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MATHEMATICAL ANALYSIS OF SOME DIFFUSIVE ENERGY BALANCE MODELS IN CLIMATOLOGY. [1]

1. Introduction.

This paper is devoted to the study of the nonlinear parabolic problem

$$(P)\begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x = R_a(x,t,u) - R_e(x,t,u) & x \in I, t > 0, \\ \rho(x)|u_x|^{p-2}u_x = 0 & x \in \partial I, t > 0, \\ u(x,0) = u_0(x) & x \in I, \end{cases}$$

where $I = (-1, 1)$.

The problem arises from climate modeling, more specifically from an energy climate model due to Held and Suarez [1974] where the case $p = 3$ was proposed. We point out that many of the results of this work will be obtained under the general assumption $1 < p < \infty$ and so they are also of application to the classical models introduced for Budyko [1969] and Sellers [1969] corresponding to the choice $p = 2$.

A list of structure assumptions is the following:

$$\rho(x) = k(1 - x^2) \text{ with } k > 0, \tag{1}$$

$$\left.\begin{array}{l} R_a(x,t,u) = Q(x,t)\beta(u) \text{ where } Q \in C([-1,1] \times \mathbb{R}_+) \text{ satisfies} \\ 0 < Q(x,t) \text{ and } \beta \text{ is a nondecreasing function such that} \\ |\beta(u)| \leq M \quad \forall u \in \mathbb{R}, \text{ for some } M > 0, \end{array}\right\} \tag{2}$$

$$\left.\begin{array}{l} R_e(x,t,u) \text{ is a continuous function on } x, \text{ Lipschitz on } t \text{ and } R_e(x,t,\cdot) \\ \text{is nondecreasing as function on } u, \text{ for any fixed } (x,t) \in \overline{I} \times \mathbb{R}_+. \end{array}\right\} \tag{3}$$

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The modelling of the problem is considered in Section 2. Two special choices of the functions $R_a(x,t,u)$ and $R_e(x,t,u)$ are of relevance in Climatology: *the Budyko model* corresponds to the case in which $\beta(u)$ is assumed to be a discontinuous function

$$\beta(u) = \begin{cases} a_f & \text{in} \quad u > -10 \\ a_i & \text{in} \quad u < -10 \end{cases} \tag{4}$$

for some positive numbers a_i, a_f such that $a_i < a_f$ and the function R_e is assumed to be linear

$$R_e(x,t,u) = A + Bu \tag{5}$$

for some $A \in I\!R$ and $B > 0$. *The Sellers model* assumes that $R_a(x,t,u)$ is a regular function of u, as for instance

$$\beta(u) = a_i + \frac{1}{2}(a_f - a_i)(1 + tanh\gamma u), \tag{6}$$

where a_i, a_f are as before, $\gamma \in (0,1)$ is fixed and $R_e(x,t,u)$ is of the forme

$$R_e(x,t,u) = \epsilon(u)|u|^3 u \tag{7}$$

for some function ϵ such that $\epsilon(u) \in (0,K)$ for any $u \in I\!R$ and some $K > 0$.

The notion of weak solutions of problem (P) is introduced in Section 3. It is proven that if $u_0 \in L^\infty(I)$ there exists at least one bounded weak solution of (P). This is obtained by two different methods: via a compactness abstract method and via a regularization argument. Due to the presence of the degenerate coefficient $\rho(x)$ the natural energy space is given by $V = \{w \in L^2(I) : w_x \in L^p(I : \rho)\}$, where $L^p(I : \rho)$ is the weighted-Lebesgue space associated to ρ.

The question of the uniqueness of bounded weak solutions is studied in Section 4. The answer is positive for the Sellers model (it is enough to require β be a locally Lipschitz continuous function). As in the case of the homogeneous model (see Díaz[1992]) the Budyko model may have more than one solution. This is explicitly shown in the Subsection 4.1 by means of the construction of a counterexample. Nevertheless in the Subsection 4.2 it is shown that there is at most one solution of the Budyko model in the class of solutions satisfying a "nondegeneracy property".

Finally, the mushy region ($M(t) = \{x \in I : u(x,t) = -10\}$) is considered in Section 5. When $p = 2$ the results of Xu [1991] allows to conclude that

$M(t)$ is a curve in t (*the free boundary*). Nevertheless, if $p > 2$ it is shown that $M(t)$ has a non empty interior set if we assume this property on $M(0)$ and t is small enough.

2. On the modelling of problem (P).

Several energy balance models have been introduced in the literature since the pionnering works by the Russian Budyko [1969] and the american Sellers [1969]. We refer the reader to the expository papers in this volume by North [1992] and Stakgold [1992] (see also Díaz [1992]). As usual $u(x,t)$ represents the mean annual temperature average on the latitude circles around the Earth (denoted by $x = \sin\phi$ where ϕ is the latitude). The degeneracy at $x = \pm 1$ of the diffusion coefficient ρ given by (1) is due to the peculiar expression of the diffusion operator on a circle. Notice that when $p = 2$ and $k = 1$ the diffusion operator becomes the *linear* Legendre diffusion operator

$$Au = -((1-x^2)u_x)_x. \tag{8}$$

The main reason argued by Held and Suarez [1974] in order to introduce the nonlinear diffusion operator (with $p = 3$) is to take into account the negative feedback inherent to the transport of energy by large eddies in the Earth's atmosphere (such a transport increases as the gradient of the temperature increases). We point out that the elliptic operator

$$A_p u = -(\rho(x)|u_x|^{p-2}u_x)_x \tag{9}$$

degenerates not only at $x = \pm 1$ (due to the assumption (1)), but also in the set of points where $u_x = 0$, assumed that $p > 2$. For $1 < p < 2$, the operator becomes singular in this set.

The right hand side of the equation in (P) stands for the mean radiation flux. $R_a(x,t,u)$ represents the fraction of the solar energy absorbed by the Earth. Clearly it depends on the solar flux $Q(x,t)$ and the planetary albedo. Very often the absorbed energy is assumed to be given by

$$R_a(x,t,u) = Q(x,t)(1-\alpha(u)) \tag{10}$$

where $\alpha(u)$ is a real function which represents *the albedo* as function of the temperature and takes constant values for temperatures far from a critical value (usually $u = -10°C$). For u near the critical value there are two different kind of assumptions: Budyko [1969] assumes that α is discontinuous at $u = -10°C$ and Sellers [1969] supposes that α is a smooth function. In both cases assumption (2) holds.

The energy emmited by the Earth to the outher space is modelled by the term $R_e(x,t,u)$. Different empirical relations have been proposed in the literature. Budyko [1969] assume R_e given by (5) and Sellers [1969] uses a Stefan-Boltzman type law leading to (7) where $\epsilon(u)$ is a regular, positive and bounded function representing an emissivity (u is this time measured in Kelvin degrees). It is understood that, in both expresions, the coefficients may depend on the amount of greenhouse gases, clouds and water vapor in the atmosphere. For more generality we can also assume that they can change with the position x and the time t. Notice thatn any case the general assumption (3) is fulfilled.

More general models have been introduced in the literature. For instance, the study of the temperature distribution as a function of latitude, longitude and time leads to a parabolic equation on the unit sphere S^2 of $\mathbb{R}^3$. In that case the diffusion operator becomes the Laplace-Beltrami operator when $p = 2$. We remark that if we write the usual p-Laplacian operator

$$Au = -\text{div}(|\nabla u|^{p-2}\nabla u)$$

on the sphere $x = \cos\theta\sin\phi$, $y = \sin\theta\sin\phi$, $z = \cos\phi$ and apply it to functions $u = u(\phi)$ we obtain an operator different of the one given by (9). Nevertheless, the associated parabolic problem can be treated in an analogous way after some technical changes.

