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# Recent advances in nonlinear elliptic and parabolic problems

Proceedings of an international conference, Nancy, France, March 1988

Longman Scientific &

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# Existence of a solution for a weaker form of a nonlinear elliptic equation

ABSTRACT. Consider the nonlinear elliptic equation

$$-\mathrm{div}(A \text{ grad } u) - \mathrm{div}(\phi(u)) = f \text{ in } \Omega,\ u \in H_0^1(\Omega), \tag{1}$$

where A is a $(L^\infty(\Omega))^{N\times N}$ coercive matrix, $f \in H^{-1}(\Omega)$ and $\phi \in (C^0(R))^N$; no growth restriction is assumed on $\phi$; thus the term $\mathrm{div}(\phi(u))$ cannot be understood in the distributional sense.

In this paper we prove the existence of a solution of

$$[-\mathrm{div}(A \text{ grad } u)]h(u) - \mathrm{div}(\phi(u))h(u)) + \phi(u)h'(u)\text{grad } u = fh(u) \text{ in } \Omega,\ \forall h \in C_c^1(R),\ u \in H_0^1(\Omega) \tag{2}$$

where in contrast with (1) every term has a meaning in $\mathcal{D}'(\Omega)$ if $u \in H_0^1(\Omega)$. Equation (2) is a weaker form of the original problem, obtained in a formal way through a pointwise multiplication of (1) by $h(u)$.

SUNTO. Consideriamo l'equazione ellittica non lineare

$$-\mathrm{div}(A \text{ grad } u) - \mathrm{div}(\phi(u)) = f \text{ in } \Omega,\ u \in H_0^1(\Omega), \tag{1}$$

con A matrice coerciva a coefficienti limitati, $f \in H^{-1}(\Omega)$e $\phi \in (C^0(R))^N$; non facciamo su $\phi$ nessuna ipotesi di crescenza; in tal caso il termine $\mathrm{div}(\phi(u))$ non è, a priori, una distribuzione.

In quest' articolo proviamo l'esistenza di una soluzione del problema

$$[-\mathrm{div}(A \text{ grad } u)]h(u) - \mathrm{div}(\phi(u))h(u)) + \phi(u)h'(u)\text{grad } u = fh(u) \text{ in } \Omega,\ \forall h \in C_c^1(R),\ u \in H_0^1(\Omega) \tag{2}$$

dove invece ogni termine ha un senso in $\mathcal{D}'(\Omega)$. L'equazione (2) è una forma indebolita del problema originale, ottenuta in modo formale moltiplicando puntualmente (1) per $h(u)$.

RESUMEN. Consideramos la ecuación elíptica no lineal siguiente

$$-\mathrm{div}(A \text{ grad } u) - \mathrm{div}(\phi(u)) = f \text{ en } \Omega, \; u \in H_0^1(\Omega), \tag{1}$$

siendo A una matriz coerciva con coeficientes en $L^\infty$, $f \in H^{-1}(\Omega)$ y $\phi \in (C^0(R))^N$; no hacemos ninguna hipóthesis sobre el crecimiento de $\phi$ en el infinito; por lo que el término $\mathrm{div}(\phi(u))$ no puede ser definido como una distribución.

En este artículo demostramos la existencia de una solución de

$$\begin{aligned}&[-\mathrm{div}(A \text{ grad } u)]h(u) - \mathrm{div}(\phi(u)h(u)) + \phi(u)h'(u)\text{grad } u \\ &= fh(u) \text{ en } \Omega, \; \forall h \in C_c^1(R), \; u \in H_0^1(\Omega)\end{aligned} \tag{2}$$

donde todos los términos están bien definidos en $\mathcal{D}'(\Omega)$. La ecuación (2) es una forma débil del problema original obtenida de manera formal al multiplicar (1) por $h(u)$ en todo punto.

RÉSUMÉ. Considérons l'équation elliptique non linéaire

$$-\mathrm{div}(A \text{ grad } u) - \mathrm{div}(\phi(u)) = f \text{ dans } \Omega, \; u \in H_0^1(\Omega), \tag{1}$$

où A est une matrice coercive à coefficients $L^\infty$, $f \in H^{-1}(\Omega)$ et $\phi \in (C^0(R))^N$; nous ne faisons aucune hypothèse sur la croissance de $\phi$ à l'infini; le terme $\mathrm{div}(\phi(u))$ n'a donc a priori aucune raison d'être une distribution.

Nous démontrons dans cet article l'existence d'une solution de

$$\begin{aligned}&[-\mathrm{div}(A \text{ grad } u)]h(u) - \mathrm{div}(\phi(u)h(u)) + \phi(u)h'(u)\text{grad } u \\ &= fh(u) \text{ dans } \Omega, \; \forall h \in C_c^1(R), \; u \in H_0^1(\Omega)\end{aligned} \tag{2}$$

où, contrairement à (1), chaque terme a un sens dans $\mathcal{D}'(\Omega)$ dès que $u \in H_0^1(\Omega)$. L'équation (2) est une forme affaiblie du problème original obtenue de façon formelle en multipliant ponctuellement (1) par $h(u)$.



