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Symmetrization of nonlinear elliptic and parabolic problems and applications: a particular overview

Introduction.

The symmetrization process will be illustrated by the consideration of different classes of nonlinear partial differential equations. In Section 1 we consider the Dirichlet problem associated to the elliptic equation

$$-\mathrm{div}(Q(|\nabla u|)\nabla u) + \beta(u) = f.$$

The parabolic formulation

$$\gamma(u)_t - \mathrm{div}(Q(|\nabla u|)\nabla u) + \beta(u) = f$$

with $Q(r) = r^{p-2}$, will be the object of Section 2. Some variational inequalities will be considered in Section 3. Among them we can mention the obstacle problem, the Stefan problem and a suitable formulation related with the study of Bingham fluids. Finally, Section 4 contains an extension of some of the precedent results to the case of systems of nonlinear equations, as, for instance, the system arising in chemical adsorption.

1. The symmetrization process for nonlinear elliptic equations.

Let Ω be a bounded regular open set of $\mathbf{R}^N$. We consider the Dirichlet (or Plateau) problem

$$-\mathrm{div}(Q(|\nabla u|)\nabla u) + \beta(u) = f \quad \text{in } \Omega \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

We assume $Q \in C^2((0,\infty))$, $Q(r)r \to 0$ if $r \to 0$, $Q(r)r^2$ convex strictly increasing and β continuous non-decreasing such that $\beta(0) = 0$.

To introduce the notion of weak solution we first define the auxiliary function

$$A(r) = \int_0^r Q(s)s\,ds$$

and the Orlicz and Orlicz-Sobolev spaces $L^A(\Omega)$, $W^{1,A}(\Omega)$ and $W_0^{1,A}(\Omega)$ associated to A as usual (see [50]). Notice that if $Q(r) = r^{p-2}$ then $W^{1,A}(\Omega) = W^{1,p}(\Omega)$ (the usual Sobolev space).

By using variational techniques it is possible to show ([43],[23]) the following existence result

Theorem 0.

Let $f \in (W_0^{1,A}(\Omega))'$ and assume

$$\max(|\Omega|^{\frac{1}{N}}, |f|_{L^N(\Omega)}) \leq N(\omega_N)^{\frac{1}{N}} \lim_{r\to\infty} \mathcal{B}(r) \tag{3}$$

where

$$\omega_N \quad \textit{is the measure of the unit ball in } \mathbf{R}^N \tag{4}$$

and

$$\mathcal{B}(r) = Q(r)r \quad \textit{for any } r > 0. \tag{5}$$

Then there exists a unique $u \in W_0^{1,A}(\Omega)$ with $\beta(u) \in L^1(\Omega)$ weak solution of (1),(2) in the sense that

$$\int_\Omega Q(|\nabla u|)\nabla u \cdot \nabla v dx + \int_\Omega \beta(u)v dx = < f, v >$$

for any $v \in W_0^{1,A}(\Omega) \cap L^\infty(\Omega)$.

Remarks.

1. If $Q(r) = r^{p-2}$ then $\lim_{r\to\infty} \mathcal{B}(r) = +\infty$ and assumption (3) is trivially satisfied for any $f \in (W_0^{1,A})'$.
2. If $Q(r) = \frac{1}{\sqrt{1+r^2}}$ assumption (3) is "almost" necessary (see [49],[33] and their references)
3. The "comparison principle" holds in this class of solutions and thus if, for instance, $f \geq 0$ in Ω then $u \geq 0$ *a.e.* in Ω ([26]).

We also consider the "symmetrized problem"

$$-\mathrm{div}(Q(|\nabla U|)\nabla U) + \beta(U) = F \quad \text{in } \Omega^* \tag{6}$$

$$U = 0 \quad \text{on } \partial\Omega^* \tag{7}$$

where Ω^* is the ball $B_R(0)$ centered at the origin and with equal measure than Ω. As we want to find solutions of (6),(7) as simple as possible, we assume that $F : \Omega^* \to \mathbf{R}$ is radially symmetric and decreasing along the radii. In this way the solution U of (6) and (7) have a simple structure: U is a function that is radially symmetric and decreasing along the radii. Our main goal is to collect several results comparing (in some sense) the solutions u and U of (1), (2) and (6), (7) respectively, assuming a suitable relation between the data f and F.

A first choice leading to comparison results is $F = f_*$: the symmetric decreasing rearrangement of f.

