delta). \tag{22}$$

Then there exists $t^* \leq t_1$ such that $y(t,\rho) = 0$ for any $0 \leq \rho \leq \rho_0$ and $t \in [0,t^*]$.

PROOF OF LEMMA 2: It is easy to see that function $\bar{y}(\rho) = \eta_\mu((\rho-\rho_0)_+)$ satisfies

$$-Ct^\omega \frac{\partial \bar{y}}{\partial \rho} + \phi(\bar{y}) = (-Ct^\omega\mu + 1)\phi(\eta_\mu((\rho-\rho_0)_+)).$$

Then, taking $\mu \geq \bar{\mu}$ and $t \leq t^*$ with t^* such that

$$t^* \leq \left(\frac{1-\varepsilon}{C\mu}\right)^{1/\omega}$$

we have that $\bar{y}$ is a supersolution of the differential inequality. In order to conclude that $y(t,\rho) \leq \bar{y}(\rho)$ for $\rho \in (0,\rho_0+\delta)$ we only need to have $y(t,\rho_1) \leq \bar{y}(\rho_1)$. This last condition holds if we take μ large enough such that

$$\frac{1}{\rho_0+\delta}\int_0^{\Pi} \frac{ds}{\phi(s)} \leq \mu$$

where $M = \sup\{y(t,\rho_0+\delta) : t \in [0,t_1]\}$. □

REMARK 5: In the special case $f \equiv 0$ the time t^* is called the waiting time. The existence of t^* was previously shown, for particular formulations of (1), by different authors (see, e.g. [14]). It is not difficult to check that our condition (18) coincides with the one in the literature for the one-dimensional porous media equation. The case of $f \quad 0$ was previously treated by an energy method in [4], where a different proof of Lemma 2 (for $\phi(s) = s^\alpha$, $\alpha \in (0,1)$) was given.

REMARK 6: The conclusion of Theorem 2 also holds when $C_3 > 0$ and we replace the condition $\beta < (p-1)$ by the assumption $q < (p-1)$. In that case the exponents in (18) are different and in fact $t^* = t_f$ (*nondiffusion of the support of the solution*). These results, applications to suitable systems, the treatment of the case in which B also depends on ∇u and $f(t,\cdot) \in W^{-1,p'}(\Omega)$, etc. will be given in [7].

REMARK 7: The existence of a waiting time for higher order nonlinear parabolic equations is the object of [11] (see [10] for the proof of the finite speed of propagation for higher order equations by an energy method).

2. ELLIPTIC EQUATIONS

The local energy method used in Section 1.2 can also be applied to the study of nonlinear elliptic equations of the form

$$- \operatorname{div} \vec{A}(x,u,\nabla u) + B(x,u) = f(x), \tag{23}$$

where $\vec{A}$ and B satisfy conditions (3), (4) and (5).

THEOREM 3: Assume $C_3 > 0$ and $q < p-1$. Assume that there exists $\rho_0 > 0$, $\delta > 0$ and $\varepsilon > 0$ (ε small enough), such that

$$\|f(x)\|^{1/(1-\lambda)}_{L^{r/(r-1)}(B_\rho(x_0))} \leq \varepsilon(\rho-\rho_0)_+^{1/(1-\sigma)} \quad \text{a.e. } \rho \in (0,\rho_0+\delta), \tag{24}$$

where

$$q+1 \leq r < \frac{Np}{N-p}, \quad \theta = \Big(\frac{1}{q+1} - \frac{1}{r}\Big)\Big/\Big(\frac{1}{q+1} - \frac{N-p}{Np}\Big),$$

$$\lambda = \Big(\frac{1-\theta}{q+1} + \frac{\theta}{p}\Big), \quad \sigma = \frac{p}{(p-1)}\Big(\frac{\theta}{p} + \frac{(1-\theta)}{q+1}\Big)).$$

Let u be any local weak solution of (23) and assume that the following a priori estimate holds

$$\|\nabla u\|_{L^p(B_{\rho+\delta}(x_0))} \leq \varepsilon^*$$

for some $\varepsilon^* > 0$ small enough. Then $u(x) = 0$ a.e. $x \in B_{\rho_0}(x_0)$.

PROOF: We introduce the energy functions

$$E(\rho) = \int_{B_\rho} \vec{A}(x,u,\nabla u)\cdot\nabla u \, dx, \quad b(\rho) = \int_{B_\rho} |u|^{q+1} \, dx.$$

Using the interpolation inequality

$$\|u\|_{L^r(B_\rho)} \leq C \, \|u\|^{1-\theta}_{L^{q+1}} \big(\|\nabla u\|_{L^p} + \|u\|_{L^{q+1}} \big)^\theta$$

with θ and r as in the statement of the theorem, and applying the interpolation-trace lemma of [13], we conclude that

$$E + b \leq C \Big(\frac{dE}{d\rho}\Big)^{(p-1)H/p} + \varepsilon(\rho-\rho_0)_+^{(1/(1-\sigma)}$$

for

$$H = \frac{1}{1 - \big(\frac{\theta}{p} + \frac{1-\theta}{q+1}\big)}.$$

The proof of Theorem 3 ends with the following lemma:

LEMMA 3: Let $y(\rho) \geq 0$ be such that

$$\phi(y(\rho)) \leq C_0 \frac{dy}{d\rho}(\rho) + G((\rho-\rho_0)_+), \text{ a.e. } \rho \in (0,\rho_0+\delta), \tag{25}$$

where ϕ is a nondecreasing continuous function such that $\phi(0) = 0$ and $1/\phi(\cdot) \in L^1(0,1)$. Given $\mu > 0$ we define θ_μ and η_μ as in Lemma 1. We also assume that

$$\exists \bar{\mu} > 0 \text{ such that } G(s) \leq (1-C_0\bar{\mu})\phi(\eta_{\bar{\mu}}(s)) \text{ a.e. } s \in (0,\delta) \tag{26}$$

and

$$\delta \geq \theta_{\bar{\mu}}(M) \quad \text{with} \quad M \geq y(\rho_0 + \rho). \tag{27}$$

Then $y(\rho) = 0$ a.e. $\rho \in [0,\rho_0]$.

PROOF OF LEMMA 3: Function $\bar{y}(\rho) = \eta_{\bar{\mu}}((\rho-\rho_0)_+)$ satisfies

$$- C_0 \frac{d\bar{y}}{d\rho} + \phi(\bar{y}) = (-C_0 \bar{\mu} + 1) \phi(\eta_{\bar{\mu}}((\rho-\rho_0)_+)),$$

and so it is a supersolution of the equation. Finally, by (26) and (27), we have that $y(\rho_0+\rho) \leq \bar{y}(\rho_0+\delta)$ and by comparison on $[0,\rho_0+\delta]$ we conclude the result. □

REMARK 8: The conclusion of Theorem 3 can be understood as a *nondiffusion of the support of u* with respect to the support of f. This property was first obtained in [12] for a special formulation of (23) and by means of a comparison argument. The case of B depending on ∇u, extensions to nonlinear systems, etc, will be given in [7].

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