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On the disappearance of the mushy region in multidimensional Stefan problems

1. Introduction. We present in this communication some of the results of the work by the authors [2],[3] concerning the disappearance in a finite time of the mushy region for the Stefan problem. Previous results in this direction was obtained in [8] for one-dimensional problems (see also [1]). For the treatment of the multidimensional problems, different cases must be introduced according the "nature" of the spatial domain Ω and the type of boundary conditions. Our results deals with this complex situation but rest on the same general program: we first consider the problem under symmetry conditions on the domain and the data and then we reduce the treatment of general formulations to the symmetry case by means of rearrangements techniques.

To fix ideas and to simplify the exposition we shall restrict ourselves to the consideration of the following one-phase problem (we send the reader to [2], [3] for the treatment of two-phases problem and other boundary conditions):

$$\begin{aligned} (1) \quad & \frac{\partial u}{\partial t} - \Delta \theta = 0 && \text{in } Q_\infty = \Omega \times (0, \infty) \\ (2) \quad & \theta(x, t) = \theta^0(t) && \text{on } \Sigma_\infty = \partial\Omega \times (0, \infty) \\ (3) \quad & u(x, 0) = u_0(x) && \text{on } \Omega, \end{aligned}$$

where Ω is a regular bounded open set of \mathbb{R}^N . As usual, u and θ represents the rescaled enthalpy and temperature, respectively, and they are related by

$$(4) \quad u = \phi(\theta)$$

where

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$$(5) \quad \begin{cases} \phi \text{ is a strictly increasing continuous function on } (0, \infty) \text{ extended} \\ \text{to } [0, \infty) \text{ by } \phi(0) = [-L, 0]. \end{cases}$$

Here $L > 0$ represents the latent heat. We assume on the data the following conditions:

$$(6) \quad u_0 \in L^\infty(\Omega), \quad u_0(x) \geq -L \quad \text{a.e. } x \in \Omega$$

$$(7) \quad \theta^0 \in W^{1,\infty}(0, \infty), \quad \theta^0(t) \geq 0 \quad \text{for any } t \geq 0.$$

Our results will be concerned with weak solutions i.e. pairs of functions $(u, \theta) \in [L^2_{loc}(0, \infty; L^2(\Omega)) \cap C([0, \infty); L^1(\Omega))] \times L^2_{loc}(0, \infty; H^1(\Omega))$ satisfying (2), (4) and such that

$$(8) \quad \int_{\Omega} u(x, t) \eta(x, t) dx - \int_0^t \int_{\Omega} u(x, \tau) \eta_t(x, \tau) dx d\tau + \int_0^t \int_{\Omega} \nabla \theta \cdot \nabla \eta dx d\tau = \int_{\Omega} u_0(x) \eta(x, 0) dx$$

for any $t > 0$ and any $\eta \in H^1(\Omega \times (0, t))$, $\eta(\cdot, s) \in H^1_0(\Omega)$ if $0 \leq s \leq t$. Existence uniqueness and many other results on weak solutions are today well-known in the literature (see e.g. [5], [7], [10]). In particular we know that $u, \theta \in L^\infty(Q_\infty)$ and that $u(x, t) \geq -L$, $\theta(x, t) \geq 0$ a.e. (x, t) in Q_∞ . So, for any $t \geq 0$, we have the domain decomposition $\Omega = l(t) \cup m(t)$ where $l(t)$ and $m(t)$ are the liquid phase and the mushy region defined by

$$l(t) = \{x \in \Omega: \theta(x, t) > 0\} \quad \text{and} \quad m(t) = \{x \in \Omega: \theta(x, t) = 0\}.$$

Our main goal is to obtain conditions on $\theta^0(t)$ and $u_0(x)$ in order to assure the disappearance (in measure) of region $m(t)$ after a finite time t^* .

