ENVIRONMENT, ECONOMICS AND THEIR MATHEMATICAL MODELS

Editors

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ON THE CONTROLLABILITY OF SOME SIMPLE CLIMATE MODELS

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1 Introduction.

Originated by the works by Budyko [6] and Sellers [31] in 1969, several energy balance models have been extensively studied in the literature as a tool for assesing qualitatively the impact of the ice-albedo feedback on the climate, i.e. on the description of the evolution of a long term average of the Earth surface temperature.

A basic two-dimensional model can be formulated in terms of the following reaction-diffusion equation

$$c(t,x)u_t(t.x) - \operatorname{div}(k(t,x)\operatorname{grad}u(t,x)) = R_a(t,x,u(t.x)) - R_e(t,x,u(t,x)) \tag{1}$$

on the Euclidean two-sphere S^2 of $\mathbb{R}^3$ or, more generally, on a compact oriented two-dimensional manifold $\mathcal{M}$ without boundary. So div and grad must be understood with respect to the given Riemannian metric. The heat capacity and the diffusion coefficient, c and k, are strictly positive functions. The right hand side of (1) stands for the mean radiation flux balance: $R_a(t,x,u)$ represents the fraction of the solar energy absorbed by the Earth and $R_e(t,x,u)$ the energy emmitted by the Earth to the outer space.

It is well accepted that $R_a(t,x,u)$ obeys to the expression

$$R_a(t,x,u) = Q(t,x)\beta(u) \tag{2}$$

where $Q(t,x)$ is a positive number representing the mean incoming solar radiation flux (the solar constant) and $\beta(u) = (1 - \alpha(u))$ is the co-albedo function obtained from the planetary albedo $\alpha(u)$ as function of the temperature. Usually $\beta(u)$ is assumed to be increasing (or more precisely non-decreasing), $\beta(u) \in [0,1]$ for any value of u and $\beta(u)$ takes a constant value $a_i \in (0,1)$ (the ice co-albedo) for small values of u, and $a_f \in (a_i,1)$ for large values of

u. The shape of β in the transition zone is a controversial question: Budyko assumes β discontinuous

$$\beta(u) = \begin{cases} a_i & \text{if } u < -10 \\ a_f & \text{if } u > -10 \end{cases} \tag{3}$$

(the transition temperature is customarily taken as $u = -10^0 C$), and Sellers assumes that β is a regular function (at least Lipschitz continuous) such as, for instance,

$$\beta(u) = a_i + \frac{1}{2}(a_f - a_i)(1 + \tanh \gamma u) \tag{4}$$

for some $\gamma \in (0, 1)$.

The mean emmitted energy flux $R_e(t, x, u)$ is determinated empirically. It is understood that R_e depends on the amount of greenhouse gases, clouds and water vapor in the atmosphere. It seems natural to assume that R_e increases with u but the increasing rate is also controversial: Budyko proposes a Newton linear type radiation ansatz

$$R_e(t, x, u) = A(t, x) + B(t, x)u \tag{5}$$

and Sellers uses a Stefan-Boltzman type law

$$R_e(t, x, u) = \sigma(t, x)(u - \tau(t, x))^4 \tag{6}$$

(in (6) u is given in Kelvin and σ and τ take positive values).

Many other energy balance models, more sophisticated that (1), have been introduced in the literature by taking also into account, for instance, the atmospheric temperature, the humidity, etc. Nevertheless, it is clear that if a simple energy balance model as (1) is embellished by adding many different factors it loses its diagnostic simplicity and may become as complicated as short-term prediction models (as, for instance, the General Circulation Model).