Definitions. Let $f : \Omega \to \mathbf{R}$ measurable. We define the functions

$$\mu(t) = |\{x \in \Omega : f(x) > t\}| \quad \text{(the distribution function)},$$

$$\tilde{f} : [0,\infty) \to \mathbf{R},\ \tilde{f}(s) = \inf\{t \geq 0 : \mu(t) \leq s\} \quad \text{(the decreasing rearrangement)},$$

and finally

$$f_* : \Omega^* \to \mathbf{R}, f_*(x) = \tilde{f}(\omega_N |x|^N) \text{ (the symmetric decreasing rearrangement)}.$$

This notion have been extensively treated in the literature in the last years. The reader can find an exhaustive treatment in the books [47], [7], [45] and [41]. Concerning the comparison between u and U, assumed $F = f_*$, the situation is quite different for the cases $\beta \equiv 0$ and $\beta \not\equiv 0$.

Theorem 1 ([50]).

Let $\beta \equiv 0$ and assume $f \in L^1(\Omega) \cap (W_0^{1,A}(\Omega))'$ and $f \geq 0$ on Ω. Then $U_* = U$ and

$$u_*(x) \leq U(x) \quad a.e. \quad x \in \Omega^*. \tag{8}$$

When $\beta \not\equiv 0$ the pointwise comparison (8) fails ([42], [45]) and the comparison between u and U must be established in a more complicated way (see [17], [42], [51], [52],[45], [19]). The following result shows that as a consequence of the "stability" of the symmetrization process:

Theorem 2 ([21]).

Let f and F be nonnegative functions in the spaces $(W_0^{1,A})' \cap L^1$ associated to the domains Ω and Ω^* respectively. We assume F symmetric and decreasing along the radii. Let u and U the solutions of (1),(2) and (6),(7). For $s \in (0, |\Omega|]$ define the auxiliary "mass" functions

$$l(s) = \int_0^s \tilde{f}(\sigma)d\sigma, \qquad L(s) = \int_0^s \tilde{F}(\sigma)d\sigma,$$

$$k(s) = \int_0^s \beta(\tilde{u}(\sigma))d\sigma, \quad K(s) = \int_0^s \beta(\tilde{U}(\sigma))d\sigma.$$

Then the following estimate holds

$$\|(k-K)_+\|_{L^\infty(0,|\Omega|)} \leq \|(l-L)_+\|_{L^\infty(0,|\Omega|)}, \tag{9}$$

where $(.)_+$ denotes the positive part of the corresponding functions. In particular, if we assume

$$\int_{B_r(0)} f_*(x)dx \leq \int_{B_r(0)} F(x)dx \quad \textit{for any } r \in [0,R] \tag{10}$$

then

$$\int_{B_r(0)} \beta(u_*(x))dx \leq \int_{B_r(0)} \beta(U(x))dx \quad \textit{for any } r \in [0,R]. \tag{11}$$

In spite of its "sophisticated" statement, the above result has many relevant applications:

Corollary 1.
Assume f and F as in Theorem 2 and satisfying (10). Then

$$\int_{\Omega} \Phi(\beta(u(x)))dx \leq \int_{\Omega^*} \Phi(\beta(U(x)))dx$$

for any convex nondecreasing real function Φ. In particular

$$\| \beta(u) \|_{L^q(\Omega)} \leq \| \beta(U) \|_{L^q(\Omega^*)} \quad \text{for any } 1 \leq q \leq \infty.$$

Corollary 2 ([21]).
Let $F = f_*$ and assume that

$$dist\ (support\ U,\ \partial\Omega^*) > 0$$

Then

$$| \{x \in \Omega :\ u(x) = 0\} | \geq | \{x \in \Omega^* :\ U(x) = 0\} | .$$

Remarks.

