THE ONE DIMENSIONAL POROUS MEDIUM EQUATION WITH A STRONG CONVENCTION: STUDY VIA LAGRANGIAN COORDINATES

J.I. DIAZ
Dpto. de Matemática Aplicada
Universidad Complutense de Madrid. 28040 Madrid, SPAIN
and
S.I. SHMAREV
Lavrentiev Ins. of Hydrodynamics
URSS Academy of Sciences. 630090 Novosibirsk, USSR

1. Introduction.

In this communication we present some of the results of the article Díaz-Shmarev (1992) concerning the Cauchy Problem

$$(P)\begin{cases} u_t = (u^m)_{xx} + (u^\lambda)_x & \text{in } Q = (-\infty, +\infty) \times (0,T) \\ u(x,0) = u_0(x) & \text{for } x \in (-\infty, +\infty), \end{cases}$$

assuming

$$m \geq 1 \quad \text{and } \lambda > 0, \tag{1}$$

$$u_0 \in C^0(-\infty, \infty), u_0 \geq 0, \text{ support of } u_0 = [0,a]. \tag{2}$$

Problems of this kind appear in many different contexts: filtration through porous media, statistics mechanics (the nonlinear Fokker-Plank equation), nonlinear heat equation, plasma physics, etc. (see references in Díaz-Kersner (1987)).

The equation of (1) is a nonlinear parabolic equation which becomes degenerate when $m > 1$, and singular when $\lambda < 1$, on the set of points where $u = 0$. Due to that, classical solutions do not exists, in general. Nevertheless, there is already a satisfactory theory on the existence and uniqueness of generalized solutions (see, *e.g.* Díaz-Kersner (1987), Gilding (1990) and their references).

A peculiar property of the degenerate equations ($m > 1$ and $\lambda \geq 1$) is the finite speed of propagation. More exactly, for any $t \in [0,T]$ there exists two free boundaries on Q defined by

$$\xi(t) = \inf\{x \in \mathbf{R} : u(x,t) > 0\}$$

and

$$\eta(t) = \sup\{x \in \mathbf{R} : u(x,t) > 0\}.$$

In that case, ξ and η are Lipschitz continuous functions on $[0,T]$ and if $T = +\infty$ they satisfy

$$\lim_{t\to\infty} \xi(t) = -\infty \quad \text{and} \quad \lim_{t\to\infty} \eta(t) = +\infty$$

(see Kalashnikov (1987) and Gilding (1988)).

It tourns out that the behaviour of the solution of (P) is rather different in the case of a **strong convection**

$$0 < \lambda < 1 \quad \text{and} \quad m > \lambda. \tag{3}$$

Indeed; it was firstly proved in Díaz-Kersner (1983) (if besides $m \geq 1$) and generalized after in Gilding (1988) that under assumption (3)

$$\xi(t) = -\infty \quad \text{for any } t \in (0, T]$$

but

$$-\infty < \eta(t) < \infty \quad \text{for any } t \in (0, T],$$

and that, if fact,

$$\lim_{t \to \infty} \eta(t) = -\infty.$$

The initial growth of the interfaces $\xi(t)$ and $\eta(t)$, when existing, have been studied in the series of works Alvárez-Díaz-Kersner (1987), Alvárez-Díaz (1987) and (1990). In particular if we assume (3) then the interface $\eta(t)$ is a wetting front ($\eta(t) \leq a$ for any t near 0) "if and only ifoo

$$u_0(x) \leq C \mid x - a \mid^{\frac{1}{(m-\lambda)}} \quad \text{for any } x \text{ near } a \text{ and some } C < \left(\frac{m-\lambda}{m}\right)^{\frac{1}{(m-\lambda)}}. \tag{4}$$

It is easy to see, from the definition of $\eta(t)$, that η is, at least, a lower semi-continuous function. The question of the continuity of η have been an open problem during some years. A first result in this direction was given by Shamarev (1990) under suitable conditions on the initial datum u_0. This result was improved in Díaz-Shmarev (1992) where other qualitative properties on $\eta(t)$ was also given.

2. Formulation in Lagrangian coordinates

One of the main difficulties in the study of $\eta(t)$ is that its location is not known. To advoid this difficulty the main idea of the so called "tracking fronts methods"is to introduce a change of variable and unknow in such a way that the free boundary becomes a fixed and know curve.

It is a curious fact that in our case the "good" new formulation comes from the interpretation of the equation of (1) as the mass conservation law of a "ficticious" fluid with density $u(x, t)$ in the Eulerian description $u(x, t)$. If $V(x, t)$ denotes the Eulerian description of the velocity of the fluids then

$$u_t + (uV)_x = 0. \tag{5}$$

By identifying (5) and the equation of (P) we get

$$V = -\frac{1}{u}\left[(u^m)_x + u^\lambda\right]. \tag{6}$$

Notice that the equality in (6) is only formal because $\lambda \in (0, 1)$ and $u = 0$ over a positive measured set.