The equation (1) has been treated from a mathematical point of view by different authors under some assumptions of different type on R_a and R_e: see, e.g. North, Mengel and Short, [30], Hertzer [22], Hertzer and Schmidt [23] and Díaz and Tello [16] for the two-dimensional model and Xu [33] and Díaz [10] for the one-dimensional model (arising when assuming that $u(t, \cdot)$ depends only on the latitude). We point out that one of the main reasons of the interest of this simple energy balance models is that it provides clear answers to the study of climate sensitivity concerning various parameters such as, for instance, the solar constant: we send the reader to the surveys Gill and Childress [20], North [29], Stakgold [32] and Díaz [11]

The main goal of this work is to study the possible human interaction on the climate from the point of view of the control theory. Roughly speaking we are interested in knowing if it is possible to act on the system in a such manner that the climate behaves (in some sense) according to *our wishes*, carrying the temperature from a given distribution $u(0, x)$ to a wished distribution $u(T, x)$, after a given period of years T. The study of such a question seems to go back to J.Fourier [19], attracted the attention of distinguished scientists (J. Von Neumann [28]) and is being currently under consideration (J.L.Lions [25], [27]). It is clear that the *world decissions* (or considerations) on greenhouse gases emmision norms follow this philosophy, trying to control the changes on the present climate. We also mention here the pionnering works by I.Langmuir (General Electric Laboratory, New York, 1946), in different scales of time and space than in the problem we are dealing with, on *cloud seeding*, showing the possibility of originating watter cristals and clouds by seeding some chemical substances as, for instance, AgI (see Dennis [7]). We will not go into this issue, but it shows the possibility for humans to interact with the climate.

To avoid tedious technical details we shall develop our study for a model which simplifies the differential equation (1) and also replace the two-dimensional sphere by an open bounded regular set Ω of $\mathbb{R}^N$, $N \geq 1$, and adding the Neumann boundary conditions. The treatment of possible discontinuous co-albedo functions $\beta(u)$ will be made in the framework of maximal monotone graphs of $\mathbb{R}^2$ (see, e.g. Brezis [5]). So, in the rest of the paper we shall concentrate our attention on the problem

$$\mathcal{P}(f, \beta) \begin{cases} y_t - \Delta y + f(y) \in Q\beta(y) + v\chi_\omega & \text{in } (0, T) \times \Omega \\ \dfrac{\partial y}{\partial \bar{n}} = 0 & \text{on } (0, T) \times \partial\Omega \\ y(0, \cdot) = y_0(\cdot) & \text{on } \Omega \end{cases}$$

where ω is an open regular subset of Ω, with $\omega \subset\subset \Omega$, which represents the spatial region where the actions will take place. We assume

$f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing locally Lipschitz continuous function, (7)

$Q > 0$ and

β is a maximal monotone graph of $\mathbb{R}^2$ such that $D(\beta) = \mathbb{R}$ and $0 \leq b \leq 1$ for any $b \in \beta(r)$ and for any $r \in \mathbb{R}$. (8)

In this simplified formulation

$$R_a(t, x, u) = Q\beta(u)$$
$$R_e(t, x, u) = -v(t, x)\chi_\omega + f(u).$$

The existence of solutions of $\mathcal{P}(f, \beta)$ assuming $y_0 \in L^2(\Omega)$ and $v \in L^2((0, T) \times \omega)$ is almost standard except for the presence of the eventually multivalued term $\beta(y)$ in the right-hand side of the equation. When β is a Lipschitz continuous function, the uniqueness of solutions is also a routine matter.

Nevertheless, if $\beta(y)$ is a multivalued graph then there is not, in general, uniqueness of solutions and suitable conditions must be supposed on y_0 in order to get a unique solution (see Díaz and Tello [16] and Díaz [10]).

The controllability question can be formulated as follows: given $y_d \in L^2(\Omega)$ is it possible to find $v \in L^2((0,T) \times \omega)$ such that if we denote by $y(t,x;v)$ to the solution of $\mathcal{P}(f,\beta)$ then $y(T,\cdot;v) \equiv y_d(\cdot)$?. When the answer is positive we say that problem $\mathcal{P}(f,\beta)$ is *controllable*. Due to the smoothing properties of parabolic partial differential equations we can not expect to verify this property in the above meaning. A relaxed version is the notion of *approximate controllability*: given $y_d \in L^2(\Omega)$ and $\varepsilon > 0$, is it possible to find $v_\varepsilon \in L^2((0,T) \times \omega)$ such that $\|y(T,\cdot;v_\varepsilon) - y_d(\cdot)\|_{L^2(\Omega)} \leq \varepsilon$?.

