Internationale Schriftenreihe zur Numerischen Mathematik
Série Internationale d'Analyse Numérique
Vol. 106

Edited by
K.-H. Hoffmann, München; H. D. Mittelmann, Tempe;
J. Todd, Pasadena

Birkhäuser Verlag
Basel · Boston · Berlin

Free Boundary Problems in Continuum Mechanics

International Conference on Free Boundary Problems in Continuum Mechanics, Novosibirsk, July 15–19, 1991

Edited by
S. N. Antontsev
K.-H. Hoffmann
A. M. Khludnev

1992

Birkhäuser Verlag
Basel · Boston · Berlin

NEW APPLICATIONS OF ENERGY METHODS TO PARABOLIC AND ELLIPTIC FREE BOUNDARY PROBLEMS

S.N. Antontsev,
Lavrentiev Institute of Hydrodynamics, Novosibirsk, RUSSIA,

J.I. Díaz
Universidad Complutense de Madrid, Madrid, SPAIN,

S.I. Shmarev,
Lavrentiev Institute of Hydrodynamics, Novosibirsk, RUSSIA.

We present some recent results on the application of different energy methods for the study of free boundary problems. Such methods offer an alternative way when the maximum principle fails. So they are of special interest for the study of systems of equations and higher order equations. They are also useful for single equations with complicated structure making difficult the construction of super and subsolutions: this is the case, for instance, when there are unbounded data; or the nonlinearities depend on x and t; when there are first order differential terms in the equation, and so on. A monograph [1] (in preparation) collects many results in this direction. We present here several different applications.

Key words: degenerate elliptic and parabolic equations, local energy method, vanishing properties of solutions.

1.FREE BOUNDARY PROBLEMS IN STATIONARY GAS DYNAMICS.

Let us consider two-dimensional flow of barotropic gas. Let $\vec{v} = (v_1, v_2)$, $\rho = \rho(q)$, $(q^2 = v_1^2 + v_2^2)$, be, correspondingly, the velocity and the density of a gas. Then on the plane of complex potential $\Omega = \left\{(\phi,\psi):\ 0 < \phi < \infty,\ 0 < \psi < 1\right\}$ the function

$$u(\phi,\psi) = \int_q^{q_s} \frac{\rho(\tau)}{\tau}\, d\tau$$

satisfies equation

$$\frac{\partial}{\partial\phi}\left(K(u)\frac{\partial u}{\partial\phi}\right)+\frac{\partial^2 u}{\partial\psi^2}=0$$

where

$$K(u)=\frac{1-M^2}{\rho^2}=\frac{1}{\rho}\frac{d}{dq}(\rho q),$$

$M^2(q)$ is the Mach number, q_s is the sonic speed, $u(q_s)=0$, $M^2(q_s)=1$, $K(0)=0$. For equation (1) we consider the boundary-value problem

$$u(\phi,1)=h(\phi)\geq 0,\ u_\psi(\phi,0)=0,\ 0<\phi<\infty;\ u(0,\psi)=u_0(\psi). \tag{2}$$

Problem (1)-(2) describes the motion of the plane gas jet moving along the given straight boundary (being the image of the line $\psi = 0$). The unknown (free) boundary of the jet, (the image of the line $\psi = 1$), is defined by the prescribed distribution of the speed modulus. It is assumed that the given functions satisfy inequality $0 \leq (h,u_0)$, implying, due to the maximum principle, that $0\leq u(\phi,\psi)$, $q=|\vec{v}|\leq q_s$, $K(u)=\frac{1-M^2}{\rho^2}\geq 0$. We study local properties of weak solutions $u(\phi,\psi)$ of the problem (1)-(2) such that

$$0\leq u(\phi,\psi)\leq M_0,\quad E(\phi)=\int_\phi^\infty\int_0^1(Ku_\phi^2+u_\psi^2)d\phi d\psi\leq M_1,\quad \phi>0. \tag{3}$$

The existence of such solutions is proved in [2].