1. The idea of the proof of Theorem 2 is the following: By using the Fleming-Rishel formula and the De Giorgi's isoperimetric theorem in a similar way to [50] we obtain that

$$a(s)\mathcal{B}\left(-a(s)\frac{\mathrm{d}}{\mathrm{d}s}\beta^{-1}\left(\frac{\mathrm{d}k}{\mathrm{d}s}(s)\right)\right) + k(s) \leq l(s) \quad \text{in } (0, | \Omega |)$$

$$a(s)\mathcal{B}\left(-a(s)\frac{\mathrm{d}}{\mathrm{d}s}\beta^{-1}\left(\frac{\mathrm{d}K}{\mathrm{d}s}(s)\right)\right) + K(s) = L(s) \quad \text{in } (0, | \Omega |)$$

$$k(0) = K(0) = 0$$

$$k'(| \Omega |) = K'(| \Omega |) = 0$$

   where $\mathcal{B}$ is given by (5) and $a(s) = N(\omega_N)^{\frac{1}{N}} s^{\frac{N-1}{N}}$. The conclusion comes by L^∞-techniques for fully nonlinear elliptic problems (see details in [21]).
2. The proof of Corollary 1 uses a classical result due to Hardy-Littlewood-Polya [40].
3. Corollary 2 is of special interest for the study of free boundary problems arising if, for instance, we assume $Q(r) = r^{p-2}$, $\beta(r) = r^q$ and $0 < q < p - 1$ (see the monograph [19]).

We end this section by the consideration of the case of nonhomogeneous Dirichlet boundary conditions. The symmetrization process can be successfully applied at least for two special cases of interest in the applications:

(i) the case of u constant on $\partial\Omega$,

(ii) capacity type problems.

The treatment of the first class of those problems can be carried out by a direct approach ([9]) or by a homogenization argument and the application of Theorem 2 ([19]). The following conclusions hold

Corollary 3 ([19]).
Let $h \in \mathbf{R}_+$ and let v, V solutions of the problems

$$-\mathrm{div}(Q(| \nabla v |)\nabla v) + \beta(v) = 0 \quad \text{in } \Omega$$

$$v = h \quad \text{on } \partial\Omega$$

and

$$-\mathrm{div}(Q(| \nabla V |)\nabla V) + \beta(V) = 0 \quad \text{in } \Omega^*$$

$$V = h \quad \text{on } \partial\Omega^*$$

Then

$$\int_{B_r(0)} \beta(v_*(x))dx \geq \int_{B_r(0)} \beta(V(x))dx \quad \text{for any } r \in [0, R].$$

Moreover, $V > 0$ in Ω^* implies that $v > 0$ in Ω and, otherwise, the following estimate holds

$$| \{x \in \Omega : v(x) = 0\} | \leq | \{x \in \Omega^* :\ V(x) = 0\} | . \tag{16}$$

The case of "capacity" type problems leads to another comparison results. Let ω be an open bounded regular set of $\mathbf{R}^N$ such that $\overline{\omega} \subset \Omega$. We consider the problem

$$-\mathrm{div}(Q(| \nabla u |)\nabla u) + \beta(u) = f \quad \text{in } \Omega - \overline{\omega} \tag{17}$$

$$u = 1 \quad \text{on } \partial\omega \tag{18}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{19}$$

and its symmetrized version

$$-\mathrm{div}(Q(| \nabla U |)\nabla U)) + \beta(U) = F \quad \text{in } \Omega^* - \overline{\omega}^* \tag{17}$$

$$U = 1 \quad \text{on } \partial\omega^* \tag{18}$$

$$U = 0 \quad \text{on } \partial\Omega^* \tag{19}$$

where ω^* is the ball centered at the origin and with equal measure than ω. We have

Theorem 3 ([20]).

Assume $f \in L^\infty(\Omega - \overline{\omega})$ and $F \in L^\infty(\Omega^* - \overline{\omega}^*)$ be nonnegative functions. Let $\underline{f}$ and $\underline{F}$ the extension of these functions to Ω and Ω^* by means of $\sup f$ and $\sup F$ on ω and ω^* respectively. Assume also that F is symmetric and decreasing along the radii and

$$\int_{B_r(0)} \underline{f}_*(x)dx \le \int_{B_r(0)} \underline{F}(x)dx \quad \text{for any } r \in [0,R]. \tag{23}$$

Then

$$\int_{B_r(0)} \beta\big(\underline{u}_*(x)\big)dx - C_u \le \int_{B_r(0)} \beta\big(\underline{U}(x)\big)dx - C_U, \tag{24}$$

where

$$C_u = \int_{\partial\omega} Q(|\nabla u|)\nabla u \cdot n d\mathcal{H}^{N-1}, \quad C_U = \int_{\partial\omega^*} Q(|\nabla U|)\nabla U \cdot n d\mathcal{H}^{N-1}. \tag{25}$$

Remark.