2. The radially symmetric case. The treatment of the problem becomes easier under symmetry assumptions. So let $\Omega = B(0, r_0) = \{x \in \mathbb{R}^N: |x| < r_0\}$ and assume u_0 as before and such that

$$(9) \quad u_0(x) = u_0(|x|) \text{ satisfying } \begin{cases} u_0(r) = -L & \text{if } 0 \leq r \leq R_0 \\ u_0(r) > -L & \text{if } R_0 \leq r \leq r_0 \end{cases}$$

for some $R_0 \in (0, r_0)$. In order to state our result we need to introduce some auxiliary notation. Given $N > 1$ we define

$$(10) \quad \varphi_N(r) = \begin{cases} r^{N-1} (N-2)^{-1} (r^{2-N} - r_0^{2-N}) & \text{if } N > 2 \\ r \log(r/r_0) & \text{if } N = 2. \end{cases}$$

Finally, given θ^0 and u_0 we define the quantities

$$(11) \quad D(t; u_0, \theta^0) = \int_0^{\Gamma_0} u_0(r) \varphi_N(r) dr + \int_0^t \theta^0(\tau) d\tau$$

and

$$(12) \quad D_\infty \equiv D(\infty; u_0, \theta^0) = \lim_{t \rightarrow \infty} D(t; u_0, \theta^0).$$

We have

Theorem.1. Let u_0 satisfying (9). Then

(i) If $D_\infty \leq 0$ the mushy region exists ($|m(t)| > 0$) for any time $t \geq 0$.

(ii) If $D_\infty > 0$ there exists a finite time $t^* > 0$ such that the mushy region disappear ($|m(t)| = 0$) for any time $t \geq t^*$.

(iii) Assume $D_\infty > \gamma$ where

$$(13) \quad \gamma = M \int_0^{\Gamma_0} \varphi_N(r) dr, \quad M = \|u_0\|_{L^\infty(\Omega)} + \phi(\sup_{0 < t} \theta^0(t)).$$

Then the estimate $t^* > t_0^*$ holds with $t_0^* \geq 0$ defined by

$$(14) \quad D(t_0^*; u_0, \theta^0) = \gamma. \quad \blacksquare$$

3. The general case. To state the result for the general case of Ω an open bounded set of \mathbb{R}^N we start by recalling the notion of increasing rearrangement of a function: Let $h \in L^1(\Omega)$; the distribution function of h is defined by $\mu(\tau) = |\{x \in \Omega : h(x) < \tau\}|$. The increasing rearrangement of h is the one-variable function $\tilde{h} : [0, |\Omega|] \rightarrow \mathbb{R}$ given by $\tilde{h}(s) = \inf\{\tau \in \mathbb{R} : \mu(\tau) > s\}$. Finally, the increasing symmetric rearrangement of h is the function $h^* : \Omega^* \rightarrow \mathbb{R}$ defined by

$$(15) \quad h^*(x) = \tilde{h}(\omega_N |x|^N)$$

where Ω^* is the ball of \mathbb{R}^N , centered at the origin and with the same measure as Ω , i.e.

$$(16) \quad \Omega^* = B(0, r_0) \text{ with } r_0 \text{ such that } |\Omega| = |\Omega^*| = \omega_N r_0^N,$$

where ω_N denotes the measure of the unit ball in \mathbb{R}^N .

Our main result concerning the mushy region is the following:

Theorem. 2. Let Ω be a regular bounded open set of \mathbb{R}^N . Let u_0 and θ^0 satisfy (6) and (7) respectively. We also assume that

$$(17) \quad \theta^0(t) \quad \text{is nondecreasing in } t$$

and

$$(18) \quad u_0(x) \leq \theta^0(0) \quad \text{a.e. } x \in \Omega.$$

Finally, let $r_0 > 0$ be given by (16) and define $u_0(r)$ in $L^\infty(0, r_0)$ by

$$(19) \quad u_0(r) := \tilde{u}_0(\omega_N r^N).$$

Then the following conclusions hold:

a) Let (U, Θ) be the solution of the Stefan problem on $Q_\infty^* \equiv \Omega^* \times (0, \infty)$ corresponding to initial value u_0^* and boundary data θ^0 . Then

$$(20) \quad |m(t)| \leq |M(t)| \quad \text{for any } t > 0,$$

where $m(t)$ and $M(t)$ denote the mushy regions corresponding to the respective solutions (u, θ) and (U, Θ) .