3. On the existence of solutions.

It is well known (see, e.g. Díaz-Herrero [1981] for the special case of $\rho = 1$ and $R_a \equiv 0$) that if $p > 2$ the degeneracy of the diffusion operator makes impossible expect the existence of a classical solution of (P) even for a regular initial datum u_0. In order to make precise the notion of solution we shall study, we start by indicating that the eventual discontinuous character of the function R_a will be treated by assuming that

$$\left.\begin{array}{l} R_a(x,t,u) = Q(x,t)\beta(u), \text{ with } Q \text{ as in (2) and } \beta \text{ a} \\ \text{maximal monotone graph of } \mathbb{R}^2 \text{ such that } |z| \leq M \\ \text{for any } z \in \beta(u), \text{ for any } u \in \mathbb{R} \text{ and some } M > 0 \end{array}\right\} \quad (11)$$

(i.e. for example, β is given by a nondecreasing real function b as $\beta(r) = \{b(r)\}$ if b is continuous in r or $\beta(r) = [b(r-), b(r+)]$ if b has a jump at the point r: see Brezis[1973]). A usual way to verify the differential equation (at least weakly) is to multiply by a test function followed by an integration by parts. In doing so we obtain

$$\int_I u(x,T)v(x,T)dx - \int_0^T \int_I u(x,t)v_t(x,t)dxdt$$

$$+ \int_0^T \int_I \rho(x)|u_x(x,t)|^{p-2}u_x(x,t)v_x(x,t)dxdt$$

$$= \int_0^T \int_I \{Q(x,t)z(x,t) - R_e(x,t,u)\}v(x,t)dxdt + \int_I u_0(x)v(x,0)dx \tag{12}$$

for some function $z(x,t)$ which satisfies that

$$z(x,t) \in \beta(u(x,t)) \text{ a.e. } x \in I \text{ and } t \in (0,T). \tag{13}$$

For several purposes it will be useful to take the solution u as a test function. So, for t fixed, the integrals

$$\int_I \rho|u_x|^p dx \text{ and } \int_I |u|^2 dx$$

must be finite. Then a natural "energy space" associated to (P) is the one defined by

$$V = \{w \in L^2(I) : w_x \in L^p(I:\rho)\},$$

where $L^p(I:\rho)$ is the weighted-Lebesgue space

$$L^p(I:\rho) = \{v : \| v \|_{L^p(I:\rho)} = [\int_I \rho(x)|v(x)|^p dx]^{\frac{1}{p}} < \infty\}.$$

It is easy to see that V is a separable and reflexive Banach space with the norm

$$\| u \|_V = \| u \|_{L^2(I)} + \| u_x \|_{L^p(I:\rho)} .$$

Any weak solution must satisfy $u(\cdot,t) \in V$ for a.e. $t \in (0,T)$. It is not difficult to see that in that case $|u_x(\cdot,t)|^{p-2}u_x(\cdot,t) \in L^{p'}(I:\rho)$, with $p' = p/(p-1)$. We also remark that because of the physical modelling of the problem we shall restrict our study to the class of bounded functions.

Definition 1 . *By a bounded weak solution of problem* (P) *we mean a function* $u \in C([0,T] : L^2(I)) \cap L^\infty(I \times (0,T))$ *such that* $u \in L^p(0,T : V)$, $R_e(\cdot,\cdot,u) \in L^1(I \times (0,T))$ *and there exist* $z \in L^\infty(I \times (0,T))$ *satisfying (13) and the identity (12) holds for any* $v \in L^p(0,T : V) \cap L^\infty(I \times (0,T))$ *such that* $v_t \in L^{p'}(0,T : V')$.

The main purpose of this section is to prove the following result

Theorem 1 *For any* $u_0 \in L^\infty(I)$ *there exist at least one bounded weak solution* u *of* (P).

The proof of the above theorem can be carried out by means of different methods. Here we shall present two different type of techniques: **(i)** a compactness abstract method, and **(ii)** a regularization method.

3.1. *Existence via a compactness abstract method.*

Problem (P) can be considered as a perturbed problem associated to

$$(P^*) \begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x = 0, & x \in (-1,1), t > 0, \\ \rho(x)|u_x|^{p-2}u_x = 0, & x = \pm 1, t > 0, \\ u(x,0) = u_0(x), & x \in (-1,1). \end{cases}$$

The abstract Cauchy problem associated to (P^*) is given by

$$(CP^*) \begin{cases} \dfrac{du}{dt}(t) + Au(t) = 0, \quad \text{in } L^2(I), \quad \text{for } t > 0, \\ u(0) = u_0 \end{cases}$$

where we are identifying $u(t) \in L^2(I)$ with $u(\cdot,t)$. The operator $A : D(A) \longrightarrow L^2(I)$, with $D(A) \subset L^2(I)$, is described in the following result giving also the existence and uniqueness of the solution of (CP*).

Proposition 1 . *(a) Consider the functional* $\varphi : L^2(I) \longrightarrow I\!R \cup \{+\infty\}$ *given by*

$$\varphi(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_I \rho(x)|u_x|^p dx & \text{if } u \in V \\ = +\infty & \text{otherwise.} \end{cases} \tag{14}$$

Then $\varphi \not\equiv +\infty$, φ *is convex and lower semicontinuous.*
(b) Let $A(u) = \partial\varphi(u)$. *Then* $D(A) \subset V$, $D(A)$ *is dense in* $L^2(I)$ *and*

$$Au = -(\rho(x)|u_x|^{p-2}u_x)_x \text{ for any } u \in D(A). \tag{15}$$

(c) For any $u_0 \in L^2(I)$ there exists a unique function $u \in C([0,T] : L^2(I))$, for $T > 0$ arbitrary, such that $u(t) \in D(A)$ for a.e. $t > 0$, $t^{\frac{1}{2}}\frac{du}{dt} \in L^2(0,T : L^2(I))$ and satisfies (CP^*). Moreover if $u_0 \in L^q(I)$ with $1 \leq q \leq +\infty$ then $u(t) \in L^q(I)$. Finally, the application $S(t)u_0 = u(t)$ is a semigroup of contractions on $L^2(I)$.

Proof. (a) To prove that $\varphi \not\equiv +\infty$ and that φ is convex is obvious. The lower semicontinuity of φ can be shown, for instance, using the reflexivity of the space $L^p(I : \rho)$, and that the norm is l.s.c. for the weak convergence.
(b) It is clear that $V = D(\varphi)(\equiv \{w \in L^2(I) : \varphi(w) < \infty\})$ is a dense subspace of $L^2(I)$ (notice that $C_0^\infty(I) \subset V$). Then as $D(\partial\varphi) \subset D(\varphi)$ and $\overline{D(\partial\varphi)} = \overline{D(\varphi)}$ (see Brezis [1973]) we have that $\overline{D(\partial\varphi)} = L^2(I)$. On the other hand it is a routin matter to see that φ is Gateux differentiable in V and that

$$< \varphi'(u), h >_{V'V} = \lim_{\lambda \searrow 0} \frac{\varphi(u + \lambda h) - \varphi(u)}{\lambda} = \int_I \rho(x)|u_x|^{p-2}u_x h_x dx.$$

As $\partial\varphi(u)$ is a maximal monotone operator we obtain (15).
(c) The existence of u with the indicated regularity is now a consequence of the abstract Hille-Yosida theorem given in Brezis [1973]. If $u_0 \in L^q(I)$ we multiply the equation by the test function $|u|^{q-1}\text{sign}u$ (more precisely, by a smooth approximation of this function) and a simple integration by parts shows that

$$\frac{d}{dt}\int_I |u|^q dx \leq 0,$$

which gives the result. ∎

Theorem 1 can be obtained from an abstract perturbation result (see Vrabie [1987] and Díaz-Vrabie [1987]) assuming that the operator $A = \partial\varphi$ generates a compact semigroup. By a result due to H.Brezis (see the reference in the book of Vrabie [1987]) this condition is equivalent to know that

$$\left.\begin{array}{l}\text{"for any } K > 0 \text{ the set } \{w \in L^2(I) : \| w \|^2_{L^2(I)} + \varphi(w) \leq K\} \\ \text{is relatively compact in } L^2(I)\text{".}\end{array}\right\} \quad (16)$$

This is proved in the following auxiliary result:

Lemma 1 . *(i) Let ρ given by (1) and assume $p > 2$. Then for any $q \in [1, p/2)$ we have that*

$$V \subset \{w \in L^2(I) : w_x \in L^q(I)\} \quad (17)$$

with continuous imbedding. Moreover, for any $r \in [1,\infty]$ we have

$$V \subset L^r(I), \tag{18}$$

where the imbedding is continuous and compact for any $r \in [1,\infty]$.
(ii) If $1 < p \leq 2$, then we have the continuous imbedding $V \subset L^q(I)$ for any $q \in [1,\infty)$ if $p = 2$ and any $q \in [1,p^*)$ with $p^* = 2p/(2-p)$.
(iii) If $1 < p \leq 2$, the imbedding $V \subset L^2(I)$ is always compact.

