ON THE MATHEMATICAL ANALYSIS OF TRANSIENT CAVITATION PROBLEMS IN HYDRODYNAMICS LUBRICATION

J.I. DIAZ
Depto. Matemática Aplicada
Universidad Complutense de Madrid
28040 Madrid, SPAIN

1. INTRODUCTION. The problem we shall consider in this work have the following general formulation

$$(1)\quad \begin{cases} \frac{\partial}{\partial t}(h\chi)-\mathrm{div}(h^3\nabla u-h\chi\underline{e})=0 & \text{in } Q=(0,T)\times\Omega \\ \text{boundary conditions} & \text{on } \Sigma=(0,T)\times\partial\Omega \\ \chi(0,.)=\chi_0(.) & \text{on } \Omega \ , \end{cases}$$

where Ω is an open bounded regular set of $\mathbb{R}^N$, $N\geq 2$, h is a $C^\infty(Q)$ given function such that

$$0<m\leq h(t,x)\leq M \quad \text{on } Q \ ,$$

$\underline{e}$ is a given vector of $\mathbb{R}^N$ and the unknowns u and χ are related by

$$\chi(t,x)\in H(u(t,x)) \qquad \forall t\in[0,T] \text{ and a.e. } x\in\Omega,$$

with H the Heaviside maximal monotone graph defined by $H(r)=\{0\}$ if $r<0$, $H(r)=\{1\}$ if $r>0$ and $H(0)=[0,1]$.

Problem (1) corresponds to the ELROD-ADAMS model for transient hydrodynamics lubrication (see [8]). There u represents the pressure and χ the relative content.

Several others fields also leads to problems formulated as (1). This is the case of the "nonsteady dam problem", "Hele-Shaw problem" and the "electrochemical machine" problem. In those formulations $h\equiv 1$ and $\underline{e}$ is either a vertical vector (0,0,.,0,1) or the null vector $\underline{0}$ (see Crank [7]).

Formulation (1) also appears as limit problem associated to the porous media equation

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$$\frac{\partial}{\partial t} v - \mathrm{div}(\underline{e}v) = \Delta v^m$$

as $m \to \infty$ (here $h \equiv 1$ and $u \sim v^m$).

In order to fix ideas we shall concentrate our attention on the "journal bearing problem" which for which $\Omega=(0,2\pi)x(0,1)$ and the boundary conditions are $u(t,x,1)=a$, $u(t,x,0)=0$, $u(t,x+2\pi,y)=u(t,x,y)$ for any $t\in(0,T)$, $x\in(0,2\pi)$ and $y\in(0,1)$.

The plan of the rest of the paper is the following: In section 2 we shall show that (1) is a well-posed problem in the sense of the accretive operator theory on the Banach space $X=L^1(\Omega)$ and we shall deduce several consequences from that fact. (Those results are a slight generalization of the ones presented in ALVAREZ-DIAZ-CARRILLO [2]). The last section (section 3) is devoted to the study of the cavitation region and extends some of the results of CARRILLO-DIAZ-GILARDI [5]. Detailed proofs will be given elsewhere.

2. BASIC THEORY FOR THE TRANSIENT JOURNAL BEARING PROBLEM.

As already said in the introduction, to fix ideas, we shall consider here "transient journal bearing problem" which following [8] is formulated in the following terms: $\Omega=(0,2\pi)x(0,1)$, $\Gamma_1=(0,2\pi)x\{1\}$, $\Gamma_0=(0,2\pi)x\{0\}$,

$$(2)\quad \begin{cases} \frac{\partial}{\partial t}(h\chi) - \mathrm{div}(h^3\nabla u - h\chi\underline{e}) = 0 & \text{in } Q \\ u = a & \text{on } (0,T)x\Gamma_1 \\ u = 0 & \text{on } (0,T)x\Gamma_0 \\ u(t,x+2\pi,y) = u(t,x,y) & \text{for } t\in(0,T), x\in(0,2\pi) \text{ and } y\in(0,1) \\ \chi(0,x) = \chi_0(x) & \text{on } \Omega\ , \end{cases}$$