b) If $D(\infty; u_0, \theta^0) > 0$ there exists a finite time t^* such that the mushy region disappears ($|m(t)| = 0$) for any $t \geq t^*$.

c) If $D(\infty; u_0, \theta^0) > \gamma$, with γ given by (13), the estimate $t^* \geq t_0^*$ holds, where t_0^* is defined by (14). \blacksquare

4. Remarks and sketch of the proofs.

A. We notice that in contrast with the radial case no assumption is made in Theorem 2 on the nature of $m(0)$ (the mushy region at $t=0$). So, for instance $m(0)$ may be a very irregular set with several connected components.

B. Assumptions (17) and (18) have a technical nature. One way to avoid them is to introduce $\theta_\#^0 \in W^{1,\infty}(0, \infty)$ be any nondecreasing function such that $0 \leq \theta_\#^0(t) \leq \theta^0(t)$ for any $t \geq 0$ and to define $\underline{u}_0(\cdot) = \min\{u_0(\cdot), \theta_\#^0(0)\}$. Then the conclusions of Theorem 2 remains true for general data u_0 and θ^0 by replacing the definition of (U, Θ) by the solution of the radial problem corresponding to the data \underline{u}_0 and $\theta_\#^0$.

C. Theorem 2 and the definition of D_∞ show that the behavior of the mushy region is influenced by an appropriate combination of the boundary temperature of the initial enthalpy. In particular the latter enters this combination as the integral of $\tilde{u}_0(\omega_N r^N) \varphi_N(r)$. We also remark that the weight function $\varphi_N(r)$ is peculiar to the boundary condition under consideration (so for the Neumann problem $\varphi_N(r)$, must be replaced by another suitable function $\psi_N(r)$, see [3]).

D. The main step of the proof of Theorem 1 is the equality

$$(21) \quad \int_0^{\Gamma_0} u(r, t) \varphi_N(r) dr = D(t; u_0, \theta^0)$$

which is obtained by taking a suitable test function η in (8). Conclusion (i) follows easily from (21) arguing by contradiction. Conclusions (ii) and (iii) are shown by regularizing the problem and by using the structure of the mushy region $m_c(t)=B(0,R_c(t))$ of the regularized problem (here $R_c(t)$ is known to be a strictly decreasing function of t).

E. The proof of Theorem 2 consists of several steps. The first and fundamental point is to prove the inequality

$$(22) \quad \int_s^{|\Omega|} (\tilde{u}(\sigma, t)) d\sigma \geq \int_s^{|\Omega|} (\tilde{U}(\sigma, t)) d\sigma$$

for any $s \in [0, |\Omega|]$. That inequality is first obtained for a regularized problem (function ϕ is replaced by a Lipschitz sequence of functions ϕ_ϵ) and then by passing to the limit. The notion of relative rearrangement is used, as in [9], extending the approach of [6] to the case of time-dependent boundary data. Finally conclusion (a) is shown using some arguments introduced in [4] (Theorem 1.28). Conclusions (b) and (c) comes easily from Theorem 1.

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Isoperimetric inequalities for the Stefan problem

1. Introduction

We consider the Stefan problem in its simplest form and in an annular space geometry : find a pair (θ, h) of functions defined in $q = \omega \times (0, T)$ such that, in some weak sense,

$$(1.1) \quad \begin{cases} \frac{\partial h}{\partial t} - \Delta \theta = 0 \text{ in } q, \\ \theta = g \text{ on } \sigma = \partial \omega \times (0, T), \\ h|_{t=0} = h_0, \\ h \in a(\theta) \text{ a.e. in } q. \end{cases}$$