If $\beta \equiv 0$ the conclusion (24) is replaced by the pointwise comparison

$$\frac{\underline{u}_*(x)}{C_u} \le \frac{\underline{U}(x)}{C_U} \quad a.e. \quad x \in \Omega^*. \tag{26}$$

([47],[45]). In the linear case, $Q(r) = 1$, the solution u represents the capacity potential in the condenser $\overline{\Omega - \omega}$, and C_u is the electrostatic capacity of ω relative to Ω (see [47],[48],[30] and [34]).

2. The symmetrization process for nonlinear parabolic equations.

The formulation of the class of parabolic problems we shall consider is the following

$$\gamma(u)_t - \Delta_p u + \beta(u) = f(t,x) \quad \text{in } (0,T)\times\Omega \tag{27}$$

$$u = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{28}$$

$$\gamma\big(u(0,x)\big) = \gamma\big(u_0(x)\big) \quad \text{on } \Omega, \tag{29}$$

where Ω is an open bounded regular set of $\mathbf{R}^N$, the diffusion operator is

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$$

and β and γ are continuous functions with γ strictly increasing. The existence and uniqueness of weak solutions have been largely studied by different authors and methods. Many references can be found in the articles [5], [12], [15], [27].

The symmetrized problem is formulated as

$$\gamma(U)_t - \Delta_p U + \beta(U) = F(t,x) \quad \text{in } (0,T)\times\Omega^*, \tag{30}$$

$$U = 0 \quad \text{on } (0,T)\times\partial\Omega^*, \tag{31}$$

$$\gamma\big(U(0,x)\big) = \gamma\big(U_0(x)\big) \quad \text{on } \Omega^*, \tag{32}$$

where, again, Ω^* is the ball $B_R(0)$ centered at the origin and with equal measure than Ω and $F(t,.)$ (for t fixed) and U_0 are real symmetric functions defined on Ω^* and decreasing along the radii.

The following result contain a comparison result as a consequence of a "stability" (or continuous dependence) estimate.

Theorem 4 ([22]).

Let f, u_0 and U_0 integrable nonnegative and bounded functions. Assume

$$\beta \circ \gamma^{-1} = \phi_1 + \phi_2 \tag{33}$$

with ϕ_1 convex and ϕ_2 concave

Let u and U the solutions of (27) (28) (29) *and* (30) (31) (32) *respectively and assume*

$$u \text{ and } U \text{ are bounded functions} \tag{34}$$

For $t \in [0,T]$ and $s \in \big[0, |\Omega|\big]$ we define the auxiliary functions

$$k(t,s) = \int_0^s \gamma(\tilde{u}(t,\sigma))d\sigma, \quad K(t,s) = \int_0^s \gamma(\tilde{U}(t,\sigma))d\sigma$$

$$l(t,s) = \int_0^s \tilde{f}(t,\sigma)d\sigma, \quad L(t,s) = \int_0^s \tilde{F}(t,\sigma)d\sigma.$$

Then, there exists a constant C such that,

$$\| [k(t,.) - K(t,.)]_+ \|_{L^\infty(0,|\Omega|)} \le e^{Ct} \| [k(0,.) - K(0,.)]_+ \|_{L^\infty(0,|\Omega|)}$$

$$+ \int_0^t e^{C(t-\tau)} \| [l(\tau,.) - L(\tau,.)]_+ \|_{L^\infty(0,|\Omega|)} d\tau$$

for any $t \in [0,T]$. In particular, if we assume

$$\int_{B_r(0)} \gamma\big(u_{0*}(x)\big)dx \le \int_{B_r(0)} \gamma\big(U_0(x)\big)dx \tag{35}$$

and (for a.e. $t \in (0,T)$)

$$\int_{B_r(0)} f_*(t,x)dx \le \int_{B_r(0)} F(t,x)dx \tag{36}$$

for any $r \in [0,R]$, *then*

$$\int_{B_r(0)} \gamma(u_*(t,x))dx \leq \int_{B_r(0)} \gamma(U(t,x))dx \qquad (37)$$

for any $t \in [0,T]$ *and any* $r \in [0,R]$.

Remark.

It is possible to show that the pointwise comparison $u_* \leq U$ is not true for parabolic equations (even for the linear heat equation). The comparison given in (37) have been established by many different authors by using different methods. See [5],[6],[51],[1],[10],[11],[46] ... It seems that the stability estimate was first obtained in [22]. Such kind of estimate appears in the treatment of fully nonlinear parabolic equations.