## 1. INTRODUCTION AND MAIN RESULTS

This paper investigates the existence of a solution for the following non-linear elliptic problem:

$$-\mathrm{div}(A \text{ grad } u) - \mathrm{div}(\phi(u)) = f \text{ in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega. \tag{1.2}$$

Here $\Omega$ denotes a bounded open subset of $R^N$, and A is a $N \times N$ coercive matrix with components in $L^\infty(\Omega)$, i.e. there exists $\alpha \in R$, $\alpha > 0$, such that

$$A \in (L^\infty(\Omega))^{N\times N}, \tag{1.3}$$

$$A(x)\xi\xi \geq \alpha|\xi|^2, \quad \forall\xi \in R^N, \text{ a.e. } x \in \Omega; \tag{1.4}$$

the right-hand side f of equation (1.1) is assumed to satisfy

$$f \in H^{-1}(\Omega). \tag{1.5}$$

Finally, let $\phi$ be a continuous function defined on R with values in $R^N$, i.e.

$$\phi \in (C^0(R))^N. \tag{1.6}$$

The main feature of the problem under consideration is that *no growth restriction is assumed on* $\phi$.

It is natural to seek a solution u of (1.1), (1.2) which belongs to $H_0^1(\Omega)$ since the right-hand side f of (1.1) belongs to $H^{-1}(\Omega)$. But when u is only in $H_0^1(\Omega)$ there is no reasonable ground for $\phi(u)$ to be in $(L^1(\Omega))^N$ since no growth restriction is assumed on $\phi$. Hence $\mathrm{div}(\phi(u))$ may be ill defined, even as a distribution.

This obstacle is bypassed by solving some weaker problem, obtained through pointwise multiplication of the original equation (1.1) by $h(u)$ where h belongs to $C_c^1(R)$, the class of the $C^1(R)$ functions with compact support.

<u>THEOREM 1.1</u>: Assume that (1.3), (1.4), (1.5) and (1.6) hold true. Then there exists a solution u of

$$u \in H_0^1(\Omega), \tag{1.7}$$

$$[-\operatorname{div}(A \operatorname{grad} u)]h(u) - \operatorname{div}(\phi(u)h(u)) + \phi(u)h'(u) \operatorname{grad} u$$
$$= fh(u) \text{ in } \mathcal{D}'(\Omega), \forall h \in C_c^1(R). \quad \square \tag{1.8}$$

In the equation (1.8) every term is meaningful in the distributional sense; indeed, for $h$ in $C_c^1(R)$ and $u$ in $H_0^1(\Omega)$, $h(u)$ belongs to $H^1(\Omega)$; thus for $f$ in $H^{-1}(\Omega)$ the product $fh(u)$ is the distribution defined by

$$\langle fh(u),\varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle f,\varphi\, h(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall\varphi \in \mathcal{D}(\Omega);$$

the same holds true for $[-\operatorname{div}(A \operatorname{grad} u)]\, h(u)$ since $-\operatorname{div}(A \operatorname{grad} u)$ belongs to $H^{-1}(\Omega)$. Further because $\phi h$ and $\phi h'$ belong to the class $C_c^0(R)$ of continuous functions with compact support, $\phi(u)\, h(u)$ and $\phi(u)\, h'(u)$ belong to $(L^\infty(\Omega))^N$ for any measurable $u$, which implies that $-\operatorname{div}(\phi(u)h(u))$ and $\phi(u)h'(u)$ grad $u$ are respectively a distribution (in $W^{-1,\infty}(\Omega)$) and a $L^2(\Omega)$ function.

Equation (1.8) follows formally from (1.1) by multiplying by $h(u)$ since

$$[-\operatorname{div}(\phi(u))]h(u) = -\operatorname{div}(\phi(u)h(u)) + \phi(u)h'(u) \operatorname{grad} u. \tag{1.9}$$

Note, however, that in contrast with the right-hand side, the left-hand side of (1.9) does not make sense when $h \in C_c^1(R)$. Thus (1.8) is to be viewed as a weaker form of (1.1).

The original equation (1.1) will be recovered whenever $h(u) \equiv 1$ (which does not belong to $C_c^1(R)$!) can be used in (1.8); such is not usually the case in general, except when stronger (regularity) requirements are met by $u$.