Here

- $\omega = \omega_0 \setminus \bar{\omega}_1$; ω_0, ω_1 bounded regular domains in \mathbb{R}^N ($N \geq 2$); $\bar{\omega}_1 \subset \omega_0$;
- g constant on each of $\sigma_j = \gamma_j \times (0, T) = \partial \omega_j \times (0, T)$ ($j = 0, 1$), let us say

$$g = \begin{cases} 0 \text{ on } \sigma_0 \\ 1 \text{ on } \sigma_1; \end{cases}$$

- a is a strictly monotone graph in \mathbb{R}^2 :

$$(1.2) \quad a(\theta) = \begin{cases} \alpha_0(\theta - \lambda) - \alpha & \text{for } \theta < \lambda, \\ [-\alpha, 0] & \text{for } \theta = \lambda, \\ \alpha_1(\theta - \lambda) & \text{for } \theta > \lambda, \end{cases}$$

$\alpha, \alpha_0, \alpha_1$ positive constants, $\lambda \in [0, 1]$;

- $h_0 \in L^\infty(\omega)$ satisfies an extra condition (see (1.6), (1.7) below), which essentially means that $\theta_0 = b(h_0) = a^{-1}(h_0)$ belongs to $H^1(\omega)$ and $0 \leq \theta_0 \leq 1$.

Our boundary and initial data, g and h_0 , are such that $0 \leq \theta \leq 1$ in all q , by the maximum principle. If $\lambda = 0$ in (1.2), the temperature θ in the solid phase (the latter generally defined as the region where $h \leq -\alpha$ ($h = -\alpha$ if $\lambda = 0$)) therefore must be constantly equal to zero. Similarly, if $\lambda = 1$, the temperature in the liquid phase $\{h \geq 0\}$ ($h = 0$ here) is constantly equal to 1. Thus, for these extreme cases, in practice we have a one-phase Stefan problem, while for $0 < \lambda < 1$, the problem really is a two-phase problem.

One standard way of making (1.1) precise is to say that (θ, h) is a weak solution of (1.1) if

$$(1.3) \quad \begin{cases} \theta \in L^\infty(q), h \in L^\infty(q), h \in a(\theta) \text{ a.e. in } q, \\ \int \int_q \left(h \frac{\partial \varphi}{\partial t} + \theta \Delta \varphi \right) dx dt = \\ = \int \int_\sigma g \frac{\partial \varphi}{\partial \nu} d\gamma dt - \int_\omega h_0(x) \varphi(x, 0) dx \end{cases}$$

for every "test function" $\varphi \in C^1(\bar{q})$ satisfying $(\partial^2 \varphi / \partial x_i \partial x_j) \in C(\bar{q})$ and $\varphi = 0$ on $\sigma \cup (\omega \times \{T\})$ (see, e.g. [1] or [2]).

One can obtain the weak solution as a limit as $\epsilon \rightarrow 0$ ($\epsilon > 0$) of the classical solutions $(\theta_\epsilon, h_\epsilon)$ of some regularized problems $(1.1)_\epsilon$ where

- a is replaced by a_ϵ , single-valued smooth function with

$$(1.4) \quad a'_\epsilon \geq \delta > 0 \text{ (}\delta \text{ independent of } \epsilon\text{)}$$

$$(1.5) \quad b_\epsilon = a_\epsilon^{-1} \text{ converges uniformly to } b = a^{-1};$$

- h_0 is replaced by smooth functions h_{ϵ_0}

$$(1.6) \quad h_{\epsilon_0} \rightarrow h_0 \text{ in } L^1(\omega),$$

and $\theta_{\epsilon_0} = b_\epsilon(h_{\epsilon_0})$ satisfies

$$(1.7) \quad \begin{cases} 0 \leq \theta_{\epsilon_0} \leq 1 \text{ in } \omega, \theta_{\epsilon_0|_{\partial \omega}} = g, \\ \int_\omega |\nabla \theta_{\epsilon_0}|^2 dx \text{ is bounded independently of } \epsilon, \text{ as } \epsilon \rightarrow 0. \end{cases}$$

(see [7], [1], [2]).

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Free boundary problems involving solids

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