Theorem 4 has many applications, some of them of special interest when solutions exhibit some peculiar behaviors.

Corollary 4.

Assume u_0, f, U_0 *and* F *as in Theorem 4 and satisfying* (35) *and* (36). *Then*

$$\| \gamma(u(t,.)) \|_{L^q(\Omega)} \leq \| \gamma(U(t,.)) \|_{L^q(\Omega^*)}$$

for any $t \in [0,T]$ *and any* $q \in [1,+\infty]$.

Corollary 5 ([6],[22]. Blow-up problems).

Let u_0, f, U, F *as in Corollary 4. Define the blow-up time* T_Ω (*resp.* T_{Ω^*}) *in* $L^r(\Omega)$ *by means of*

$$\lim_{t \nearrow T_\Omega} \| u(t,.) \|_{L^r(\Omega)} = +\infty$$

and

$$\| u(t,.) \|_{L^\infty(\Omega)} < +\infty \quad \text{if} \quad t \in [0,T_\Omega),$$

(analogously for T_{Ω^*} *by replacing* Ω *and* u *by* Ω^* *and* U*). Then* $T_\Omega \geq T_{\Omega^*}$.

Corollary 6 ([22]. Finite time extinction problems).

Let u_0, f, U, F *as in Corollary 4. Define the extinction time* $T_\Omega^\#$ (*resp.* $T_{\Omega^*}^\#$) *by means of*

$$\| u(t,.) \|_{L^1(\Omega)} = 0 \quad \text{for any} \quad t \geq T_\Omega^\#$$

and

$$\| u(t,.) \|_{L^1(\Omega)} > 0 \quad \text{if} \quad t \in [0,T_\Omega^\#),$$

(analogously for $T_\Omega^\#$ *by replacing* Ω *and* u *by* Ω^* *and* U *). Then* $T_\Omega^\# \leq T_{\Omega^*}^\#$.

Corollary 7 ([10],[22]. Free boundary problems).

Assume u_0, f, U *and* F *as in Corollary 4. Then*

$$| \{x \in \Omega : u(t,x) = 0\} | \leq | \{x \in \Omega^* : U(t,x) = 0\} |$$

for any $t \in [0,T]$.

Remark.

The phenomena mentioned in Corollaries 5-7 arisés under suitable conditions on the nonlinear terms of the equation (27). So, for instance, the blow-up property holds if, for instance, $F = f = 0$, $p = 2$, $\gamma(u) = u$ and $\beta(u) = -u^q$ with $q > 1$. The existence of a finite extinction time can be proved, for instance, if $p = 2$, $\gamma(u) = u^{\frac{1}{m}}$ with $0 < q < 1$ and $m > 0$. The existence of a free boundary is typical of slow diffusion problems ($\gamma(u) = u^{\frac{1}{m}}$, $\beta \equiv 0$ and $(p-1)m > 1$) and also when the absorption is stroug enough (e.g. $p = 2$, $\gamma(u) = u^{\frac{1}{m}}$, $\beta(u) = u^q$ and $0 < q < m$).

The case of nonhomogeneous boundary conditions has been also considered in the literature. So, if the solution is constant on the boundary $(0,T) \times \partial\Omega$ conclusions similar to Corollary 3 have been obtained in [10] and [11]. The parabolic capacity problems was considered in [39] (for $p = 2$ and $\beta \equiv 0$) where it was proved that the comparison (24) must be now stated in terms of

$$\int_{B_r(0)} \gamma(u^*(t,x))dx - \int_0^t C_u(\tau)d\tau \leq \int_{B_r(0)} \gamma(U(t,x))dx - \int_0^t C_U(\tau)d\tau \qquad (38)$$

where $t \in [0,T]$, $r \in [0,R]$ and

$$C_u(t) = \int_{\partial\omega} \nabla u(t,x) \cdot n dx \quad \text{and} \quad C_U(t) = \int_{\partial\omega^*} \nabla U(t,x) \cdot n dx$$

3. The symmetrization process for variational inequalities.

Many different problems of natural sciences are formulated in terms of variational inequalities instead of equations ([28]). Here we shall comment some result illustrating the symmetrization process in that context.