THEOREM 1.2: Assume that (1.3), (1.4), (1.5) and (1.6) hold true and define $\tilde\psi \in C^1(R)$ by

$$\tilde\psi(t) = \int_0^t |\phi(s)|ds. \tag{1.10}$$

Let $u$ be a solution of (1.7), (1.8) such that:

$$\phi(u) \in (L^1_{loc}(\Omega))^N, \tag{1.11}$$

$$\tilde\psi(u) \in L^1_{loc}(\Omega). \tag{1.12}$$

Then $u$ is a (usual weak) solution of the original problem (1.1), (1.2). $\square$

We do not know if (1.12) is a necessary condition for Theorem 1.2; (1.11) seems to be necessary to lend a distributional meaning to $\operatorname{div}(\phi(u))$.

Consider now a solution $u$ of (1.1). Formal multiplication of (1.1) by $u$ and integration by parts yields

$$\int_\Omega A \operatorname{grad} u \operatorname{grad} u \, dx + \int_\Omega \phi(u) \operatorname{grad} u \, dx = \langle f,u\rangle. \tag{1.13}$$

Define $\tilde\phi \in (C^1(R))^N$ as

$$\tilde\phi(t) = \int_0^t \phi(s)ds.$$

Then, formally, $\operatorname{div}(\tilde\phi(u)) = \phi(u) \operatorname{grad} u$ and since $\tilde\phi(0) = 0$

$$\int_\Omega \phi(u) \operatorname{grad} u \, dx = \int_\Omega \operatorname{div}(\tilde\phi(u))dx = \int_{\partial\Omega} \tilde\phi(0)n \, ds = 0; \tag{1.14}$$

thus

$$\int_\Omega A \operatorname{grad} u \operatorname{grad} u \, dx = \langle f,u\rangle. \tag{1.15}$$

Let us stress that most of the operations performed before are purely formal. However, relation (1.15) (and even an extension of (1.15)) can be proved whenever $u$ is a solution of (1.7), (1.8).

THEOREM 1.3: Assume that (1.3), (1.4), (1.5) and (1.6) hold true and that $u$ is a solution of (1.7), (1.8). Then

$$\int_\Omega s'(u) A \operatorname{grad} u \operatorname{grad} u \, dx = \langle f,s(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \tag{1.16}$$

for any Lipschitz continuous, piecewise $C^1(R)$ function $s$ such that $s(0) = 0$. $\square$

Note that Theorem 1.3 holds true for any solution of (1.7), (1.8), and not only for the solution that we will construct by approximation in the proof of Theorem 1.1; for the latter, (1.16) follows immediately from Theorem 2.1 below.

Existence results for (1.1), (1.2) were proved by Boccardo and Giachetti [4], [5] under a smoothness assumption on the right-hand side of (1.1) as well as a convenient growth condition on $\phi$: indeed a regularity result permits constructing a solution u in $L^{p^*}(\Omega)$ whenever f lies in $W^{-1,p}(\Omega), p \geq 2$; the condition $|\phi(t)| \leq c(t^{p^*}+1)$ then implies that $\phi(u)$ belongs to $(L^1(\Omega))^N$, and that u is a solution of the original equation (1.1). The uniqueness of the solution for equations of the type of (1.1) was investigated by Carrillo and Chipot [9], Carrillo [8] and Chipot and Michaille [10]; in these papers the solution u belongs to $L^\infty(\Omega)$ or $\phi$ is assumed to grow at most linearly at infinity.

The weaker form (1.8) of the problem (1.1) is very similar to the idea of "*renormalized solution*" introduced by Di Perna and Lions in their important papers [12], [13] when investigating the existence of solutions for the Boltzmann equation. It is also reminiscent of the introduction by Benilan et al. [1] of the space $T^{1,p}(\Omega)$ in the study of the existence and uniqueness of a solution for $-\mathrm{div}(|\mathrm{grad}\, u|^{p-2}\, \mathrm{grad}\, u) = f$ with f in $L^1(\Omega)$. Finally, it should be mentioned that this weaker form is related to the idea of entropy solutions for scalar nonlinear hyperbolic equations of the Burgers' type.

Our proof of Theorem 1.1 starts with an approximation $\phi^\varepsilon$ of $\phi$ which is bounded on R. In this case the above performed formal operations are licit and (1.15) provides a $H_0^1(\Omega)$ bound for the corresponding solution $u^\varepsilon$. The key point is then to prove that $u^\varepsilon$ is actually a compact sequence for the strong topology of $H_0^1(\Omega)$; this is achieved through the use of nonlinear (with respect to $u^\varepsilon$) test functions, in a spirit closely related to Bensoussan, Boccardo and Murat [2]. Passing to the limit to obtain Theorem 1.1 is then easy.

The proof of Theorem 1.2 consists in observing that assumptions (1.11), (1.12) allow us to pass to the limit in (1.8) for a sequence of functions $h_\varepsilon$ that converges to 1. In a similar manner the proof of Theorem 1.3 consists in approximating s(u) by s(u) $h_\varepsilon(u)$ with $h_\varepsilon$ converging to 1. (Use of such functions h (u) has already been made in Boccardo, Murat and Puel [7].)