Proof. (i) Let $w \in L^p(I:\rho)$ and $q \in [1,p/2)$. By the Hölder inequality with $p_1 = p/q$ and $p_1' = p/(p-q)$

$$\int_I |w(x)|^q dx = \int_I |w(x)|^q \rho(x)^{q/p} \rho(x)^{-q/p} dx \leq$$

$$\leq \left(\int_I |w(x)|^p \rho(x) dx\right)^{q/p} \left(\int_I \frac{dx}{\rho(x)^{q/(p-q)}}\right)^{(p-q)/p}.$$

But
$$\int_I \frac{dx}{\rho(x)^{q/(p-q)}} \leq \frac{1}{K_0^{q/p-q}} \int_{-1}^{1} \frac{dx}{(1-x^2)^{q/p-q}} < \infty$$

since $(1-x^2) \geq Cd(x,\partial I)$ and $q/(p-q) < 1$. This proves the first part of the statement. This also shows the continuous imbedding $V \subset W^{1,1}(I)$ and so (17) holds by a well-known result (see, e.g., Brezis [1983], Theorem VIII.7). Then $V \subset W^{1,q}(I)$ for any $q \in [1,p/2)$ and by the mentioned result the imbedding (17) is also compact for $r = +\infty$. The proof of (ii) can be found in Adams [1980] or Rakotoson-Simon [1993]. Part (iii) is shown in Meyer [1967] for $p = 2$. His proof can be extended to any $p \in (1,2)$ using part (ii). ∎

Corollary 1 . *Assume (1), (3), (11), (14) and $p \geq 2$. Then for any $u_0 \in L^2(I)$ there exists a function $u \in C([0,T]:L^2(I))$ such that $u(t) \in D(A)$ a.e. $t \in (0,T]$, $t^{\frac{1}{2}}\frac{du}{dt} \in L^2(0,T:L^2(I))$, $\varphi(u) \in L^1(0,T:\mathbb{R})$ and it satisfies (P) a.e. $t \in (0,T)$ on $L^2(I)$ as well as in the sense of (12). Moreover, if $u_0 \in V$ then $\frac{du}{dt} \in L^2(0,T:L^2(I))$ and $u \in C([0,T]:V)$. Finally, if $u_0 \in L^\infty(I)$ then $u \in L^\infty(I \times (0,T))$.*

Proof. The existence of u satisfying (P) a.e. $t \in (0,T)$ on $L^2(I)$ is a consequence of the application of a suitable fixed theorem for a compact operator (see, e.g., Vrabie [1987], Corollary 2.3.2). The application of such results is guaranteed by Proposition 1, Lemma 1 and the assumptions (3) and (11). This function obviously satisfies trivially (12) (take integrals on $(\tau, T) \times I$ and make $\tau \searrow 0$). The boundedness of u, assumed $u_0 \in L^\infty(I)$, is proved as in Proposition 1 if the right hand side of the equation is a bounded term. ■

Remark 1. The above method can be applied to two-dimensional problems (on a compact Riemannian manifold without boundary): see Hetzer [1990] (for the Sellers type model) and Díaz-Tello [1993] (for the Budyko model).

3.2. *Existence via a regularization method.*

The existence of a bounded weak solution of (P) can be also obtained by approximating the multivalued (discontinuous) term $\beta(\cdot)$ by a regular function $\beta_\epsilon \in C^\infty(I\!R)$ with the properties

$$\beta_\epsilon'(s) \geq 0 \text{ and } |\beta_\epsilon(s)| \leq M \quad \forall s \in I\!R. \tag{19}$$

It is also usefull to remove the degeneracy at ∂I by replacing $\rho(x)$ by

$$\rho_\epsilon(x) = \rho(x) + \epsilon. \tag{20}$$

In order to approximate u by classical solutions of a related problem we also replace the data u_0, Q and R_e by C^∞ functions $u_{0,m}$, Q_n, $R_{e,k}$ such that

$$u_{0,m}(\pm 1) = 0, \quad \| u_{0,m} \|_{L^\infty(I)} \leq \| u_0 \|_{L^\infty(I)},$$

and

$$u_{0,m} \longrightarrow u_0 \text{ in } L^2(I), \text{ as } m \to \infty,$$

$$Q_n \longrightarrow Q \text{ in } C(\overline{I} \times [0,T]),$$

$$\left.\begin{array}{l} R_{e,k} \text{ satisfies (2), } R_{e,k}(\cdot,\cdot,u) \longrightarrow R_e(\cdot,\cdot,u) \text{ in } C(\overline{I} \times [0,T]) \\ \text{for any fixed } u \in I\!R \text{ and } R_{e,k}(x,t,\cdot) \longrightarrow R_e(x,t,\cdot) \text{ in } C(J) \text{ for} \\ \text{any compact } J \subset I\!R \text{ and any fixed } (x,t) \in \overline{I} \times [0,T]. \end{array}\right\}$$

Given ϵ, m, n and k positive numbers we consider the problem (P_ϵ)

$$\begin{cases} u_t - [\rho_\epsilon(x)|u_x|^{p-2}u_x]_x - \epsilon u_{xx} = Q_n(x,t)\beta_\epsilon(u) - R_e(x,t,u), & x \in I \times (0,T), \\ \rho_\epsilon(x)(|u_x|^{p-2}u_x + \epsilon u_x) = 0 & \text{on } \partial I \times (0,T), \\ u(x,0) = u_{0,m}(x) & \text{on } I. \end{cases}$$

The partial differential equation is now uniformly parabolic and so by well-known results (see e.g. Ladyzenskaja-Solonnikov-Ural'ceva [1968], Chapt.V) there exists a unique classical solution $U = u_{\epsilon,m,n,k}$. In order to study the convergence, when $\epsilon \searrow 0$ and $m, n, k \to +\infty$ we need some a priori estimates.

Lemma 2 . *The solution U of (P_ϵ) satisfies (for n and k large enough)*

$$\| U \|_{L^\infty(I\times(0,T))} \leq C, \tag{21}$$

$$\| \rho_\epsilon U_x \|_{L^p(0,T:L^p(I))} \leq C, \tag{22}$$

where C denotes a positive constant independent of ϵ, m, n and k.

Proof. Estimate (21) is derived from the maximum principle (see e.g. Ladyzenskaja-Solonnikov-Ural'ceva [1968]). To obtain (22) we multiply the equation by U. Integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\int_I U^2(x,t)dx + \int_I \rho_\epsilon |U_x|^p dx + \epsilon \int_I |U_x|^2 dx \leq C$$

(where we have used (19) and (21)). ∎

Using the a priori estimates and the assumption (11) the proof of the convergence $U \longrightarrow u$, $\beta_\epsilon(U) \longrightarrow z$ with $z \in \beta(u)$ and that u is a bounded weak solution of (P) is standard (notice that this is not the case if we want obtain more regularity on u_t as, for instance, that given in Corollary 1).

Remark 2. The regularization of the multivalued term $\beta(u)$ was already carryed out in Xu [1991] for $p = 2$ (see also Feireisl-Norbury [1991] for some related problems). We also point out that the existence of a weak solution can be obtained by the method of upper and lower solutions combined with monotone iteration arguments (see e.g. Carl [1989] and Díaz-Stakgold [1989] for other related problems).

4. On the uniqueness of solutions: positive and negative answers.

The type of answer to the question of the uniqueness of solutions to problem (P) is rather different in the cases of the *Sellers model* (where $R_a(x,t,u)$ is a smooth function) and the *Budyko model* (where $R_a(x,t,u)$ is a discontinuous function of u).

4.1. *The Sellers model.*

The following result shows the uniqueness and others properties of solutions for the Sellers model.

Theorem 2 *Let $p > 1$, and assume that*

$$R_a \text{ satisfies (11) with } \beta \text{ a locally Lipschitz function of } u. \tag{23}$$

Then given $u_0 \in L^\infty(I)$ there exists at most one bounded weak solution of (P).