We start by the consideration of the Obstacle Problem. As we shall see the treatment is different according inferior or superior obstacles, $u \geq \psi$ or $u \leq \psi$, respectively. To simplify the exposition le us consider the obstacle problem associated to $u \geq 0$ on Ω. Let $f \in L^{p'}(\Omega)$ and $u \in K(\Omega)$ be the solution of

$$\inf_{w \in K(\Omega)} \left\{ \frac{1}{p} \int_\Omega |\nabla w|^p \, dx - \int_\Omega f w dx \right\} \qquad (39)$$

where

$$K(\Omega) = \{w \in W_0^{1,p}(\Omega) : w \geq 0 \quad \text{a.e. on } \Omega\}. \qquad (40)$$

The symmetrized problem is

$$\inf_{w\in K(\Omega^*)} \left\{ \frac{1}{p}\int_\Omega |\nabla w|^p\, dx - \int_\Omega Fw dx \right\} \tag{41}$$

where $K(\Omega^*)$ is given by (40) replacing Ω by Ω^* and F is a radially symmetric and decreasing along the radii function. We have

Theorem 5 ([8]).
Assume $f \in L^{p'}(\Omega)$ and $F = f_*$. Then

$$u*(x) \le U(x) \quad \text{a.e. } x \in \Omega^* \tag{42}$$

Moreover

$$|\{x\in\Omega:\ u(x)=0\}| \ge |\{x\in\Omega^*:\ U(x)=0\}|, \tag{43}$$

and

$$|\{x\in\Omega^*:\ U(x)=0\}| > 0$$

if and only if there exists $s_0 \in (0, |\Omega|]$ such that

$$\int_0^{s_0} \tilde{f}(\sigma)d\sigma = 0.$$

An "equivalent" formulation of problem (39)-(40) is the given by

$$-\Delta_p u + \beta(u) \ni f \quad \text{in } \Omega \tag{44}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{45}$$

which has a great similarity with the formulation (1)-(2) but where now β is the maximal monotone graph of $\mathbf{R}^2$ given by

$$\beta(r) = \begin{cases} 0, & \text{if } r > 0, \\ (-\infty, 0], & \text{if } r = 0, \\ \emptyset, & \text{if } r < 0. \end{cases}$$

(see [16],[19]).

As an example of the superior obstacle problem we can take the associated to the condition $u \le 1$. We introduce the closed and convex sets

$$K(\Omega) = \{w \in W_0^{1,p}(\Omega):\ w \le 1 \quad a.e. \text{ on } \Omega\}. \tag{46}$$

The variational formulation is given by (39), (46) and the symmetrized version by (41), (46). When, $f \in L^{p'}(\Omega)$ and $F \in L^{p'}(\Omega^*)$ it is possible to show ([19]) that the

corresponding solutions $u \in W_0^{1,p}(\Omega)$ and $U \in W_0^{1,p}(\Omega^*)$ are "characterized" by the existence of two functions $b \in L^1(\Omega)$ and $B \in L^1(\Omega^*)$such that

$$-\Delta_p u + b = f \quad \text{in } \Omega \qquad \text{and} \qquad b(x) \in \beta(u(x)) \quad a.e. \quad x \in \Omega \tag{47}$$

$$-\Delta_p U + B = F \quad \text{in } \Omega^* \qquad \text{and} \qquad B(x) \in \beta(U(x)) \quad a.e. \quad x \in \Omega^* \tag{48}$$

where β is now the maximal monotone graph of $\mathbf{R}^2$ given by

$$\beta(r) = \begin{cases} 0, & \text{if } r < 1, \\ [1, +\infty), & \text{if } r = 1, \\ \emptyset, & \text{if } r > 1. \end{cases} \tag{49}$$

The comparison result is now stated in terms similar to Theorem 2:

Theorem 6 ([21]).
Assume f and F nonnegative functions in $L^{p'}(\Omega)$ and $L^{p'}(\Omega^*)$ satisfying (10) *and F symmetric and decreasing along the radii. Then*

$$\int_{B_r(0)} b_*(x)dx \le \int_{B_r(0)} B(x)dx \tag{50}$$

Moreover

$$|\{x\in\Omega:\ u(x)=1\}| \le |\{x\in\Omega^*:\ U(x)=1\}|. \tag{51}$$

Remarks.

1. The study of the measure of the coincidence set $\{x \in \Omega^* : U(x) = 1\}$ (for $p = 2$) was carried out in [44] where the superior obstacle problem was connected with the inferior obstacle problem $u \ge 0$ but under the nonhomogeneous boundary condition $u = 1$ on $\partial\Omega$. An extension of Theorem 5 to more general maximal monotone graphs was given in [19] (see Theorem 2.22).
2. The parabolic (inferior) obstacle problem was considered in [25]. By discretization in the time variable similar conclusion to the stability inequality of Theorem 4 was proved.