The main goal of this work is to show the difference appearing in the study of the approximate controllability property for the Budyko and Sellers type models: the answer is positive for the Budyko case and negative (at least for a suitable class of wanted states y_d) for the Sellers model. An analysis on the understanding of this dichotomy is also presented.

2 The Budyko type model.

As we shall see, the behaviour at infinity of the function $f(y)$ is the relevant difference among the Budyko and Sellers models. A class of functions f which include the choice of Newton linear emmision radiation can be defined in the following terms:

$$\left. \begin{array}{l} f \text{ satisfies (7) and there exist some positive constants} \\ C_1, C_2 \text{ and M such that } |f(s)| \leq C_1 + C_2|s| \text{ if } |s| > \text{M} \end{array} \right\} \quad (9)$$

The following result give a positive answer to the controllability of this class of problems. It follows closely the results of Fabré, Puel and Zuazua [17], [18] and Díaz and Ramos [14] (stated for Dirichlet boundary conditions and the special case $\beta \equiv 0$).

Theorem 1
Assume (8) and (9). Then the problem $\mathcal{P}(f,\beta)$ is approximately controllable.

As usual in the study of the controllability of nonlinear problems (see, e.g. Henry [21], Fabré, Puel and Zuazua [17], [18]) the proof of Theorem 1 will be obtained through the application of a fixed point theorem for an operator defined in terms of the control associated to a linear problem. More precisely, let $s_0 \in \mathbb{R}$ such that f is globally Lipschitz, with constant $K > 0$ on $(s_0 - \delta, s_0 + \delta)$ (such a s_0 exists by (7)). Define

$$g(s) = \left\{ \begin{array}{ll} \dfrac{f(s) - f(s_0)}{s - s_0} & \text{if } s \neq s_0 \\ K & \text{if } s = s_0. \end{array} \right.$$

Given $z \in L^2((0,T) \times \Omega)$ we start by considering the linear control problem

$$(\text{LP}) \left\{ \begin{array}{ll} y_t - \Delta y + g(z)y = v\chi_\omega & \text{in } (0,T) \times \Omega \\ \dfrac{\partial y}{\partial \vec{n}} = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0,\cdot) = y_0(\cdot) & \text{on } \Omega. \end{array} \right.$$

The approximate controllability for the problem (LP) can be proved by using the Hahn-Banach theorem and a unique continuation theorem (see Lions [24] and Díaz, Henry and Ramos [13] for the case of nonnegative controls). Here we shall borrow another approach due to Fabré, Puel and Zuazua [17], [18] which allows to obtain some additional information on the construction of the controls.

Theorem 2 ([17],[18])
Let $a \in L^\infty((0,T) \times \Omega)$, $\varepsilon > 0$ and $\hat{y}_d \in L^2(\Omega)$, with $\|\hat{y}\|_{L^2(\Omega)} > \varepsilon$. Given $\varrho_0 \in L^2(\Omega)$ we consider the problem

$$\left. \begin{array}{ll} -\varrho_t - \Delta\varrho + a\varrho = 0 & \text{in } (0,T) \times \Omega \\ \dfrac{\partial\varrho}{\partial\vec{n}} = 0 & \text{on } (0,T) \times \partial\Omega \\ \varrho(T,\cdot) = \varrho_0(\cdot) & \text{on } \Omega. \end{array} \right\} \quad (10)$$