Let the functions $K(u)$, $h(\phi)$ satisfy the conditions

$$0\leq K(u)\leq K_0u^\alpha,\qquad \forall\ u\geq 0,\ \alpha>0. \tag{4}$$

$$0\leq h(\phi)\leq h_0\equiv\varepsilon\left(1-\frac{\phi}{T}\right)_+^\sigma,\ u_+=max\,(u,0). \tag{5}$$

$$\int_\phi^\infty h_\phi^2\,d\phi\leq\varepsilon^2C\left(1-\frac{\phi}{T}\right)_+^{2\sigma-1},\ \sigma>1/2. \tag{6}$$

Here K_0, α, ε, σ, T, $C(\sigma,T)$ are some positive constants. Let us remark that (6) is valid for h_0 with $C=\dfrac{\sigma^2}{T(2\sigma-1)}$.

THEOREM 1. *(waiting distance).* Let $u(\phi,\psi)$ be a weak solution of the problem (1)-(2) and the conditions (3)-(6) hold with $\sigma>max(1/2,2/\alpha)$. Then

$$u(\phi,\psi)\equiv 0\quad\text{as}\quad\phi>T\ \text{for any}\ 0<\psi<1$$

if ε and M_1 are small enough.

Proof. Any weak solution $u(\phi,\psi)$ of the problem (1)-(2) possesses the following energy equality

$$E(\phi)=\int_\phi^\infty\int_0^1(Ku_\phi^2+u_\psi^2)d\phi d\psi=I_1+I_2 \tag{7}$$

where

$$I_1=-\int_0^1 Ku_\phi(u-h)d\psi,\qquad I_2=\int_\phi^\infty\int_0^1 Ku_\phi h_\phi\,d\psi d\phi.$$

Using (3)-(6) and relations

$$(u-h)^2=\left[\int_1^\psi u_\psi(\phi,\sigma)d\sigma\right]^2\leq -E',\ E'=\frac{dE}{d\phi}\leq 0,\ u(\phi,\psi)\leq 2(h^2-E')$$

one may estimate I_1, I_2 as follows

$$|I_1|\leq\left(\int_0^1 Ku_\phi^2\,d\psi\right)^{1/2}\left(\int_0^1|Ku^{-\alpha}|u^\alpha(u-h)^2d\psi\right)^{1/2}\leq$$

$$\leq 2^{\alpha/4}K_0^{1/2}(-E')^{1/2}(h^2-E')^{\alpha/4}(-E')^{1/2}\leq C(h^2-E')^{\alpha/4}. \tag{8}$$

$$|I_2|\leq 2^{\alpha/4}K_0^{1/2}E^{1/2}(h^2-E')^{\alpha/4}\left(\int_\phi^\infty h_\phi^2d\phi\right)^{1/2}\leq$$

$$\leqslant \delta E^{(4+\alpha)/8}\Big(\int_\phi^\infty h_\phi^2 d\phi\Big)^{(4+\alpha)/8} + C_\delta (h^2-E')^{(\alpha+4)/4}, \ \delta > 0. \quad (9)$$

If $\alpha \geqslant 4$ then, applying (3)-(6), choosing δ so that

$$\delta E^{\frac{4+\alpha}{8}-1}\Big(\int_\phi^\infty h_\phi^2 d\phi\Big)^{(4+\alpha)/8} \leqslant 1/2$$

and using (7)-(9), we get inequality

$$E \leqslant C\,(h^2-E')^{(\alpha+4)/4}.$$

Here and elsewhere later we denote by C different constants depending only on α, T, M_1, K_0. If $\alpha < 4$ then we obtain, applying Young inequality to the first addend in the right-side part of (9):

$$|I_2| \leqslant E/2 + C\Big[(h^2-E')^{(\alpha+4)/4} + \Big(\int_\phi^\infty h_\phi^2 d\phi\Big)^{(4+\alpha)/(4-\alpha)}\Big]$$

and, respectively,

$$E \leqslant C\Big[(h^2-E')^{(\alpha+4)/4} + \Big(\int_\phi^\infty h_\phi^2\, d\phi\Big)^{(4+\alpha)/(4-\alpha)}\Big].$$

Hence, in the both cases the energy function E satisfies ordinary differential inequality

$$E' + CE^{\alpha/(4+\alpha)} \leqslant C\Big[h^2 + \iota(\alpha)\Big(\int h_\phi^2\, d\phi\Big)^{4/(4-\alpha)}\Big] \leqslant H_0,$$

where $H_0 = C\varepsilon^2\Big(1-\frac{\phi}{T}\Big)_+^{4/\alpha}$, $\iota(\alpha) = 0$ if $\alpha \geqslant 4$, $\iota(\alpha) = 1$ if $\alpha < 4$.