Extensions of the present work to general Leray-Lions operators as well as to parabolic equations will be given in our forthcoming paper [3].

## 2. PROOF OF THEOREM 1.1

Define $T_m$ to be the truncation to level $m > 0$, i.e.

$$T_m(t) = \begin{cases} t & \text{if } |t| \leq m, \\ m\,\dfrac{t}{|t|} & \text{if } |t| \geq m, \end{cases} \tag{2.1}$$

and $\phi^\varepsilon$, to be the following approximation $\phi^\varepsilon$ of $\phi$:

$$\phi^\varepsilon(t) = \phi(T_{1/\varepsilon}(t)). \tag{2.2}$$

Consider the nonlinear elliptic equation

$$\begin{aligned} &-\mathrm{div}(A\, \mathrm{grad}\, u^\varepsilon) - \mathrm{div}(\phi^\varepsilon(u^\varepsilon)) = f \text{ in } \Omega, \\ &u^\varepsilon \in H_0^1(\Omega). \end{aligned} \tag{2.3}$$

Since $\phi^\varepsilon$ lies in $(C^0(R) \cap L^\infty(R))^N$, a simple application of Schauder's fixed point theorem in $L^2(\Omega)$ implies that (2.3) has (at least) one solution.

Recall now the following:

<u>LEMMA 2.1</u>: Let $\theta$ be a $(L^\infty(R))^N$, piecewise continuous function and v belong to $H_0^1(\Omega)$. Define the Lipschitz continuous, piecewise $(C^1(R))^N$ function $\tilde{\theta}$ by:

$$\tilde{\theta}(t) = \int_0^t \theta(s)\, ds. \tag{2.4}$$

Then

$$\tilde{\theta}(v) \in (H_0^1(\Omega))^N,$$

$$\mathrm{grad}\, v = 0 \text{ a.e. on the set } \{x \in \Omega | v(x) = c\} \text{ for any } c \in R, \tag{2.5}$$

$$\mathrm{div}(\tilde{\theta}(v)) = \theta(v)\, \mathrm{grad}\, v \text{ in } \Omega,$$

$$\int_\Omega \theta(v) \text{ grad } v \, dx = 0. \quad \square \tag{2.6}$$

This lemma is a classical result (see e.g. Kinderlehrer and Stampacchia [11, p. 54] or Boccardo and Murat [6, Theorem 4.2]). The assertion (2.6) follows from Stokes' theorem

$$\int_\Omega \theta(v) \text{ grad } v \, dx = \int_\Omega \text{div}(\tilde{\theta}(v))dx = 0,$$

since $\tilde{\theta}(v) \in (H_0^1(\Omega))^N$.

Multiplication of (2.3) by $u^\varepsilon$ and integration by parts yields in view of (2.5), (2.6)

$$\langle -\text{div}(\phi^\varepsilon(u^\varepsilon)), u^\varepsilon \rangle = \int_\Omega \phi^\varepsilon(u^\varepsilon) \text{ grad } u^\varepsilon \, dx = 0, \tag{2.7}$$

since $\phi^\varepsilon$ lies in $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$ [contrast (2.7), which is licit since $\phi^\varepsilon$ belongs to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$ with (1.14), which is formal since $\phi$ only belongs to $(C^0(\mathbb{R}))^N$]. Thus we have

$$\int_\Omega A \text{ grad } u^\varepsilon \text{ grad } u^\varepsilon \, dx = \langle f, u^\varepsilon \rangle \tag{2.8}$$

and the coersiveness (1.4) implies that

$$\|u^\varepsilon\|_{H_0^1(\Omega)} \le \frac{1}{\alpha} \|f\|_{H^{-1}(\Omega)}.$$

At the possible expense of extracting a subsequence (still denoted by $\varepsilon$), we conclude that

$$u^\varepsilon \longrightarrow u \text{ in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{2.9}$$

We shall prove the following:

<u>THEOREM 2.1</u>: The subsequence $u^\varepsilon$ tends *strongly* to $u$ in $H_0^1(\Omega)$. □

The proof of Theorem 2.1 is based on two lemmas.



<u>LEMMA 2.2</u>: Define for $k > 0$ the set

$$E_k^\varepsilon = \{x \in \Omega \,|\, |u^\varepsilon(x)| \ge k\}. \tag{2.10}$$

Then for any fixed $k > 0$

$$\limsup_{\varepsilon \to 0} \int_{E_k^\varepsilon} |\text{grad } u^\varepsilon|^2 \, dx \le \frac{1}{\alpha} \langle f, u - T_k(u) \rangle. \quad \square$$

<u>PROOF</u>: Consider the test function

$$v^\varepsilon = u^\varepsilon - T_k(u^\varepsilon) \in H_0^1(\Omega).$$

Defining $\chi_k$ as

$$\chi_k(t) = \begin{cases} 0 & \text{if } |t| < k, \\ 1 & \text{if } |t| \ge k, \end{cases} \tag{2.12}$$

we obtain by virtue of Lemma 2.1:

$$\text{grad } v^\varepsilon = \chi_k(u^\varepsilon) \text{ grad } u^\varepsilon.$$