Idea of the proof. First of all we point out that $u_t \in L^{p'}(0,T:V')$. This can be obtained from the definition of bounded weak solutions and the characterization of the dual space V' (see e.g. Ivanov [1981], Lemma V.2.1). Moreover, if we define $w = e^{-Ct}u$, w satisfies (in a weak sense) the equation

$$w_t - e^{-C(p-2)t}(\rho(x)|w_x|^{p-2}w_x)_x = e^{-Ct}Q(x,t)\beta(we^{Ct}) - e^{-Ct}R_e(x,t,we^{Ct}) - Cw.$$

Since β is assumed locally Lipschitz we can choose C large enough such that the function

$$F(x,t,w) = e^{-Ct}Q(x,t)\beta(we^{Ct}) - Cw$$

is a strictly decreasing function of v for fixed (x,t). Now assume that we have another solution u^* of (P) corresponding to the same datum u_0. We take $w-w^*$ ($w^* = e^{-Ct}u^*$) as test function in the difference of the identities satisfied by w and w^* (see the definition of bounded weak solution). We have that

$$< w_t(t) - w_t^*(t), w(t) - w^*(t) >_{V'V} = \frac{d}{dt}\int_I |w(t) - w^*(t)|^2 dx$$

(see e.g. Temam [1988]). Moreover, there exists $K > 0$ such that if $p \geq 2$

$$\int_I \rho(x)(|w_x|^{p-2}w_x - |w_x^*|^{p-2}w_x^*)(w_x - w_x^*)dx \geq K\int_I \rho(x)|w_x - w_x^*|^p dx \tag{24}$$

For $1 < p < 2$ the right-hand side term must be replaced by

$$K\int_I \rho(x)|w_x - w_x^*|^2(|w_x|^{2-p} + |w_x^*|^{2-p})dx$$

(see, e.g., Díaz[1985] Lemma 4.10). Using the monotonicity of $R_e(\cdot,\cdot,u)$ and $F(\cdot,\cdot,w)$ we obtain that

$$\frac{d}{dt}\int_I |w(t) - w^*(t)|^2 dx \leq 0$$

and so necessarily $u = u^*$.

■

Corollary 2 . *Assume (23). Let* $u_0, \hat{u}_0 \in L^\infty(I)$ *and let* $u, \hat{u}$ *be weak solutions of (P) corresponding to the energy emmision functions* $R_a(x,t,u) = \gamma(u) + f(x,t)$ *and* $\hat{R}_a(x,t,u) = \gamma(u) + \hat{f}(x,t)$ *satisfying the condition (11). Then there exists a constant* $K = K(T) \geq 0$ *such that*

$$\| [u(t) - \hat{u}(t)]_+ \|_{L^2(I)} \leq$$

$$\leq e^{Kt} \left(\| [u_0 - \hat{u}_0]_+ \|_{L^2(I)} + \int_0^t e^{-Kt} \| [f(s) - \hat{f}(s)]_+ \|_{L^2(I)} \, ds \right). \tag{25}$$

In particular $u_0 \leq \hat{u}_0$, $f \leq \hat{f}$ *imply* $u \leq \hat{u}$.

Proof. It suffices to use now $(w - w^*)_+$ $(= max(w - w^*, 0))$ as a test function. Indeed, by a variant of a result due to Stampacchia we know that $(w - w^*)_+ \in L^p(0, T : V)$. Moreover

$$< w_t(t) - w_t^*(t), (w(t) - w^*(t))_+ >_{V'V} = \frac{d}{dt} \int_I |[w(t) - w^*(t)]_+|^2 dx$$

and inequality (25) follows. ■

4.2. *A non uniqueness result for the Budyko model.*

The discontinuity of the coalbedo function $\beta(u)$ and its role as a source term in the equation may lead to the existence of multiple (even infinite) solutions of the problem. This has already been shown in Díaz [1992] for the case of the homogeneous (zero-dimensional) balance model

$$\frac{du}{dt} = R_a(u) - R_e(u).$$

The main purpose of this subsection is to show that this situation may also occurs for problem (P). Our presentation is inspired in the work of Feireisl-Norbury [1991] (see also Feireisl [1991]). We fix our attention in the special case of Budyko model i.e., R_a and R_e are given by (2), (4) and (5) respectively. We shall also assume that

$$Q(x,t) \equiv Q \text{ and } Qa_i < A - 10B. \tag{26}$$

Consider a function u_0 such that

$$\left.\begin{array}{l} u_0 \in C^\infty(I),\ u_0(x) = u_0(-x) \text{ for all } x \in [0,1], \\ u_0^{(k)}(0) = 0 \text{ for } k = 1,2,\ u_0(0) = -10 \\ u_0'(x) < 0 \text{ if } x \in (0,1),\ u_0'(1) = 0 \end{array}\right\} \qquad (27)$$

(in this hypothetical case the maximum of the distributed temperature is $-10°C$ and it is only attained at the equator). We first show the existence of a "completely ice covered" solution u^*.

Proposition 2 . *Let R_a, R_e given by (2),(4) and (5) respectively. Assume that (26) and (27) holds. Then there exist at least one solution u^* of (P) such that $u^*(x,t) < -10$ for any $x \in (-1,1)$ and $t \in (0,T]$.*

Proof. Let u^* be the unique solution of the problem

$$(P)\left\{\begin{array}{ll} u_t - (\rho(x)|u_x|^{p-2}u_x)_x + Bu = -A + Qa_i, & x \in I, t > 0, \\ \rho(x)|u_x|^{p-2}u_x = 0 & x \in \partial I, t > 0, \\ u(x,0) = u_0(x) & x \in I. \end{array}\right.$$

The existence and uniqueness can be shown again by different methods (for instance, it is a trivial consequence of Proposition 1). The function $z = -10 - u^*$ satisfies that

$$z_t - (\rho(x)|z_x|^{p-2}z_x)_x + f(z) = 0$$

with

$$f(z) = Bz + 10B - A + Qa_i.$$

Moreover $z(x,0) > 0$ and $z(0,0) = 0$. Then from (26) and the strong maximum principle (see Vazquez [1984]) we deduce that $z(x,t) > 0$ [i.e. $u^*(x,t) < -10$] for all $(x,t) \in (-1,1) \times (0,T]$. ∎

The nonuniqueness of the solutions will be a consequence of the existence of solutions which exhibit the presence of "free-ice zones".

Theorem 3 . *Under the assumptions of Proposition 2 there exists at least one weak solution u of (P) such that $\{(x,t) : u(x,t) > -10\}$ is not empty for any $t > 0$ small enough.*

To carry out the proof of Theorem 3 we shall construct a family of auxiliary functions v^λ depending on a parameter $\lambda > 0$ in the following way. We first introduce the partition $(-1,1) \times [0,\lambda] = Q_1^\lambda \cup Q_2^\lambda \cup Q_3^\lambda$ by

$$\begin{aligned} Q_1^\lambda &= \{(x,t) \in (0,1) \times [0,\lambda], x > t/\lambda\} \\ Q_2^\lambda &= \{(x,t) \in (-1,1) \times [0,\lambda], -t/\lambda \le x \le t/\lambda\} \\ Q_3^\lambda &= \{(x,t) \in (-1,0) \times [0,\lambda], x < -t/\lambda\}. \end{aligned}$$

Now we define v^λ on Q_1^λ as the unique solution of the problem

$$P(Q_1^\lambda) \begin{cases} v_t - (\rho(x)|v_x|^{p-2}v_x)_x + Bv = -A + Qa_i, & (x,t) \in Q_1^\lambda, \\ v_x(1,t) = 0, \quad v(\frac{t}{\lambda},t) = -10, & t \in [0,\lambda], \\ v(x,0) = u_0(x) & x \in [0,1]. \end{cases}$$

The existence and uniqueness of a solution of $P(Q_1^\lambda)$ is an easy modification of the results of Friedman [1964] (see also Idrissi [1983]). Finally

$$v^\lambda(x,t) = -10 + C^\lambda(t)(x - t/\lambda)(x + t/\lambda) \text{ for all } (x,t) \in Q_2^\lambda, \tag{28}$$

$$v^\lambda(x,t) = v^\lambda(-x,t) \text{ if } (x,t) \in Q_3^\lambda.$$

We have

Proposition 3 . *It is possible to choose $C^\lambda(t)$ in (28) such that*
(i) $v^\lambda \in C([-1,1] \times [0,\lambda])$, $v_x^\lambda \in C((-1,1) \times [0,\lambda])$.
(ii) v^λ is a bounded weak solution of the associated problem

$$\begin{cases} v_t - (\rho(x)|v_x|^{p-2}v_x)_x + Bv = -A + h^\lambda(x,t) & \text{in } I \times (0,\lambda), \\ \rho(x)|v_x|^{p-2}v_x = 0 & \text{on } \partial I \times (0,\lambda), \\ v(x,0) = u_0(x) & \text{on } I, \end{cases}$$

where $h^\lambda \in L^\infty(I \times (0,\lambda))$ satisfies that $h^\lambda \equiv Qa_i$ in $Q_i^\lambda \cup Q_3^\lambda$ and

$$h(x,t) \le Q(a_f - a_i)/2 \text{ for } x \in I \text{ and } t \in (0,T_\lambda) \text{ with } T_\lambda \text{ small enough.} \tag{29}$$

(iii) $v^\lambda(x,t) > -10$ on Q_2^λ and $v^\lambda < -10$ on $Q_1^\lambda \cup Q_3^\lambda$.