where $a>0$ is a given number (representing the atmospheric pressure). Our first goal is to formulate (2) as a Cauchy Problem

$$(3)\quad \begin{cases} \frac{dv}{dt}(t) + A(t)v(t) \ni 0 & \text{a.e. } t\in(0,T), \text{ in } X \\ v(0) = v_0 \end{cases}$$

where X is the Banach space $L^1(\Omega)$ and A is a suitable operator on X, $A:D(A) \subset X \longrightarrow P(X)$. To do that we start by denoting $\alpha=H^{-1}$ (in the sense of maximal monotone graphs) i.e. $\alpha(r)=\phi$ (the empty set) $\alpha(1)=[0,+\infty)$. if $r\in(-\infty,0)\cup(1,\infty)$, $\alpha(r)=\{0\}$ if $r\in[0,1]$, $\alpha(0)=(-\infty,0]$ and $\alpha(1)=[0,+\infty)$.

If we define

$$v(t,x)=h(t,x)\chi(t,x) \quad , \; x=(x,y)$$

we easily check that

$$\chi\in H(u) \Leftrightarrow u\in\alpha(\frac{v}{h}) .$$

So, (2) can be equivalently formulated in the following terms

$$(4) \begin{cases} \frac{\partial v}{\partial t} - \mathrm{div}(h^3\nabla u - v\underline{e})=0 & \text{in } Q \\ u=a & \text{on } (0,T)\times\Gamma_1 \\ u=0 & \text{on } (0,T)\times\Gamma_0 \\ u \text{ is } 2\pi\text{-x periodic} & \text{on } (0,T)\times\{0\}\times(0,1)\cup\{2\pi\}\times(0,1) \\ v(0,x)=v_0(x) & \text{on } \Omega , \end{cases}$$

where $v_0(\underline{x})\chi_0(\underline{x})$. In this way, (4) corresponds to a formulation in terms of (3) if we introduce

$v:[0,T]\longrightarrow L^1(\Omega)$, $v(t)(\underline{x})=v(t,x)$,

$A(t)w=-\mathrm{div}(h^3(t,.)\nabla\theta(t,.)-w\underline{e})$ for any $w\in D(A)$ where $D(A)=\{w\in L^1(\Omega)$ such that $\exists\ \theta(t,\underline{x})\in\alpha(\frac{w(x)}{h(t,\underline{x})})$ a.e. $x\in\Omega$, $t\in(0,T)$, $\mathrm{div}(\nabla\theta(t,.)-w\underline{e})\in L^1(\Omega)$ in the sense of $D'(\Omega)$, for a.e. $t\in(0,T)$, $\theta(t,.)=a$ on $(0,T)\times\Gamma_1$, $\theta(t,.)=0$ on $(0,T)\times\Gamma_0$ and θ is 2π-x-periodic$\}$.

The main difficulties in our study come from the fact that α is multivalued, from that $D(\alpha)=[0,1]$ and from the t-dependence of $A(t)$.

2.1. Definitions and results of the abstract theory.

We recall some facts from the abstract theory: Let X be a Banach space and $B:D(B)\subset X\longrightarrow P(X)$. We say that B is accretive (in X) if there exists λ_0 such that $\forall\lambda\geq\lambda_0$

$$\|w_1-w_2\|\leq\|w_1-w_2+\lambda(z_1-z_2)\| \qquad \forall z_1\in Bw_1, z_2\in Bw_2,$$

or, equivalently, if the resolvent operator

$$J_\lambda:=(I+\lambda B)^{-1}$$

is a contraction ($\|J_\lambda w_1-J_\lambda w_2\|\leq\|w_1-w_2\|$). When X is a Hilbert Space the accretive operators are called as monotone operators. Finally, we say that B is a m-accretive operator (in X) if it is accretive and $R(I+\lambda B)=X$.