Application of Lemma 2.1 to the $(L^\infty(\mathbb{R}))^N$, piecewise continuous function $\theta(s) = \phi^\varepsilon(s)\chi_k(s)$ yields

$$\langle -\text{div}(\phi^\varepsilon(u^\varepsilon)), v^\varepsilon \rangle = \int_\Omega \phi^\varepsilon(u^\varepsilon)\chi_k(u^\varepsilon) \text{ grad } u^\varepsilon = 0.$$

Thus multiplication of (2.3) by $v^\varepsilon$ and integration by parts yields:

$$\int_\Omega \chi_k(u^\varepsilon) A \text{ grad } u^\varepsilon \text{ grad } u^\varepsilon \, dx = \langle f, u^\varepsilon - T_k(u^\varepsilon) \rangle. \tag{2.13}$$

Since the right-hand side of (2.13) tends to $\langle f, u - T_k(u) \rangle$ as $\varepsilon \to 0$, the coerciveness assumption (1.4) implies Lemma 2.2.

<u>LEMMA 2.3</u>: Define for $i > 0$ and $j > 0$ the set

$$F_{ij}^{\varepsilon} = \{x \in \Omega |\ |u^{\varepsilon}(x) - T_j(u(x))| \le i\}. \tag{2.14}$$

Then for any fixed $i > 0$ and $j > 0$

$$\limsup_{\varepsilon \to 0} \int_{F_{ij}^{\varepsilon}} |\text{grad}(u^{\varepsilon} - T_j(u))|^2 \, dx \le$$

$$\le \frac{1}{\alpha} \langle f, T_i(u - T_j(u)) \rangle - \frac{1}{\alpha} \int_{\Omega} A \text{ grad } T_j(u) \text{ grad } T_i(u - T_j(u)) dx. \quad \square \tag{2.15}$$

PROOF: Consider the test function

$$w^{\varepsilon} = T_i(u^{\varepsilon} - T_j(u)) \in H_0^1(\Omega),$$

and define

$$X^{\varepsilon} = \langle -\text{div}(\phi^{\varepsilon}(u^{\varepsilon})), w^{\varepsilon} \rangle = \int_{\Omega} \phi^{\varepsilon}(u^{\varepsilon}) \text{ grad } w^{\varepsilon} \, dx;$$

we claim that

$$X^{\varepsilon} \to 0 \text{ as } \varepsilon \to 0 \text{ for any fixed } i > 0 \text{ and } j > 0. \tag{2.16}$$

Indeed by Lemma 2.1

$$\text{grad } w^{\varepsilon} = \begin{cases} \text{grad } (u^{\varepsilon} - T_j(u)) & \text{on } F_{ij}^{\varepsilon}, \\ 0 & \text{on } \Omega \setminus F_{ij}^{\varepsilon}. \end{cases} \tag{2.17}$$

Since

$$|u^{\varepsilon}(x)| \le |u^{\varepsilon}(x) - T_j(u(x))| + |T_j(u(x))| \le i+j \text{ on } F_{ij}^{\varepsilon}$$

we have

$$\phi^{\varepsilon}(u^{\varepsilon}(x)) = \phi(T_{1/\varepsilon}(u^{\varepsilon}(x))) = \phi(T_{i+j}(u^{\varepsilon}(x))) \text{ on } F_{ij}^{\varepsilon} \text{ for } 1/\varepsilon \ge i+j.$$

Thus whenever $1/\varepsilon \ge i + j$

$$X^{\varepsilon} = \int_{\Omega} \phi^{\varepsilon}(u^{\varepsilon}) \text{ grad } w^{\varepsilon} \, dx = \int_{F_{ij}^{\varepsilon}} \phi^{\varepsilon}(u^{\varepsilon}) \text{ grad } w^{\varepsilon} \, dx$$

$$= \int_{\Omega} \phi(T_{i+j}(u^{\varepsilon})) \text{ grad } w^{\varepsilon} \, dx,$$

which tends to

$$X = \int_{\Omega} \phi(T_{i+j}(u)) \text{ grad } T_i(u - T_j(u)) dx, \tag{2.18}$$

since $w^{\varepsilon}$ tends to $T_i(u-T_j(u))$ weakly in $H_0^1(\Omega)$ while $\phi(T_{i+j}(u^{\varepsilon}))$, which is bounded in $(L^{\infty}(\Omega))^N$ and converges a.e., tends to $\phi(T_{i+j}(u))$ strongly in $(L^2(\Omega))^N$ through application of Lebesgue's dominated convergence theorem. Applying Lemma 2.1 to the $(L^{\infty}(R))^N$ piecewise continuous function $\theta(t) = \phi(T_{i+j}(t))\,[1 - \chi_i(t-T_j(t))]\chi_j(t)$ proves that the right-hand side of (2.18) is zero, hence (2.16).