Proof.(i) The continuity of v^λ follows from the continuity of the solution of $P(Q_1^\lambda)$ (any $w \in L^\infty(J)$ such that $\rho(x)w' \in L^p(J)$ satisfies $w \in C^0(J)$, for any open interval $J \subset (0,1)$). Moreover, by (27), the solution v^λ of $P(Q_1^\lambda)$ is regular on the segment $\{(t/\lambda, t) : t \in (0,\lambda)\}$ and the function

$$g^\lambda(t) = v_x^\lambda(t/\lambda, t)$$

satisfies that $g^\lambda \in C^1((0,\lambda))$, $g^\lambda(0) = (g^\lambda)'(0) = 0$ and from (26) and the strong maximum principle (see e.g. Vazquez [1984]) $g^\lambda(t) < 0$ if $t \in (0,\lambda]$. Then choosing

$$C^\lambda(t) = \frac{g^\lambda(t)\lambda}{2t}$$

we obtain that $v_x^\lambda \in C((-1,1) \times [0,\lambda])$. From the strong maximum principle and (27) we deduce (iii). To complete the proof we only need to show that the (multivalued) equation also holds on Q_2^λ. So it suffices to show that if u^λ is given by (27) then the function

$$h^\lambda(x,t) = v_t^\lambda - (\rho(x)|v_x^\lambda|^{p-2}v_x^\lambda)_x + Bv^\lambda$$

satisfies (29). A strightforward computation yields

$$h^\lambda(x,t) = \frac{\lambda(x - t/\lambda)(x + t/\lambda)}{2t^2}\left[g(t)(Bt+2) - g'(t)t\right]$$

$$- \left(\frac{g(t)\lambda}{2t}\right)^{p-1} 2^{p-1}kx^{p-2}\left[(p-1) - (p+1)x^2\right] - \frac{g(t)}{\lambda}$$

(where g denotes g^λ). The bound

$$\left|\frac{(x - t/\lambda)(x + t/\lambda)}{2t^2}\right| \leq C(\lambda) \text{ on } Q_2^\lambda$$

with $C(\lambda)$ independent of, allows to choose T_λ so small such that the function h^λ satisfies (29).

■

Proof of Theorem 3. We consider a regular approximation β_ϵ of β (e.g. $\beta_\epsilon \in C^\infty(\mathbb{R})$) satisfying (19) and also

$$a_i + \frac{a_f - a_i}{2} \leq \beta_\epsilon(s) \leq a_f \text{ if } s \geq -10 \text{ and } a_i \leq \beta_\epsilon(s) \leq a_i + \frac{a_f - a_i}{2} \text{ if } s < -10. \quad (30)$$

By theorems 1 and 2 we know the existence and uniqueness of a solution u_ϵ of the problem

$$\begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x + Bu = -A + Q\beta_\epsilon(u) & \text{in } I \times (0,T), \\ \rho(x)|u_x|^{p-2}u_x = 0 & \text{on } \partial I \times (0,T), \\ u(x,0) = u_0(x) & \text{on } I. \end{cases}$$

On the other hand, from Proposition 3, (29) and (30) we know that v^λ satisfies

$$\begin{cases} v_t - (\rho(x)|v_x|^{p-2}v_x)_x + Bv \leq -A + Q\beta_\epsilon(v) & \text{in } I \times (0,T_\lambda), \\ \rho(x)|v_x|^{p-2}v_x = 0 & \text{on } \partial I \times (0,T_\lambda), \\ v(x,0) = u_0(x) & \text{on } I. \end{cases}$$

and then by Theorem 2 we conclude that $u^\epsilon \geq v^\lambda$ on $\overline{I} \times [0,T_\lambda]$. Using the same kind of a priori estimates as in Lemma 2 we have that $u^\epsilon \longrightarrow u$ (weakly in $L^p(0,T:V)$ and weakly in $L^\infty(0,T:V)$) as $\epsilon \downarrow 0$, with u a bounded weak solution of (P) such that

$$u \geq v^\lambda \text{ on } \overline{I} \times [0,T_\lambda], \text{ for any } \lambda > 0, \tag{31}$$

and the conclusion follows from (28). ∎

Remark 3. It is not difficult to show (see Feireisl-Norbury [1991]) that (27) implies that the solution u of Theorem 3 satisfies $u_x(x,t) > 0$ for any $x \in (-1,0) \cup (0,1)$ and $t > 0$. Then by the *Implicit Function Theorem* there exists a continuous function $\zeta : [0,T] \longrightarrow [0,1]$, defining completely *the free boundary* associated to u i.e. such that for any fixed $t \in [0,T]$

$$\{x \in \overline{I} : u(x,t) = 1\} = \{-\zeta(t)\} \cup \{\zeta(t)\}. \tag{32}$$

Clearly $\zeta \in C^1((0,T])$. Moreover (31) implies that

$$\zeta(t) \geq t/\lambda \text{ for any } \lambda > 0.$$

As $\zeta(0) = 0$ we deduce that necessarily $\zeta'(t) \uparrow +\infty$ as $t \downarrow 0$.

4.3. *On the uniqueness of solutions of the Budyko model.*

We have proved that the mere presence of a "bad point" x_0 where $u(t_0, x_o) = -10$ and $u_x(t_0, x_0) = 0$ can be the reason of multiple solutions for $t \geq t_0$. The following result shows that if the initial datum u_0 leads to a solution u never flat at the level $u = -10$ then in fact u is the unique solution. We introduce the following notation:

Definition 2 . *Let $w \in L^\infty(I)$. We say that w satisfies the strong (resp. weak) p-nondegeneracy property if there exists $C > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$|\{x \in I : |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$$

(resp. $|\{x \in I : 0 < |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$).

Theorem 4 *Assume $p \geq 2$. Let R_e satisfying (3) and R_a given by (2) and (4). Let $u_0 \in L^\infty(I)$.*
(i) Assume that there exists a solution $u(\cdot, t)$ satisfying the strong p-nondegeneracy property for any $t \in [0, T]$. Then u is the unique bounded weak solution of (P).
(ii) At most there is a unique solution among the class of bounded weak solutions satisfying the weak p-nondegeneracy property.

We start by proving that under the nondegeneracy property the multivalued term generates a continuous operator from $L^\infty(I)$ into $L^q(I)$, for any $q \in [1, \infty)$.

Lemma 3 *(i) Let $w, \hat{w} \in L^\infty(I)$ and assume that w satisfies the strong p-nondegeneracy property. Then for any $q \in [1, \infty)$ there exists $\tilde{C} > 0$ such that for any $z, \hat{z} \in L^\infty(I)$, $z(x) \in \beta(w(x))$, $\hat{z}(x) \in \beta(\hat{w}(x))$ a.e. $x \in I$ we have*

$$\| z - \hat{z} \|_{L^q(I)} \leq (a_f - a_i) \min\{\tilde{C} \| w - \hat{w} \|_{L^\infty(I)}^{(p-1)/q} , 2^{1/q}\}. \tag{33}$$

(ii) If $w, \hat{w} \in L^\infty(I)$ and satisfy the weak p-nondegeneracy property then

$$\int_I (z(x) - \hat{z}(x))(w(x) - \hat{w}(x))dx \leq (a_f - a_i)C \| w - \hat{w} \|_{L^\infty(I)}^p . \tag{34}$$

Proof of Lemma 3. If $\| w - \hat{w} \|_{L^\infty(I)} > \epsilon_0$ then

$$\| z - \hat{z} \|_{L^q(I)} \leq (a_f - a_i)2^{1/q} \leq (a_f - a_i)\frac{2^{1/q}}{(\epsilon_0)^{(p-1)/q}} \| w - \hat{w} \|_{L^\infty(I)}^{(p-1)/q} .$$

Assume now that $\| w - \hat{w} \|_{L^\infty(I)} \leq \epsilon_0$. Define the coincidence sets

$$A = \{x \in I : w(x) = -10\} \quad \hat{A} = \{x \in I : \hat{w}(x) = -10\},$$

as well as the descomposition

$$\Omega = A \cup \Omega_+ \cup \Omega_- \quad \Omega = \hat{A} \cup \hat{\Omega}_+ \cup \hat{\Omega}_-,$$

where

$$\Omega_+ = \{x \in I : w(x) > -10\} \quad \Omega_- = \{x \in I : w(x) < -10\}$$

and $\hat{\Omega}_+, \hat{\Omega}_-$ are defined similarly replacing w by $\hat{w}$. Let $z, \hat{z}$ defined as in the statement. Then

$$\begin{array}{lll} |z(x) - \hat{z}(x)| \leq (a_f - a_i) & \text{on} & A \cup \hat{A} \cup (\Omega_+ \cap \hat{\Omega}_-) \cup (\Omega_- \cap \hat{\Omega}_+) \\ z(x) = \hat{z}(x) & \text{on} & (\Omega_+ \cap \hat{\Omega}_+) \cup (\Omega_- \cap \hat{\Omega}_-) \end{array}$$