The importance of the accretive operators comes from a well-know result due to CRANDALL and LIGGETT (1971) showing that the Cauchy Problem (3) (with $A(t)\equiv B$) is well posed in X if B is m-accretive.

In the case of time-dependent operators we extend the above notions in a natural way. Nevertheless the study of the Cauchy Problem now requires some additional hypothesis on the t-dependence of A(t).

$$
(5)\quad \begin{cases} \|J_\lambda(t)w-J_\lambda(s)w\| \le |f(t)-f(s)|\,L \\ \text{for any } t,s\in[0,T],\ \forall\lambda>0,\ \forall w\in X \text{ and for some} \\ L>0 \text{ and some function } f:[0,T]\longrightarrow\mathbb{R}, \end{cases}
$$

where f(t) is assumed to be either Lipschitz-continuous (CRANDALL-LIGGETT and EVANS-MASSEY), Riemann integrable (PLANT) or Lebesgue integrable (EVANS). See references in [9]. We recall, in particular, the main result of [9]:

Theorem. *Let* A(t) *be a family of m-accretive operators satisfying* (5) *for some* f *Lipschitz continuous. Then for any* $v_0\in\overline{D(A(0))}$ *there exists a unique solution of the problem* (3). *Moreover* v *is the (uniform) limit of the step functions* $v^n(t)=w^n_k$ *on* $(t^n_{k-1},t^n_k]$, *where*

$$P^n\equiv\{0=t^n_0<t^n_1<\ldots<t^n_{N(n)}\equiv T(n)\}$$

is any partition such that

$$\lim_{n\to\infty}\ \max_{1\le k\le N(n)} (t^n_t-t^n_{k-1})=0$$

and

$$\frac{w^n_k-w^n_{k-1}}{t^n_k-t^n_{k-1}}+A(t^n_k)w^n_k\ni 0 .$$

<u>2.2. Application to the journal bearing problem.</u> Our program in this subsection will be: (i) Definition of an operator $A_2(t)$ on $L^2(\Omega)$ such that it is accretive in $L^1(\Omega)$ and $R(I+\lambda A_2(t))\supset L^\infty(\Omega)$; (ii) Definition of A(t) as the closure of $A_2(t)$ in $L^1(\Omega)$ [and so by (1) m-accretive in $L^1(\Omega)$]; (iii) Checking assumption (5).

The definition of $A_2(t)$ is similar to the already anticipated in a formal way:

$D(A_2(t))=\{w\in L^\infty(\Omega): 0\le w(x)\le h(t,x)$ a.e. $x\in\Omega$, $t\in(0,T)$ such that there exist $u(t)(x)\in\alpha(\frac{w(x)}{h(t)(x)})$ satisfying $u\in L^1(0,T:H^1(\Omega))$, $\mathrm{div}(h^3(t)\nabla u(t)-w\underline{e})\in L^2(\Omega)$ and $u(t)=a$ on Γ_1, $u(t)=0$ on Γ_2 and u(t) is 2π-x-periodic$\}$,

$$A_2(t)w=-\mathrm{div}(h^3(t)\nabla u(t)-w\underline{e}) \text{ if } w\in D(A_2(t)).$$

We have

Theorem. *There exists a positive* λ_0 *such that* $\forall\lambda\geq\lambda_0$ *and* $g\in L^{\infty}(\Omega)$ *there exists a unique* w, $w\in D(A_2(t))$ *(t fixed) satisfying*

(6) $$w+\lambda A_2(t)w\ni g \ .$$

Moreover, if w_1 *and* w_2 *are solutions of* (6) *corresponding to* g_1 *and* g_2 *respectively one has*

(7) $$\|(w_1-w_2)^+\|_{L^1(\Omega)}\leq\|(g_1-g_2)^+\|_{L^1(\Omega)}$$

Finally, A(t) *satisfies* (5) *for some Lipschitz-continuous function* f.