Multiplying now (2.3) by $w^{\varepsilon}$, integrating by parts and using (2.17) and (2.16) we obtain

$$\lim_{\varepsilon \to 0} \int_{F_{ij}^{\varepsilon}} A \text{ grad } u^{\varepsilon} \text{ grad}(u^{\varepsilon} - T_j(u)) dx = \langle f, T_i(u - T_j(u)) \rangle. \tag{2.19}$$

Substracting to both sides of (2.19) the quantity

$$\lim_{\varepsilon \to 0} \int_{F_{ij}^{\varepsilon}} A \text{ grad } T_j(u) \text{ grad } (u^{\varepsilon} - T_j(u)) dx$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega} A \text{ grad } T_j(u) \text{ grad } T_i(u^{\varepsilon} - T_j(u)) dx$$

$$= \int_{\Omega} A \text{ grad } T_j(u) \text{ grad } T_i(u - T_j(u)) dx$$

and using the coerciveness (1.4) complete the proof of Lemma 2.3. $\square$

PROOF OF THEOREM 2.1: From

$$|u^{\varepsilon}(x)| \ge |u^{\varepsilon}(x) - T_j(u(x))| - |T_j(u(x))| \ge i-j \text{ on } \Omega \setminus F_{ij}^{\varepsilon},$$

We deduce that $\Omega\setminus F^{\varepsilon}_{ij} \subset E^{\varepsilon}_{i-j}$. Then:

$$\int_{\Omega} |\mathrm{grad}(u^{\varepsilon}-u)|^2\, dx \leq \int_{F^{\varepsilon}_{ij}} |\mathrm{grad}(u^{\varepsilon}-u)|^2\, dx + \int_{E^{\varepsilon}_{i-j}} |\mathrm{grad}(u^{\varepsilon}-u)|^2\, dx$$

$$\leq 2\int_{F^{\varepsilon}_{ij}} |\mathrm{grad}(u^{\varepsilon}-T_j(u)|^2 dx + 2\int_{F^{\varepsilon}_{ij}} |\mathrm{grad}(T_j(u)-u)|^2 dx \qquad (2.20)$$

$$+ 2\int_{E^{\varepsilon}_{i-j}} |\mathrm{grad}u^{\varepsilon}|^2 dx + 2\int_{E^{\varepsilon}_{i-j}} |\mathrm{grad}\ u|^2\, dx = I^{\varepsilon}_{ij} + II^{\varepsilon}_{ij} + III^{\varepsilon}_{ij} + IV^{\varepsilon}_{ij}.$$

We now estimate the lim sup, for $i$ and $j$ fixed and $\varepsilon$ converging to zero, of each of the four terms of the right-hand side of (2.20).

In view of Lemma 2.3, $\limsup_{\varepsilon\to 0} I^{\varepsilon}_{ij}$ is small provided $j$ is sufficiently large; indeed, since

$$\|\mathrm{grad}\ T_k(v)\|_{(L^2(\Omega))^N} \leq \|\mathrm{grad}\ v\|_{(L^2(\Omega))^N},\ \forall k > 0,\ \forall v \in H^1_0(\Omega),$$

the right-hand side of (2.15) is bounded by

$$\frac{1}{\alpha}\left[\ \|f\|_{H^{-1}(\Omega)} + \|A\|_{(L^{\infty}(\Omega))^{N\times N}} \|\mathrm{grad}\ u\|_{(L^2(\Omega))^N}\right] \|\mathrm{grad}(u-T_j(u)\|_{(L^2(\Omega))^N}$$

which tends to zero when $j$ tends to infinity.

Since $F^{\varepsilon}_{ij}$ is a subset of $\Omega$

$$\limsup_{\varepsilon\to 0} II^{\varepsilon}_{ij} \leq 2\int_{\Omega} |\mathrm{grad}(T_j(u)-u)|^2\, dx,$$

which is small when $j$ is large.

In view of Lemma 2.2, $\limsup_{\varepsilon\to 0} III^{\varepsilon}_{ij}$ is small provided $i-j$ is sufficiently large.

Finally, splitting $\Omega$ into the union of the sets $\{x|\ |u(x)| \neq i-j\}$ and $\{x|\ |u(x)| = i-j\}$ (note that grad $u = 0$ a.e. on the second set by virtue of (2.5)), and using Lebesgue's dominated convergence theorem, it is easy to prove that

$$\limsup_{\varepsilon\to 0} IV^{\varepsilon}_{ij} \leq 2\int_{\{x|\,|u(x)|>i-j\}} |\mathrm{grad}\ u|^2\, dx,$$

which is small when $i-j$ is large.