Thus as $|I| = 2$

$$\| z - \hat{z} \|_{L^q(I)} \leq (a_f - a_i) \min\{|A \cup \hat{A} \cup (\Omega_+ \cap \hat{\Omega}_-) \cup (\Omega_- \cap \hat{\Omega}_+)|^{1/q}, 2^{1/q}\}. \quad (35)$$

But we have

$$\Big(A \cup \hat{A} \cup (\Omega_+ \cap \hat{\Omega}_-) \cup (\Omega_- \cap \hat{\Omega}_+)\Big) \subset B_\epsilon \equiv \{x \in \Omega : -10-\epsilon \leq w(x) \leq -10+\epsilon\}.$$

Indeed; it is clear that $A \subset B_\epsilon$. Moreover,

$$\hat{w}(x) - \| w - \hat{w} \|_{L^\infty(I)} \leq w(x) \leq \| w - \hat{w} \|_{L^\infty(I)} + \hat{w}(x) \text{ a.e. } x \in I.$$

Then the inclusion $\hat{A} \subset B_\epsilon$ is obvious. If $x \in \Omega_+ \cap \hat{\Omega}_-$, $-10 < w(x) \leq \epsilon + \hat{w}(x) < -10 + \epsilon$ and so $x \in B_\epsilon$. Finally if $x \in \Omega_- \cap \hat{\Omega}_+$, $-10 - \epsilon \leq -10 - |w(x) - \hat{w}(x)| \leq \hat{w}(x) + w(x) - \hat{w}(x) \leq w(x) < -10$ and $x \in B_\epsilon$. Consequentely, inequality (33) follows from the strong p-nondegeneracy assumption on w.
Let $w, \hat{w}$ satisfying the weak p-nondegeneracy property. As before we can assume that $\| w - \hat{w} \|_{L^\infty(I)} \leq \epsilon_0$. Then remarking that

$$(z(x) - \hat{z}(x))(w(x) - \hat{w}(x)) = 0 \text{ if } x \in A \cap \hat{A}$$

and that if $w(x) \neq -10$ (resp. $\hat{w}(x) \neq -10$) and $x \in \hat{A}$ (resp. $x \in A$) we have that

$$x \in \{x \in I : 0 < |w(x) + 10| \leq \epsilon\} \text{ (resp. } \{x \in I : 0 < |\hat{w}(x) + 10| \leq \epsilon\})$$

we obtain (34). ∎

Proof of Theorem 4. Let $\hat{u}$ be any other bounded weak solution of (P). Then, as in the proof of Theorem 2, using the monotonicity of R_e

$$\frac{d}{dt}\int_I |u(t)-\hat{u}(t)|^2dx+\int_I \rho(x)(|u_x(t)|^{p-2}u_x(t)-|\hat{u}_x(t)|^{p-2}\hat{u}_x(t))(u_x(t)-\hat{u}_x(t))dx$$

$$\leq Q\int_I (z(x,t)-\hat{z}(x,t))(u(x,t)-\hat{u}(x,t))dxdt$$

for some $z, \hat{z} \in L^\infty(I\times(0,T))$ with $z(x,t)\in\beta(u(x,t))$, $\hat{z}(x,t)\in\beta(\hat{u}(x,t))$ for a.e. $(x,t)\in I\times(0,T)$. Now assume $p>2$. Then by (24) we obtain that

$$\frac{d}{dt}\int_I |u(t)-\hat{u}(t)|^2dx+\parallel (u(t)-\hat{u}(t))_x \parallel^p_{L^p(I:\rho)}\leq$$

$$\leq Q\parallel z(t)-\hat{z}(t)\parallel_{L^1(I)}\parallel u(t)-\hat{u}(t)\parallel_{L^\infty(I)}.$$

From Theorem 4 of Rakotoson-Simon [1993] we have the estimate

$$\parallel v\parallel_{L^\infty(I)}\leq C_1\parallel v_x\parallel_{L^p(I:\rho)}+|I|_\rho^{-1}\parallel v\parallel_{L^1(I:\rho)},\quad \forall v\in V \tag{36}$$

where

$$C_1=|I|_p^{(p-2)/2p}C_0$$

with $C_0>0$ independent of I and

$$|I_\rho|=\int_{-1}^1\rho(x)dx=k\int_{-1}^1(1-x^2)dx=4k/3.$$

Then by Lemma 3 and using $(a+b)^p\leq 2^p(a^p+b^p)$ we get

$$Q\parallel z(t)-\hat{z}(t)\parallel_{L^1(I)}\parallel u(t)-\hat{u}(t)\parallel_{L^\infty(I)}-\parallel (u(t)-\hat{u}(t))_x\parallel^p_{L^p(I:\rho)}\leq$$

$$\leq\parallel u(t)-\hat{u}(t)\parallel^p_{L^\infty(I)}\left(QC(a_f-a_i)-\frac{1}{2^pC_1^p}\right)+(C_1|I|_\rho)^{-p}\parallel u(t)-\hat{u}(t)\parallel^p_{L^1(I:\rho)}\leq$$

$$\parallel u(t)-\hat{u}(t)\parallel^p_{L^\infty(I)}\left(QC(a_f-a_i)-\frac{1}{2^pC_1^p}\right)+C_2\parallel u(t)-\hat{u}(t)\parallel^2_{L^2(I)}$$

where

$$C_2=\frac{(\int_I\rho(x)^2dx)^{p/2}}{C_1^p(\int_I\rho(x)dx)^p}\parallel u-\hat{u}\parallel^{p-2}_{L^\infty((0,T):L^2(I))}\leq C_3$$

for some C_3 independent of u and $\hat{u}$ (that can be obtained from the estimates as (25) in terms of the data, $\| u_0 \|_{L^2(I)}$, $Q(a_f - a_i)$ and $\| R_e(x,t,\cdot) \|_{L^\infty(0,T:L^2(I))}$). Assume now that

$$QC(a_f - a_i) - \frac{1}{2^p C_1} \leq 0.$$

Then we conclude that

$$\frac{d}{dt} \| u(t) - \hat{u}(t) \|^2_{L^2(I)} \leq C_3 \| u(t) - \hat{u}(t) \|^2_{L^2(I)} . \tag{37}$$

Setting $U(t) = \| u(t) - \hat{u}(t) \|^2_{L^2(I)}$ we obtain that $U(t) \leq U(0)e^{C_3 t}$ but as $U(0) = 0$ we deduce that $u(t) = \hat{u}(t)$ for any $t \in [0,T]$. If (37) does not hold we introduce the rescaling $y = \alpha x$ with $\alpha > 0$. Given a function $h(x,t)$ we define $h(y,t)$ by $h(y,t) = h(\alpha x, t)$. Then the functions $u(y,t)$ and $\hat{u}(y,t)$ satisfy

$$\frac{\partial u}{\partial t} - \alpha^p(\rho_\alpha(y)|u_y|^{p-2}u_y)_y = Qz(y,t) - R_e(\frac{y}{\alpha}, t, u)$$
$$\frac{\partial \hat{u}}{\partial t} - \alpha^p(\rho_\alpha(y)|\hat{u}_y|^{p-2}\hat{u}_y)_y = Q\hat{z}(y,t) - R_e(\frac{y}{\alpha}, t, \hat{u})$$

in $(-\alpha, \alpha) \times (0,T)$, where

$$\rho_\alpha(y) = K\left(1 - \frac{y^2}{\alpha^2}\right).$$

Arguing as in the case $\alpha = 1$ we have

$$\frac{d}{dt} \| u(t) - \hat{u}(t) \|^2_{L^2(-\alpha,\alpha)} + \alpha^p \| (u(t) - \hat{u}(t))_y \|^p_{L^p((-\alpha,\alpha):\rho_\alpha)} \leq$$
$$\leq Q \| z(t) - \hat{z}(t) \|_{L^1(-\alpha,\alpha)} \| u(t) - \hat{u}(t) \|_{L^\infty(-\alpha,\alpha)} .$$