Idea of the proof. First step: Let α_ε be a strictly increasing Lipschitz continuous function such that $\alpha_\varepsilon\to\alpha$ (in the sense of maximal monotone graphs). Then the first part of the statement holds by replacing α by α_ε (this uses the pseudo-monotonicity of the operator $A_2(t)$). Estimate (7) follows by multiplying by $\text{sign}_+ (w_1^\varepsilon-w_2^\varepsilon)$ (first we regularize the sign function and later we use that $\text{sign}_+(w_1^\varepsilon-w_2^\varepsilon)=\text{sign}_+(u_1^\varepsilon(t)-u_2^\varepsilon(t))$.

Second step. The following "a priori" estimates holds

(8) $$\|w^\varepsilon\|_{L^\infty}\leq\|g\|_{L^\infty}$$

and

(9) $$\|u^\varepsilon(t)\|_{H^1}\leq C \text{ (independently of } \varepsilon).$$

(Estimate (8) is obtained by multiplying by $\text{sign}_+(w^\varepsilon-k)$ for some suitable k>0, and (9) comes by multiplying by u^ε). Then $u^\varepsilon(t)\longrightarrow u(t)$ weakly in $H^1(\Omega)$ and $w^\varepsilon\longrightarrow w$ weakly in $L^2(\Omega)$. Using a result of BENILAN-CRANDALL-SACHS [3] we have that $u(t)\in\alpha(w/h(t))$ and that w satisfies (6).

Third step Property (5) is obtained by using (7) and an estimate on $\|\nabla u(t)\|$ in $L^1(\Omega)$ ∎

Remark. It is important to remark that by a recent result of ALVAREZ-CARRILLO [1] the solution w of (6) is unique. That allows to have (7) from the similar estimate for the approximate problem replacing α by α_ε. A similar program of proof for the Stefan Problem can be found in RULLA [10].

2.3. Other consequences of the accretiveness of A(t).

Once we have proved that the operator A(t) is m-accretive in $L^1(\Omega)$ many results from the abstract theory can be applied (see BENILAN-CRANDALL-PAZY [4]). For instance:

(a) The assumption $v_0\geq 0$ on Ω implies $w\geq 0$ on Q.

(b) The equation in (4) is satisfied in distributional sense.

(c) If (χ,u) ,$(\hat{\chi},\hat{u})$ are the associated solutions of problem (2) corresponding to the initial data χ_0 and $\hat{\chi}_0$ then, for any $t\in[0,T]$, we have the estimate

$$\int_\Omega h(t,x)\left[\chi(t,x)-\hat{\chi}(t,x)\right]^+ dx\le\int_\Omega h(0,x)\left[\chi_0(x)-\hat{\chi}_0(x)\right]^+ dx \qquad (10)$$

In particular,

$\chi_0\le\hat{\chi}_0$ implies $\chi(t,.)\le\hat{\chi}(t,.)$ for any $t\in[0,T]$.

(d) Some Trotter-Kato (or "pas fractionaire") formula holds by writing (formally) (4) as

$$\frac{dv}{dt} + A^1(t)v + A^2(t)v = 0$$

with

$$A_1(t)v=-\mathrm{div}(h^3\nabla\alpha(\frac{v}{h}))$$

$$A_2(t)v=\mathrm{div}(v\underline{e}).$$

The solution $v(t)$ can be obtained as "product" of solutions of

$$\frac{dv^1}{dt} + A_1(t)v^1=0 \quad , \quad \frac{dv^2}{dt} +A^2(t)v^2=0 \quad .$$

(e) Results on the continuous dependence of v with respect the operator (and so with respect h and $\underline{e}$) can be also obtained.