Define the functional

$$\mathcal{J}(\varrho_0, \hat{y}_d, \varepsilon) = \frac{1}{2} \left(\int_{(0,T) \times \omega} |\varrho| dt dx \right)^2 + \varepsilon\|\varrho_0\|_{L^2(\Omega)} - \int_\Omega \hat{y}_d\varrho_0 dx. \quad (11)$$

Then:

i) $\mathcal{J}(\varrho_0, \hat{y}_d, \varepsilon)$ is a strictly convex continuous and coercive functional in $L^2(\Omega)$. In particular, it achieves its minimum at a unique point $\hat{\varrho}_0 \in L^2(\Omega)$.

ii) If $\hat{\varrho}$ denotes the solution of (10) for $\varrho_0 = \hat{\varrho}_0$ there exists $w \in sign(\hat{\varrho})\chi_\omega$ such that the solution of

$$\left. \begin{array}{ll} y_t - \Delta y + ay = |\hat{\varrho}|_{L^1((0,T)\times\Omega)}w\chi_\omega & \text{in } (0,T) \times \Omega \\ \dfrac{\partial y}{\partial\vec{n}} = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0,\cdot) = 0 & \text{on } \Omega. \end{array} \right\} \quad (12)$$

satisfies that

$$\|y(T,\cdot) - \hat{y}_d\|_{L^2(\Omega)} \leq \varepsilon. \quad \Box \quad (13)$$

We point out that although the conclusions of Theorem 2 were obtained in [17], [18] for zero Dirichlet boundary conditions, obvious modifications lead to the proof for the case of Neumann boundary conditions.

Proof of Theorem 1.

Given $z \in L^2((0,T) \times \Omega)$ we define by u the (unique) solution of the problem

$$
\left.
\begin{aligned}
&u_t - \Delta u + g(z)u = -f(s_0) + g(z)s_0 + Q\beta_0(z) &&\text{in } (0,T) \times \Omega \\
&\frac{\partial u}{\partial \vec{n}} = 0 &&\text{on } (0,T) \times \partial\Omega \\
&u(0,\cdot) = y_0(\cdot) &&\text{on } \Omega,
\end{aligned}
\right\}
\quad (14)
$$

where $\beta_0 : \mathbb{R} \to \mathbb{R}$ is, for instance, the so-called *main section* of the graph β. It is defined by

$$
\beta_0(r) = b_0 \quad \text{if} \quad |b_0| \le |b| \quad \forall b \in \beta(r). \quad (15)
$$

From assumptions (7), (8) and (9) we know that $g(z)$ and $\beta_0(z)$ are bounded functions and so $u \in \mathcal{C}([0,T] : L^2(\Omega))$. Applying Theorem 2 with $a = g(z)$ and $\widehat{y}_d := y_d - u(T, \cdot)$, given $\varepsilon > 0$ we know that the functional $\mathcal{J}(\varrho_0, \widehat{y}_d, \varepsilon)$ possees a unique minimum $\widehat{\varrho}_0 \in L^2(\Omega)$ (notice that this $\widehat{\varrho}_0$ and the solution $\widehat{\varrho}$ of (10) depend on z, y_d and y_0) and that there exists $w \in \text{sign}(\widehat{\varrho})\chi_\omega$ (and so w also depends on z) such that the solution $\mathcal{Y}$ of (12) satisfies

$$
\|\mathcal{Y}(T,\cdot) + u(T,\cdot) - y_d(\cdot)\|_{L^2(\Omega)} \le \varepsilon. \quad (16)
$$