Hence $\limsup_{\varepsilon\to 0} \int_{\Omega} |\mathrm{grad}(u^{\varepsilon}-u)|^2\, dx$ is small if we choose $j$, next $i-j$, sufficiently large. Theorem 2.1. is proved. □

<u>END OF THE PROOF OF THEOREM 1.1</u>: Let $h$ belong to $C^1_c(\mathbb{R})$ and $\varphi$ belongs to $\mathcal{D}(\Omega)$. Multiplying (2.3) by $h(u^{\varepsilon})\varphi$ which belongs to $H^1_0(\Omega)$ and integrating by parts we obtain

$$\int_{\Omega} (A\ \mathrm{grad}\ u^{\varepsilon} + \phi^{\varepsilon}(u^{\varepsilon}))(h'(u^{\varepsilon})\varphi\ \mathrm{grad}\ u^{\varepsilon} + h(u^{\varepsilon})\ \mathrm{grad}\ \varphi)\, dx = \qquad (2.21)$$

$$= \langle f, h(u^{\varepsilon})\varphi\rangle$$

Since $h$ and $h'$ have compact support on $\mathbb{R}$, we have for $\varepsilon$ sufficiently small

$$\phi^{\varepsilon}(t)h(t) = \phi(T_{1/\varepsilon}(t))h(t) = \phi(t)h(t),$$
$$\phi^{\varepsilon}(t)h'(t) = \phi(T_{1/\varepsilon}(t))h'(t) = \phi(t)h'(t), \qquad (2.22)$$

and the right-hand sides in (2.22) are $(C^0(\mathbb{R}) \cap L^{\infty}(\mathbb{R}))^N$ functions. Because $u^{\varepsilon}$ converges to $u$ strongly in $H^1_0(\Omega)$ (see Theorem 2.1) it is easy to pass to the limit successively in each term of (2.21); this yields

$$\int_{\Omega} (A\ \mathrm{grad}\ u + \phi(u))(h'(u)\varphi\ \ \mathrm{grad}\ u + h(u)\ \mathrm{grad}\ \varphi)\, dx$$

$$= \langle f, h(u)\varphi\rangle,\ \forall h \in C^1_c(\mathbb{R}),\ \forall\varphi \in \mathcal{D}(\Omega)$$

which is equivalent to (1.8). Theorem 1.1 is proved. □

## 3. <u>PROOFS OF THEOREMS 1.2 AND 1.3</u>

<u>PROOF OF THEOREM 1.2</u>: Let $u$ in $H^1_0(\Omega)$ be a solution of (1.8). Consider a function $H \in \mathcal{D}(\mathbb{R})$ such that

$$H(t) = 1 \text{ if } |t| \leq 1,\ H(t) = 0 \text{ if } |t| \geq 2,$$

$$|H(t)| \leq 1,\ |H'(t)| \leq 2,\ \forall t \in R,$$

and define for $\varepsilon > 0$ $h_\varepsilon$ by:

$$h_\varepsilon(t) = \begin{cases} H(t + 1/\varepsilon) & \text{if } t + 1/\varepsilon \leq 0, \\ 1 & \text{if } |t| \leq 1/\varepsilon, \\ H(t - 1/\varepsilon) & \text{if } t - 1/\varepsilon \geq 0. \end{cases} \tag{3.1}$$

The function $h^\varepsilon$ belongs to $C_c^1(R)$ and can be used in (1.8); multiplying (1.8) by $\varphi \in \mathcal{D}(\Omega)$ and integrating by parts we obtain

$$\int_\Omega A \operatorname{grad} u\,(h_\varepsilon'(u)\varphi \operatorname{grad} u + h_\varepsilon(u) \operatorname{grad} \varphi)\, dx$$

$$+ \int_\Omega \phi(u) h_\varepsilon(u) \operatorname{grad} \varphi\, dx + \int_\Omega \varphi\phi(u) h_\varepsilon'(u) \operatorname{grad} u\, dx \tag{3.2}$$

$$= \langle f, h_\varepsilon(u)\varphi\rangle.$$

Since for any $t \in R$,

$$|h_\varepsilon(t)| \leq 1, \quad h_\varepsilon(t) \to 1 \quad \text{as } \varepsilon \to 0, \tag{3.3}$$

$$|h_\varepsilon'(t)| \leq 2, \quad h_\varepsilon'(t) \to 0 \quad \text{as } \varepsilon \to 0, \tag{3.4}$$

Lebesgue's dominated convergence theorem implies that

$$h_\varepsilon(u) \to 1 \text{ in } H^1(\Omega) \text{ strongly.}$$

We can thus pass to the limit in the first and last terms of (3.2).