Remark. We end this section by pointing out that other boundary conditions in (1) can be treated in a similar way assumed that the operator $A_2(t)$ is well-determinated, but that this is not always the case. If for instance we take $\Omega\subset\mathbb{R}^N$ and $u=0$ on $\partial\Omega$ then it is well-know that there is not necessarily uniqueness of the solution of the stationary problem

$$\begin{cases} h\chi-\mathrm{div}(h^3\nabla u-h\chi\underline{e})=g(x) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega . \end{cases}$$

In that case we must take in the definition of $D(A_2(t))$ functions satisfying the additional entropy condition

$$(h^3\nabla u-h\chi\underline{e}).\underline{n}\le 0 \quad \text{on } \partial\Omega$$

where $\underline{n}$ is the outward unit normal to $\partial\Omega$ •

3. ON THE CAVITATION REGION.

We consider in this section the Dirichlet problem

$$(11)\qquad \begin{cases} \frac{\partial}{\partial t}(h\chi)-\mathrm{div}(h^3\nabla u-h\chi\underline{e})=0 & \text{in } Q=(0,T)\mathrm{x}\Omega \\ u=k & \text{on } \Sigma=(0,T)\mathrm{x}\partial\Omega \\ \chi(0,.)=\chi_0(.) & \text{on } \Omega \end{cases}$$

with the usual state condition $\chi\in H(u)$. Following the above remark we must made precise the notion of solution:

Definition. A couple (u,χ) is called entropy solution (supersolution) of (11) if:

$u\in L^2(0,T:H^1(\Omega))$, $\chi\in L^\infty(Q)$,

$u\geq0$, $0\leq\chi\leq1$ and $u(1-\chi)=0$ on Q,

$u=k$ $(u\geq k)$ on Ω,

$\chi(0,.)=\chi_0(\chi(0,.)\geq\chi_0)$ on Ω, and

$$\int_Q [-h\chi\frac{\partial\zeta}{\partial t}+(h^3\nabla u+h\chi\underline{e}).\nabla\zeta]dxdt\leq(\geq)0 ,$$

for every $\zeta\in H^1(Q)$ such that $\zeta(0,.)=\zeta(T,.)=0$ on Ω, $\zeta\geq0$ on Σ and $\zeta=0$ on $\Sigma\cap\{k>0\}$ (resp. $\zeta\geq0$ on Q, $\zeta=0$ on Σ).

Remark. We notice that

$$\frac{\partial}{\partial t}(h\chi)\in L^2(0,T:H^{-1}(\Omega))$$

and so by well known results $h\chi\in C([0,T]:H^{-1}(\Omega))$, which made sense to the initial condition.

In order to study the cavitation region (which we shall take as $\{(t,x)\in Q:\chi(t,x)<1\}$) we shall need to construct suitable supersolutions of (11). As in [5], we start by stating an useful criteria for functions u with a interface. Let $F\in C^1(\overline{Q})$ such that $|F|+|\nabla F|+|\partial_t F|\neq0$ in $\overline{Q}$. Let us denote

$$Q^+=Q\cap\{F>0\},\ Q^-=Q\cap\{F<0\},\ \mathcal{F}=Q\cap\{F=0\}.$$

The following results represents the "Rankine-Hugoniot type conditions" for the front $\mathcal{F}$.

Proposition. *Let* u_+ *and* χ_- *satisfy*

$$u_+\in L^2(Q^+)\cap C^2(\overline{Q}\cap\overline{Q}^+),\ \nabla u_+\in L^2(Q^+)^N,$$

$$\chi_-\in C^0(\overline{Q^-}),\ \frac{\partial}{\partial t}(h\chi)-\mathrm{div}(h\chi\underline{e})\in L^2(Q^-),\ \chi_-\in[0,1).,$$

Define

$$u=\begin{cases} u_+ & \text{in } Q^+ \\ 0 & \text{in } Q^- \end{cases}, \qquad \chi=\begin{cases} 1 & \text{in } Q^+ \\ \chi_- & \text{in } Q^- \end{cases}$$