By (3) $E(T) = 0$ if only M_1 and ε are small enough. Thus, $u_\psi(\phi,\psi) = 0$, $u(\phi,\psi) = 0$ as $\phi \geqslant T$ and the Theorem 1 is proven.

2. THE FLOW OF IMMISCIBLE FLUIDS THROUGH A POROUS MEDIUM.

Consider the system of equations

$$\begin{cases} \phi(x)\dfrac{\partial s}{\partial t} - div(k_0(x)a(s)\nabla s) = div(k_0(x)b(s)\nabla p) + f(x,t), \\ div(k_0(x)d(s)\nabla p)=0\ , \end{cases} \quad (10)$$

under the structural assumptions: $0 < C_1 \leqslant f(x) \leqslant C_2$,

$$C_3|\xi|^2 \leqslant (k_0(x)\xi,\xi) < C_4|\xi|^2,\ \forall\ \xi \in \mathbb{R}^N-(0),\ 0 < C_5 \leqslant d(s)$$

and $C_6 s^\alpha(1-s)^\beta \leqslant a(s) \leqslant < C_7 s^\alpha(1-s)^\alpha$. This system arises in the study of immiscible fluids flow through a porous medium. References on the physical derivation of the system and on the basic theory of the existence of weak solutions can be found in [3]. We make emphasis in the absence of the maximum principle for the system (10). To illustrate the application of energy methods we concentrate our attention in the degenerate case $\alpha > 0$.

THEOREM 2. *(finite speed of propagation).* Let (s,p) be any local weak solution of (10) such that $p \in L^\infty(0,T;W^{1,q}(B_{r_1}(x_0)))$ for some $q > 2$, $r_1 > 0$, $x_0 \in \mathbb{R}^N$. Assume $\alpha > 0$ and $(b')^2 \leqslant Ms^\alpha$. Let $s(x,0)$ and $f(x,t)$ vanishing on $B_{r_1}(x_0)$ and $B_{r_1}(x_0)\times(0,T)$ respectively. Then there exist $t_0 \in (0,T)$ and $0 < r(t) < r_1$ such that $s(x,t) = 0$ on $B_{r(t)}(x_0)$ for any $t \in [0,t_0]$.

THEOREM 3. *(waiting time).* Assume (for simplicity) $f\equiv 0$ and the assumptions of Theorem 1. If in addition

$$\int_{B_r(x_0)} |s(x,0)|^2 dx \leqslant C(r-r_1)_+^q$$

for any $r \in [0,r_2]$, $r_2 > r_1$ and a suitable q>0 then there exists $t^* > 0$ such that $s(x,t) = 0$ on $B_r(x_0)$ for any $t \in [0,t^*]$

The proofs as well as other qualitative properties for the case $\alpha \in (-1,0)$ can be found in [3].

3. ON THE BOUNDARY LAYER FOR DILATANT FLUIDS.

The study of the boundary layer for a dilatant fluid of viscosity n>0 leads (after the formulation as a Prandtl' system in von Mises new unknowns) to the problem

$$\begin{cases} \frac{\nu}{2^{n-1}} \sqrt{w} \frac{\partial}{\partial \phi}\left(\left|\frac{\partial w}{\partial \psi}\right|^{n-1}\frac{\partial w}{\partial \psi}\right) - \frac{\partial w}{\partial x} - v_0(x)\frac{\partial w}{\partial \psi} + 2UU_x = 0, \quad 0<x<X, \quad 0<\psi<\infty, \\ w(0,\psi) = w_0(\psi) \quad w(x,0)=0 \ , \ w(x,\psi) \to U^2(x) \text{ as } \psi \to \infty, \end{cases}$$

where U, w_0 and v_0 are given functions satisfying $v_0(x) < 0$, $U_x > 0$, $U(x) > 0$, $w_0(0) = 0$, $w_0(\psi) > 0$ if $\psi > 0$. The case $n = 1$ has been studied by O.A. OLEINIK in a series of important works. Here we assume n>1 and use some technical results that allow us to assume that $0 < C_1 < w(x,\psi) < C_2$ for any $x \in (0,X)$ and $\psi \geq \psi_0$, for some $\psi_0 > 0$ (see [4]). The application of energy methods allows one to improve the results of [4]on the localization of the coincidence set where $W(x,\psi) = U^2(x)$:

THEOREM 4. Assume $w_0(\psi) = U^2(0)$ for any $\psi \geq \psi_1$, for some $\psi_1 \geq \psi_0$. Then there exist $C > 0$, and $\alpha > 0$ such that

$$w(x,\psi) = U^2(x) \text{ for any } \psi \geq \psi_1 + Cx^\alpha \text{ and any } x \in [0,X].$$

THEOREM 5 *(waiting distance).* Assume in addition

$$\int_\psi^\infty (U^2(0) - w_0(\sigma))^2 d\sigma \leq C(\psi_0 - \psi)_+^q$$

for some suitable q>0 and any $\psi \in (\psi_3, +\infty)$ for some $\psi_0 \leq \psi_3 \leq \psi_1$. Then there exists $x^* > 0$ such that $w(x,\psi) = U^2(x)$ for any $\psi \in [\psi_1, \infty)$ and any $x \in [0, x^*]$.

4. FORMATION OF "DEAD CORES" IN REACTION-DIFFUSION EQUATIONS UNDER STRONG ABSORPTION.

By introducing of new domains of integration in the definition of the energy functions it is possible to consider not vanishing initial data in the study of the formation of "dead cores" ([5,6]). Consider equation

$$\frac{\partial}{\partial t}\left(|u|^{\alpha-1}u\right) - div\, \vec{A}(x,t,u,\nabla u) + B(x,t,u) = 0 \qquad (11)$$

where $\alpha > 0$, $(\vec{\xi}, \vec{A}(x,t,u,\vec{\xi}) \geq C_0|\vec{\xi}|^p$, $(\vec{\xi}, \vec{A}(x,t,u,\vec{\xi}) \leq C_1|\vec{\xi}|^p$, $p > 1$ and $B(x,t,u)u \geq C_2|u|^\gamma$.

THEOREM 6. Assume $(p-1)/\alpha \geq 1 > \gamma/\alpha$. Let

$$u \in C^0(B_{r_1}(x_0) \times [0,\infty)) \cap L^\infty(B_{r_1}(x_0) \times [0,\infty))$$

be any local weak solution of (11). Then there exist $0 \leq T_0 < \infty$ and $r: [T_0,\infty) \to \mathbb{R}^+$, $r(T_0) = 0$, such that $u(x,t) \equiv 0$ on $B_{r(t)}(x_0)$ $\forall$ $t > T_0$.

Rigorous proofs of this assertion with different function r(t) are given in [6,7].

REFERENCES

[1] S.N. Antontsev, J.I. Diaz: Book in Birkhuser, 1992.

[2] S.N. Antontsev, Dokl.Akad nauk SSSR, 216, 1974, n.3 p.p.473-476. (Translation in Soviet Math. Doklady, 15, 1974 n.3, p.p.803-807).

[3] S.N. Antontsev, J.I. Diaz: Nonlinear Analysis, Theory, Methods Applications, 16, 4, pp.299-313.

[4] V.N. Samokhin, Trudy Seminara imeni I.G. Petrovskogo N^{0}14 pp. 89-108 (1989) (English translation) pp.2358-2373, 1990.

[5] J.I. Diaz, J. Hernandez: In Trends in theory and practice o Nonlinear differential equations, V. Lakshmikantham ed. Marcel Dekker, 1984, pp. 149-156.

[6] S.I. Shmarev, Dinamika Sploshnoy Sredi. Novosibirsk 1990, V 95,97.

[7] S.N.Antontsev, S.I.Shmarev Dokl. Akad. Nauk SSSR, 318, 1991 n.4, p.p.777-781. (Translation in Soviet Math. Doklady, v.43 1991)