Another example of variational inequality for which the symmetrization process leads to interesting conclusions is the one-phase Stefan problem

$$\gamma(u)_t - \Delta u \ni 0 \quad \text{in } (0,T)\times\Omega$$

$$u = h(t) \quad \text{on } (0,T)\times\partial\Omega$$

$$\gamma(u(0,.)) = \gamma(u_0(.)) \quad \text{on } \Omega$$

where γ is any maximal monotone graph such that $\gamma(0) = [-L, 0]$ (see [39] and [24]). A related formulation corresponds to the Hele-Shaw flows ([38]).

Our last example of variational inequality arises in the study of special formulations of Bingham fluids. We define $V = H_0^1(\Omega)$,

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad j(v) = \int_\Omega \mid \nabla v \mid dx. \tag{52}$$

Given two real positive constants μ and g we consider the problem of finding $u \in V$ such that

$$\mu a(u, v-u) + g(j(v) - j(u)) \geq < f, v-u > \quad \forall v \in V, \tag{53}$$

where $f \in V'$ is given. The existence and uniqueness of a solution u of (53) was shown in [28]. Let U be the solution of the symmetrized problem, *i.e.* (53) but replacing Ω and f as in above problems. We have

Theorem 7 ([18]).

Assume $f \in L^2(\Omega)$, $f \geq 0$ and $F = f_*$. Then

$$u_*(x) \leq U(x) \quad a.e. \quad x \in \Omega^*. \tag{54}$$

Remark.

The idea of the proof is to start by showing

$$u = \lim_{p \to 1} u_p,$$

where $u_p \in H_0^1(\Omega)$ satisfies

$$-\mu \Delta u_p - g \Delta_p u_p = f \quad \text{in } \Omega.$$

Then, the conclusion holds by applying Theorem 1 to u_p. We point out that (54) is one of the main ingredients in order to get estimates on the location and measure of the "rigid region" $\{x \in \Omega : \nabla u(x) = 0\}$ (see [28] and [37]).

4. On the application of the symmetrization process to systems of equations.

A very simple nonlinear system to which the symmetrization process can be applied is the following

$$u_t - \Delta u + f_1(u) + g_1(v) = 0 \quad \text{in } (0,T) \times \Omega$$

$$v_t - d\Delta v + f_2(u) + g_2(v) = 0 \quad \text{in } (0,T) \times \Omega,$$

$$u = v = 0 \quad \text{on } (0,T) \times \partial\Omega$$

$$u(0,x) = u_0(x) \quad \text{on } \Omega$$

$$v(0,x) = v_0(x) \quad \text{on } \Omega.$$

Theorem 8 ([22]).

Assume f_1 and g_2 Lipschitz (or nondecreasing) functions satisfying the property (33), f_2 and g_1 nonincreasing and concave functions.
Let $u_0, v_0 \in L^\infty(\Omega)$ be nonnegative functions and let (U,V) be the solution of the symmetrized system replacing Ω by Ω^* and u_0, v_0 by u_{0*} and v_{0*}. Then

$$\int_{B_r(0)} u_*(t,x)dx \leq \int_{B_r(0)} U(t,x)dx$$

and

$$\int_{B_r(0)} v_*(t,x)dx \leq \int_{B_r(0)} V(t,x)dx$$

for any $t \in [0,T]$ and any $r \in [0,R]$.

Remarks.

1. The idea of the proof is to show that $u = \lim u_n$, $v = \lim v_n$ with (u_n, v_n) given by the iterative algorithm

$$u_t^n - \Delta u^n + f_1(u^n) = -g_1(v^{n-1})$$

$$v_t^n - d\Delta v^n + g_2(v^n) = -f_2(u^{n-1}).$$

   After that the conclusion comes from the application of Theorem 4 to u_n and v_n.
2. By making $d \to 0$ and showing that $u_1 \to u$ as $d \to 0$ the conclusion of Theorem 8 remains true for system of PDE-ODE equations. Such is the case of the system arising in chemical adsorption for which $d = 0$, $f_1(u) = -f_2(u)$ and $g_1(v) = -g_2(v) = -v$ (see [29]).

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