Estimate (36) remains true when one replaces I by $I_\alpha (= (-\alpha,\alpha))$ and ρ by ρ_α. So a simple computation leads to $|I_\alpha|_{\rho_\alpha} = \alpha |I|_\rho$ and thus

$$\| v \|_{L^\infty(-\alpha,\alpha)} \leq \alpha^{(p-2)/2p} C_1 \| v_y \|_{L^p((-\alpha,\alpha):\rho_\alpha)} + (\alpha|I|_\rho)^{-1} \| v \|_{L^1((-\alpha,\alpha):\rho_\alpha)} .$$

Then by Lemma 3

$$Q \| z(t) - \hat{z}(t) \|_{L^1(-\alpha,\alpha)} \| u(t) - \hat{u}(t) \|_{L^\infty(-\alpha,\alpha)} - \alpha^p \| (u(t) - \hat{u}(t))_y \|^p_{L^p((-\alpha,\alpha):\rho_\alpha)} \leq$$

$$\leq \| u(t)-\hat{u}(t) \|^p_{L^\infty((-\alpha,\alpha))} (QC(a_f-a_i)\alpha - \frac{\alpha^{p-(p-2)/2}}{2^p C_1^p}) +$$

$$+ C_4(\alpha) \| u(t)-\hat{u}(t) \|^2_{L^2(-\alpha,\alpha)} .$$

Taking α large enough we obtain that $U_\alpha(t) = \| u(t)-\hat{u}(t) \|$ satisfies $U_\alpha \leq U_\alpha(0)e^{C_4(\alpha)t}$ and so again $u(t)=\hat{u}(t)$ for any $t \in [0,T]$.
If $p=2$ the estimate (36) must be replaced by

$$\| v \|_{L^r(I,\rho)} \leq C_1 \| v_x \|_{L^p(I:\rho)} + |I|_\rho^{(1/r)-1} \| v \|_{L^1(I:\rho)} \tag{38}$$

for any $r \in [1,\infty)$ where

$$C_1 = |I|_\rho^{1/r} C_0$$

with $C_0 > 0$ independent of I (see Rakotoson-Simon [1993]). But as $u(t) - \hat{u}(t) \in L^\infty(I)$ we know that for any $\delta > 0$ there exists $n(\delta) > 0$ such that for any $r \in [n(\delta), +\infty)$

$$\Big| \| u(t)-\hat{u}(t) \|_{L^\infty(I)} - \| u(t)-\hat{u}(t) \|_{L^r(I:\rho)} \Big| \leq \delta \tag{39}$$

and so

$$\| u(t)-\hat{u}(t) \|^p_{L^\infty(I)} \leq 2^p \| u(t)-\hat{u}(t) \|^p_{L^r(I:\rho)} + 2^p\delta^p \leq$$

$$\leq 2^p C_1^p \| (u(t)-\hat{u}(t))_x \|^p_{L^p(I:\rho)} + 2^p |I|^{[(1/r)-1]p} \| u(t)-\hat{u}(t) \|_{L^1(I:\rho)} + 2^p\delta^p.$$

Arguing as in the case $p > 2$ we obtain

$$\frac{d}{dt} \| u(t)-\hat{u}(t) \|^2_{L^2(I)} \leq \| u(t)-\hat{u}(t) \|^p_{L^\infty(I)} (QC(a_f-a_i) - \frac{1}{2^p C_1^p})$$

$$+ C_3 |I|^{p/r} + 2^p \delta^p.$$

Making $\delta \downarrow 0$ as C_3 is independent of r we obtain (37) and the proof of *(i)* ends. Part *(ii)* is obtained in a similar way by using now *(ii)* of Lemma 3. ∎

To complete the study of the uniqueness of solutions of (P) we concentrate our attention on the nondegeneracy properties. The local character of those conditions is pointed in the next result.

Proposition 4 *.(i) Let $w \in C^0(I)$. Assume that the set $A = \{x \in I : w(x) = -10\}$ has a finite number of connected components and that there exists $\epsilon > 0$ and a positive constant K such that for any $\epsilon \in (0, \epsilon_0)$ and $x \in \hat{B}_\epsilon \equiv \{x \in I : 0 < |w(x) + 10| \leq \epsilon\}$*

$$|w(x) + 10| \geq K|x - x_i|^{1/(p-1)}, \quad \forall x_i \in \partial A. \tag{40}$$

Then w satisfies the weak p-nondegeneracy property. Furthermore, if $|A| = 0$ then w satisfies the strong p-nondegeneracy property.

(ii) Let $W^{1,\infty}_{loc}(I)$ and assume that A has a finite number of connected components and that there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ $\exists \delta = \delta(\epsilon)$ such that

$$|w_x(x)| \geq \delta \text{ a.e. } x \in \{x \in I : |w(x) + 10| \leq \epsilon\} \tag{41}$$

then w satisfies the strong 2-nondegeneracy property.

Proof. From (40) we deduce that if $x \in \hat{B}_\epsilon$ then $|x - x_i| \leq \epsilon^{p-1}/K$. Thus $|B_\epsilon| \leq (N/K)\epsilon^{p-1}$ where N is the number of points of ∂A.
(ii) It is clear that (41) implies that meas $|A| = 0$. Let $[a, b] \subset \overline{I}$ a connected component of $B_\epsilon = \{x \in I : |w(x) + 10| \leq \epsilon\}$. Assume that $w_x(x) \geq \delta$ on (a, b) [the other case $w_x(x) \leq -\delta$ on (a, b) is treated in a similar way]. Then $w(a) = -10 - \epsilon$, $w(b) = -10 + \epsilon$ and there exists $x_0 \in (a, b)$ such that $w(x_0) = -10$. Then for any $x \in [x_0, b]$ we have

$$\epsilon \geq w(x) + 10 = \int_{x_0}^{x} w_x(s)ds \geq \delta(x - x_0).$$

Analogously, for any $x \in [a, x_0)$,

$$\epsilon \geq -10 - w(x) = \int_{x}^{x_0} w_x(s)ds \geq \delta(x_0 - x)$$

and thus (40), with $p = 2$, holds. ∎

Remark 4. The nondegeneracy properties of the solutions of (P) can be obtained under some additional assumptions on the initial datum. Let $u_0 \in C^1(\overline{I})$ such that

$$A_0 = \{x \in I : U_0(x) = -10\} \text{ has a finite number of connected components,} \tag{42}$$

and

$$\left.\begin{array}{l} \exists\epsilon_0 > 0 \text{ and } K > 0 \text{ such that } \forall\epsilon \in (0,\epsilon_0) \text{and any} \\ x \in \hat{B}_{\epsilon,0} = \{x \in I : 0 < |u_0(x) + 10| \leq \epsilon\} \\ \text{we have } |u_0(x) + 10| > K|x - x_i|^{1/(p-1)} \quad \forall x \in \partial A \end{array}\right\} \tag{43}$$

Then there exists a $T_u \in (0, T]$ such that $u(t)$ satisfies the weak non-degeneracy property for any $t \in [0, T_u)$ where u is any continuous weak solution of (P). In particular if u and $\hat{u}$ are continuous weak solutions of (P) there exists a $T^* \in (0, T]$ such that $u = \hat{u}$ on $[0, T^*) \times I$. Indeed; let $u, \hat{u}$ be continuous bounded weak solutions of (P), by the continuity near $t = 0$ we deduce that there exist T_u, $T_{\hat{u}} \in (0, T]$ such that $u(t)$, $\hat{u}(t)$ satisfy (40) and that the set where they take the value -10 has the same (finite) nomber of connected components for any $t \in [0, T_u)$, $[0, T_{\hat{u}})$ respectively. Taking $T^* = \min\{T_u, T_{\hat{u}}\}$ the conclusion follows from part (ii) of Theorem 4.

Remark 5. Let $u_0 \in C^1(\bar{I})$ such that u_0 is an even function, $u_{0x}(x) > 0$ for any $x \in (-1, 0)$, $u_0(0) > -10$, $u(-1) < -10$. Then (42) and (43) holds for $p = 2$. Moreover, if u is the solution built in the section *3.2* for $p = 2$ then $u(t)$ satisfies the strong 2-nondegeneracy property for any $t \in [0, T]$. Finally, if $p = 2$ problem (P) has a unique bounded solution on $[0, T] \times I$. Indeed; it is an easy modification of Lemma 6.2 and Corollary 6.3 of Feireisl-Norbury [1991].

Remark 6. It should be interesting to know if the techniques on non-degeneracy properties for the parabolic obstacle problem (see,e.g., Pietra-Verdi [1985]) can be applied to obtain the p-nondegeneracy properties for the solutions of (P).