Then (u,χ) *is an entropy solution (resp. supersolution) assumed*

$$-\mathrm{div}(h^3\nabla u)=0 \ \textit{(resp.}\geq 0\textit{) in } Q_+,$$

$$\frac{\partial}{\partial t}(h\chi_-)-\mathrm{div}(h\chi\underline{e})=0 \ \text{(resp.}\geq 0\text{) in } Q_-,$$

$$h(1-\chi_-)(-\frac{\partial}{\partial t}F-\nabla F.\underline{e})-h^3\nabla u_+.\nabla F=0 \ \text{(resp.}\geq 0\text{) on } \mathcal{F},$$

$$(h^3\nabla u_+ +\underline{e}).\underline{n}\leq 0 \qquad \text{on } \Sigma\cap\partial Q^-$$

$$\chi_-=0 \qquad \text{on } \Omega\cap\partial Q^-\cap\{\underline{n}.\underline{e}>0\}.$$

The proof of this result reduces to apply the divergence theorem. As an application we shall construct a local supersolution defined on $B_R(0)$ the ball of radius R centrated at the origin 0 of $\mathbb{R}^N$. Let M be a given positive number and define the couple $(u,\bar{\chi})$, by

$$\bar{u}(t,x)=Mz(r(t),|x|) \qquad \text{if } r(t)<|x|\leq R$$

$$\bar{u}(t,x)=0 \qquad \text{if } |x|\leq r(t)$$

where r(t) is such that the following conditions holds

$$\mathrm{div}(h^3\nabla\bar{u})=0 \qquad \text{in } B_R(0)-B_{r(t)}(0),$$

$$\bar{u}=M \qquad \text{on } \partial B_R(0)\ .$$

We also define

$$\bar{\chi}_-(t,x)=c+(1-c)\chi_{\{u>0\}}$$

for some constant $c\in[0,1)$. By applying the above Proposition to $F(t,x)=|x|^2-r^2(t)$ it is not difficult to show that $(\bar{u},\bar{\chi})$ is an entropy supersolution and that r(t) must satisfy the (singular) Cauchy Problem

$$\text{(12)} \qquad \begin{cases} \rho'(t)=-1-\dfrac{M}{(1-c)}\,\rho(t)^{1-\gamma}\left(\displaystyle\int_{\rho(t)}\tau^{1-\gamma}d\tau\right)^{-1} \\ \rho(0)=R \end{cases}$$

where

$$\text{(13)} \qquad \gamma=(N-1)-\frac{R}{m}\,\|\nabla h^3\|_{L^\infty((0,T)xB(0))} \qquad \text{(recall that } h\geq m).$$

Here we must assume h such that

$$\text{(14)} \qquad \gamma>0\ .$$

<u>Remark</u>. The existence and uniqueness of r(t) local solution of (12) was obtained in CARRILLO-GILARDI [6] when $\gamma=N$ (the dimension of the space).

A careful revision of their proof shows that it can be adapted to the general case of (14). In any case, it is proved that the existence only takes place in an maximal interval $[0,T^*]$.

Following the ideas of [5] it is possible to give some estimates on the location of the cavitation region.

Theorem. *Assume* (14) *and fix* $\varepsilon\in(0,1)$. *Let* G *be the projection on* R^N *of the support of* $k(t,x)$ *and define*

$$E=\bar{G}\cup\mathrm{supp}(\chi_0-(1-\varepsilon))^+\cup\mathrm{supp}\,\chi_0.$$

Then there exists $t^*\in(0,T)$ *and a positive constant* C *such that* $\chi(t,x_0)<1$ *in* $B_{d(x_0,E)-C\sqrt{t}}(\underline{x}_0)$ assumed $x_0\in\bar{\Omega}$ such that $d(x_0,E)\geq C\sqrt{t}$.

Further results (as for instance for Neumann boundary conditions on a part of Σ) can be also treated by adapting the results of [5].

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