If we assume now that $\phi(u) \in (L^1_{loc}(\Omega))^N$ (hypothesis (1.11)), Lebesgue's dominated convergence and (3.3) allow to pass to the limit in the second term of (3.2), since in this case

$$\phi(u) h^\varepsilon(u) \to \phi(u) \text{ in } (L^1_{loc}(\Omega))^N \text{ strongly.}$$

Define now

$$\tilde{\psi}_\varepsilon(t) = \int_0^t \phi(s) h_\varepsilon'(s)\, ds;$$

then by virtue of (3.4) and of the definition (1.10) of $\tilde{\psi}$

$$|\tilde{\psi}_\varepsilon(t)| \leq 2|\tilde{\psi}(t)| \quad \tilde{\psi}_\varepsilon(t) \to 0 \text{ as } \varepsilon \to 0,\ \forall t \in R.$$

If we assume that $\tilde{\psi}(u) \in L^1_{loc}(\Omega)$ (hypothesis (1.12)), this implies (using once again Lebesgue's dominated convergence theorem) that

$$\tilde{\psi}_\varepsilon(u) \to 0 \text{ in } L^1_{loc}(\Omega) \text{ strongly.}$$

Thus the third term of (3.2), which is equal to $\langle -\operatorname{div}(\tilde{\psi}_\varepsilon(u)), \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$ (see Lemma 2.1), tends to zero, when (1.12) holds true.

We have proved that for any $\varphi$ in $\mathcal{D}(\Omega)$

$$\int_\Omega A \operatorname{grad} u \operatorname{grad} \varphi\, dx + \int_\Omega \phi(u) \operatorname{grad} \varphi\, dx = \langle f, \varphi\rangle$$

which is equivalent to (1.1). Theorem 1.2 is proved. □

PROOF OF THEOREM 1.3: Let u be a solution of (1.7), (1.8) and s be a Lipschitz continuous, piecewise $C^1$ function from R to R such that $s(0) = 0$.

First step. We assume here that s is bounded. Then $s(u)$ belongs to $H^1_0(\Omega) \cap L^\infty(\Omega)$. In this case there exists a sequence $\varphi_n$ such that

$$\begin{cases} \varphi_n \in \mathcal{D}(\Omega), & \|\varphi_n\|_{L^\infty(\Omega)} \leq C, \\ \varphi_n \to s(u) \text{ in } H^1_0(\Omega) \text{ strongly as } n \to +\infty. \end{cases}$$

Then for any $h \in C_c^1(R)$

$$h(u)\varphi_n \to h(u)s(u) \text{ in } H^1_0(\Omega) \text{ strongly as } n \to \infty.$$

Using $\varphi_n \in \mathcal{D}(\Omega)$ as a test function in (1.8) and passing to the limit yields:

$$\langle -\mathrm{div}(A \text{ grad } u), h(u)s(u)\rangle + \int_\Omega \phi(u)h(u) \text{ grad } s(u)\, dx + \int_\Omega s(u)\phi(u)h'(u) \text{ grad } u\, dx = \langle f, h(u)s(u)\rangle, \quad \forall h \in C_c^1(R). \tag{3.5}$$

Lemma 2.1, applied first to the $(L^\infty(R))^N$, piecewise continuous function $\phi(t) = \phi(t)h(t)s'(t)$, next to the $(C^0(R) \cap L^\infty(R))^N$ function $\theta(t) = s(t)\phi(t)h'(t)$, proves that the second and third terms in (3.5) are zero. Thus

$$\langle -\mathrm{div}(A \text{ grad } u), h(u)s(u)\rangle = \langle f, h(u)s(u)\rangle, \quad \forall h \in C_c^1(R). \tag{3.6}$$

Use now in (3.6) the functions $h_\varepsilon$ defined in (3.1); from (3.3) and (3.4) it is easy to prove that for any Lipschitz continuous, piecewise $C^1$ function s such that $s(0) = 0$ *which is bounded*, one has:

$$h_\varepsilon(u)s(u) \to s(u) \text{ in } H_0^1(\Omega) \text{ strongly as } \varepsilon \to 0.$$

Thus passing to the limit in (3.6) proves that

$$\langle -\mathrm{div}(A \text{ grad } u), s(u)\rangle = \langle f, s(u)\rangle \tag{3.7}$$

for any Lipschitz continuous, piecewise $C^1$ function s such that $s(0) = 0$, $s \in L^\infty(R)$.

Second Step. Consider the general case where the function s is not assumed to be bounded, and let for $m > 0$ $s_m(t) = T_m(s(t))$ be the truncation of s at the level m (see (2.1)); we can use $s_m$ in (3.7). On the other hand, it is easy to prove that

$$s_m(u) = T_m(s(u)) \to s(u) \text{ in } H_0^1(\Omega) \text{ strongly as } m \to \infty.$$

Passing to the limit in (3.7) gives

$$\langle -\mathrm{div}(A \text{ grad } u), s(u)\rangle = \langle f, s(u)\rangle$$

for any Lipschitz continuous, piecewise continuous s, such that $s(0) = 0$,

which is equivalent to (1.16). Theorem 1.3 is proved. □

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