5. On the free boundary and the mushy region.

The discontinuity of the albedo function assumed in the Budyko model generates a natural *free boundary* or interface $\zeta(t)$ between the ice-covered area ($\{x \in I : u(x, t) < -10\}$) and the ice-free area ($\{x \in I : u(x, t) > -10\}$). The free boundary is then given as $\zeta(t) = \{x \in I : u(x, t) = -10\}$. In Xu [1991] the Budyko model for $p = 2$ is considered. He shows that if the initial datum u_0 satisfies

$$\begin{array}{l} u_0(x) = u_0(-x),\ u_0 \in C^3([-1, 1]),\ u_0'(x) < 0 \text{ for any } x \in (0, 1) \\ \text{and there exists } \zeta(0) \in (0, 1) \text{ such that } (u_0(x) + 10)(x - \zeta(0)) < 0 \\ \text{for any } x \in [0, \zeta(0)) \cup (\zeta(0), 1], \end{array}$$

then there exists a bounded weak solution u of (P) for which the set $\zeta(t) = \{\zeta_+(t)\} \cup \{\zeta_-(t)\}$ with $x = \zeta_+(t)$ a smooth curve, $\zeta_-(t) = \zeta_+(t)$ and $\zeta_+(\cdot) \in C^\infty([0, T^*))$ where T^* is characterized as the first time t for which $\zeta_+(t) = 1$. He also gives an expression for the derivative $\zeta'_+(t)$ (some related results for a model corresonding to $\rho(x) = 1$ can be found in Feireisl-Norbury [1991]). We point out that the uniqueness result (Theorem 4) can be applied for such an initial datum (see Remark 4).

The size of the separating zone $\zeta(t)$ for other models is in fact a controversial question. So, some satellite pictures (Image of the Weddell sea taken by the satellite Spot on December 10, 1987: Lions [1991]) show that the separating region between the ice-free and the ice-covered zones is not a simple line on the Earth (i.e. a point in $(-1, 0)$ or $(0, 1)$) but a narrow zone where ice and water are mixed. Mathematically it corresponds to say that the set

$$M(t) = \{x \in I : u(x,t) = -10\}$$

is a positively measured set. In the following we shall denote this set as the *mushy region* (since it plays the same role than in changing phase problems, see e.g. Díaz-Fasano-Meirmanov [1992]).

Using the strong maximum principle (see e.g. Vazquez [1984]) it is possible to show that if $p = 2$ (or more in general if $1 < p \leq 2$) the interior set of the mushy region $M(t)$ is empty even if the interior of $M(0)$ is a nonempty open set. The main goal of the next result is to show that this is not the case when $p > 2$ (as it happens for the Held-Suarez model : $p = 3$). A necessary condition for $\overset{\circ}{M}(t) \neq \emptyset$ is that $R_a(x, t, -10) - R_e(x, t, -10) \ni 0$ for any $x \in \overset{\circ}{M}(t)$ and $t \in [0, T]$. In the case of the Budyko model R_a is defined by (2) and (4), R_e by (5) and the necessary condition can be written in the following terms

$$A - 10B \in [a_i Q(x,t), a_f Q(x,t)] \text{ for a.e. } x \in I, \text{ a.e. } t \in [0, T] \qquad (44)$$

We shall show that if $p > 2$ this condition is also sufficient.

Theorem 5 . *Let $p > 2$, R_a given by (2) and (4) and R_e given by (5). Assume (44) and $u_0 \in L^\infty(I)$ such that there exist $x_0 \in I$ and $R_0 > 0$ satisfying*

$$M(0) = \{x \in I : u_0(x) = -10\} \supset B(x_0, R_0) (= \{x \in I : |x - x_0| < R_0\}).$$

If u is the bounded weak solution of (P) satisfying the weak p-nondegeneracy property then there exists $T^* \in (0, T]$ and a nonincreasing function $R(t)$ with

$R(0) = R_0$ *such that*

$$M(t) = \{x \in I : u(x,t) = -10\} \supset B(x_0, R(t))$$

for any $t \in [0, T^*)$.

Proof. We shall use an energy method as developped in Díaz-Veron [1985]. Given u bounded weak solution of (P) we define $v = u + 10$. As in Lemma 3.1 of the above reference multiplying the partial differential equation by v we obtain that for a.e. $R \in (0, R_0)$ and $t \in (0,T)$ we have

$$\frac{1}{2}\int_{B(x_0,R)} |v(x,t)|^2 dx + \int_0^t \int_{B(x_0,R)} \rho(x)|v_x|^p dx d\tau + B\int_0^t \int_{B(x_0,R)} |v(x,\tau)|^2 dx d\tau \leq$$

$$\leq \int_0^t \int_{S(x_0,R)} \rho(x)|v_x|^{p-2} v_x \cdot \vec{n} v ds d\tau + \int_0^t \int_{B(x_0,R)} \{Q(x,\tau)z(x,\tau) - A + 10B\} v dx d\tau =$$

$$= I_1 + I_2. \tag{45}$$

where $S(x_0,R) = \partial B(x_0,R) = \{x_0 - R\} \cup \{x_0 + R\}$ and $z(x,t) \in \beta(u(x,t))$ for a.e. $x \in B(x_0,R)$ and $t \in (0,T]$. We introduce the energy functions

$$E(R,t) = \int_0^t \int_{B(x_0,R)} \rho(x)|v_x|^p dx d\tau$$

$$b(R,t) = \sup_{0 \leq \tau \leq t} \text{ ess} \int_{B(x_0,R)} |v(x,\tau)|^2 dx.$$

Using Holder's inequality and the interpolation-trace Lemma of Díaz-Veron [1985] (since $p > 2$) we get

$$I_1 \leq \left(\frac{\partial E}{\partial R}(R,t)\right)^{(p-1)/p} \left(\int_0^t \int_{S(x_0,R)} |v|^p dx d\tau\right)^{1/p} \leq$$

$$\leq Ct^{(1-\theta)/p} \left(\frac{\partial E}{\partial R}(R,t)\right)^{(p-1)/p} \left(E(R,t)^{1/p} + R^{\delta} t^{1/p} b(R,t)^{1/2}\right)^{\theta} b(R,t)^{(1-\theta)/2},$$

where

$$\theta = p/(3p-2) \text{ and } \delta = -(3p-2)/2p.$$

Using the assumption (44) we have that

$$\hat{z}(\cdot) = [(A - 10B)/Q(\cdot, t)] \in \beta(-10). \tag{46}$$

Then applying Lemma 3 to $w(\cdot) = u(\cdot,t)$, $z(\cdot) = B(\cdot,t)$, $\hat{w}(\cdot) = -10$ and $\hat{z}(\cdot)$ given by (46) we get that

$$I_2 \leq (a_f - a_i) \parallel Q \parallel_{L^\infty(I\times(0,T))} C \int_0^t \parallel v(\tau) \parallel^p_{L^\infty(B(x_0,R))} d\tau.$$

Using the inequality (36) on $B(x_0, R)$ we obtain

$$I_2 \leq (a_f - a_i) \parallel Q \parallel_{L^\infty(I\times(0,T))} C(C_1 E(R,t) + tC_2(R)b(R,t)),$$

where now

$$C_2(R) = \frac{\left(\int_{B(x_0,R)} \rho(x)^2 dx\right)^{p-2}}{C_1^p \left(\int_{B(x_0,R)} \rho(x) dx\right)^p} \parallel u + 10 \parallel^p_{L^\infty((0,T):L^2(I))} \cdot$$

As in the proof of Theorem 4, without loss of generality we can assume C_1 small enough. Then, there exists $T^* \in (0,T]$ and $\lambda \in (0,1]$ such that

$$\lambda(E(R,t) + b(R,t)) \leq I_1$$

which implies that

$$\lambda E^\mu \leq t^{(1-\theta)/p} \frac{\partial E}{\partial R}$$

for some $\mu \in (0,1)$ and for any $t \in [0, T^*)$ and the proof ends as in Díaz-Veron [1985] (proof of Theorem 3.1). ∎

Remark 7. The existence of the mushy region (for any value of $p \in (1,\infty)$) can be proved for a different class of models by taking into account a discontinuous diffusivity (see Held-Linder-Suarez [1981]). In that case the problem is a variant of the Stefan problem (see, e.g., Díaz-Fasano-Meirmanov [1992]). We also point out that if we define the mushy region associated to a temperature u_c, with $u_c \neq -10$, by

$$M(t : u_c) = \{x \in I : u(x,t) = u_c\},$$

then the results of Díaz-Veron [1985] and Antonsev-Díaz [1989] allows to obtain the same type of conclusions than in Theorem 5 (but without the non-degeneracy assumption on the solution) for suitable functions $Q(x,t)$.

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