As Lagrangian coordinates we use the parameterization

$$x = X(p, t),$$

where p represents a general particle of the fluid. In our case it is useful to take as system of coordinates as the one introduced for the study of compressible fluids (see, *e.g.* Courant-Friedrich (1948))

$$p = \int_{X(p_0,t)}^{X(p,t)} u(x,t)dt \tag{7}$$

of inverse transformation

$$p(x,t) = \int_{-\infty}^{x} u(y,t)dy. \tag{8}$$

Thus, $p(x,t)$ represents the total mass of fluid in the interval $(-\infty, x)$. If we define

$$l = \int_{-\infty}^{\infty} u_0(x)dx$$

then $p \in [0,l]$ as $x \in (-\infty, \infty)$ because any solution u of (P) satisfies

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u_0(x)dx.$$

The trayectory of any particle p is $X(p,t)$ and then the function

$$U(p,t) := u(X(p,t),t)$$

represents the material description of the density and

$$V(p,t) := V(X(p,t),t)$$

the material description of the velocity. They satisfy the following equations

$$U(p,t)X_p(p,t) = 1 \tag{9}$$

(the mass conservation: it comes by differenciating (7) with respect to p) and

$$X_t(p,t) = V(X(p,t),t)$$

(the definition of the velocity). After some manipulations we see that $U(p,t)$ is characterized as solution of the boundary value problem

$$(PL) \begin{cases} U_t - U^2[(U^m)_p + U^{\lambda-1}]_p = 0 & \text{in } (0,1)\times(0,T), \\ U(0,t) = 0 = U(1,t) & \text{for } t \in (0,T), \\ U(p,0) = u_0(X(p,0)) := U_0(p) & \text{for } p \in (0,1). \end{cases}$$

Problem (PL) is a quasilinear parabolic problem and now there is not any free boundary at $t = 0$ because $U_0(p) > 0$ for $p \in (0,l)$. Nevertheless, *the principle of conservation of the difficulties* applies: the equation degenerate where $U = 0$ (and so, in particular, on the lateral boundaries $(\{0\} \cup \{l\})x \times (0, T)$). The new argument to advoid this misfortune is to approach U by U_n satisfying, essentially, (PL) but replacing the boundary conditions by

$$U_n(0,t) = U_n(l,t) = \frac{1}{n}, \quad n \in \mathbf{N}.$$

Using this general idea and sharp "a priori"estimate we obtain the results that are stated in the following section.

3. The main results.

We assume the conditions (2), (3), (4) and

$$\left(u_0^{m-\lambda}\right)' \in C([0,a]) \tag{10}$$

$$u_0 \quad \text{is strictly decreasing on } (a-\epsilon, a), \text{ for some } \epsilon > 0. \tag{11}$$

The following list of conclusions are proved in Diaz-Shmarev (1992) by using the Lagrangian formulation described in the above section:

I. There exists a unique classical solution U of (PL) such that $U(p,t) > 0$ for any $p \in (0,l)$ and any $t \in [0,T]$.

II. The function

$$u_*(x,t) = \begin{cases} U(p,t)_0 & \text{if } x \in \Omega(t), t \in [0,T], \\ 0 & \text{if } x \notin \Omega(t), t \in [0,T], \end{cases}$$

is the unique generalized solution of (P). Here $\Omega(t)$ is defined by

$$\Omega(t) = \{x \in \mathbf{R} : x = X(p,t), p \in (0,l)\}$$

and

$$X(p,t) = X(p,0) - \int_0^t U^{\lambda-1}(p,\tau)\left[1 + \frac{1}{m-\lambda+1}\left(U^{m-\lambda+1}\right)_p(p,\tau)\right]d\tau$$

$$X(0,0) = 0 \quad \text{and} \quad p = \int_0^{X(p,0)} u_0(y)dy.$$

III. The free boundary $\eta(t)$ satisfies $\eta(t) = X(l,t)$ and $\eta \in C[0,T] \cap \mathrm{Lip}(0,T]$.

IV. If condition (11) holds on $(0,l)$ [instead on $(a-\epsilon, a)$] then

$$\eta'(t) \leq 0 \quad \text{a.e. } t \in (0,T)$$

V. We have

$$\lim_{x \to \eta(t)-0} (u^{m-\lambda})_x(x,t) = \frac{\lambda - m}{m}$$

[Notice that as conclusion $(u^{m-\lambda})_x$ is a discontinuous function near $x = \eta(t)$ even for initial data $u_0 \in C^\infty(\mathbf{R})$. Property V was previously obtained "formally" in Kamin-Rosenau (1984)].

VI. The free boundary $\eta(t)$ satisfies

$$\eta'(t) = -\lim_{x \to \eta(t)-0} \left\{ \frac{m}{m-1} \left(u^{m-1}\right)_x + u^{\lambda-1} \right\}.$$

Further results have been obtained recently in Shmarev (1992) by assuming some additional conditions on u_0.

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