We then deduce that the function $y := \mathcal{Y} + u$ satisfies

$$
\left.
\begin{aligned}
&y_t - \Delta y + g(z)y = F + |\widehat{\varrho}|_{L^1((0,T)\times\Omega)} v\chi_\omega &&\text{in } (0,T) \times \Omega \\
&\frac{\partial y}{\partial \vec{n}} = 0 &&\text{on } (0,T) \times \partial\Omega \\
&y(T,\cdot) = y_0(\cdot) &&\text{on } \Omega \\
&\|y(T,\cdot) - y_d(\cdot)\|_{L^2(\Omega)} \le \varepsilon,
\end{aligned}
\right\}
\quad (17)
$$

for

$$
F = -f(s_0) + g(z)s_0 + Qb,
$$

$v = w(z)$ and $b = \beta_0(z)$. Now it becomes clear that the conclusion is reduced to show the existence of a fixed point for the operator

$$
\Lambda : L^2((0,T) \times \Omega) \to \mathcal{P}(L^2((0,T) \times \Omega))
$$

$$
z \mapsto \Lambda(z) = \{y \text{ satisfying (17) for some } v \in \text{sign}(\widehat{\varrho})\chi_\omega \text{ and some } b \in \beta(z)\}
$$

The multivalued character of Λ comes from the non-uniqueness of the control w in the problem (12) and the multivalued nature of $\text{sign}(\cdot)$ and β. On the

other hand, we know that $\Lambda(z)$ contains the point $\mathcal{Y} + u$ and so it is always a non-empty subset of $L^2((0,T) \times \Omega)$. In order to apply the Kakutani fixed point theorem (in the weak form presented in Aubin [1]) we need to check that

1. $\forall z \in L^2((0,T) \times \Omega)$, the set $\Lambda(z)$ is non-empty convex and compact in $L^2((0,T) \times \Omega)$,

2. Λ is upper hemicontinuous on $L^2((0,T) \times \Omega)$ (see the definition below).

The proof of both properties can be obtained by adapting the arguments of [17], [18] to our framework. As many of those adaptations are routine matter we shall indicate only those concerning the presence of the term $\beta(z)$. The convexity of $\overline{\Lambda}(z)$ is consequence of the linearity of the problem (17) and the convexity of the sets $\beta(z)$ and $\{v \in L^2((0,T) \times \Omega) : v \in \text{sign}(\widehat{\varrho})\chi_\omega\}$. The boundedness of β (see (8) and Proposition 2.2 of [18] lead to the existence of a compact subset $\mathcal{X} \subset L^2(\Omega)$ such that $\Lambda(z) \subset \mathcal{X}$ for any $z \in L^2((0,T) \times \Omega)$. Thus $\Lambda(z)$ is compact if it is a closed set. Let (y_n) be a sequence in $\Lambda(z)$ which converges in $L^2((0,T) \times \Omega)$ to $y \in \mathcal{X}$. Then there exists some functions $v_n \in \text{sign}(\widehat{\varrho})$ and $b_n \in \beta(z)$ such that

$$
\left.
\begin{aligned}
&y_{n,t} - \Delta y_n + g(z)y_n = F_n + |\widehat{\varrho}|_{L^1((0,T)\times\Omega)} v_n\chi_\omega &&\text{in } (0,T) \times \Omega \\
&\frac{\partial y_n}{\partial \vec{n}} = 0 &&\text{on } (0,T) \times \partial\Omega \\
&y_n(0,\cdot) = y_0(\cdot) &&\text{on } \Omega \\
&\|y_n(T,\cdot) - y_d(\cdot)\|_{L^2(\Omega)} \le \varepsilon,
\end{aligned}
\right\}
$$

where

$$
F_n = -f(s_0) + g(z)s_0 + Qb_n.
$$

Using that $\text{sign}(\cdot)$ and β are bounded maximal monotone graphs we deduce that $v_n \rightharpoonup v$ and $b_n \rightharpoonup b$ in the weak*-topology of $L^\infty((0,T) \times \Omega)$ and since any maximal monotone graph is strongly-weakly closed (see Brezis [5, p. 27]) we have that $v \in \text{sign}(\widehat{\varrho})$ and $b \in \beta(z)$. Using the compactness of the Green function associated to the problem (17) and the smoothing effect we get that $y_n \to y$ in $\mathcal{C}([0,T] : L^2(\Omega))$ with y solving (17), which proves $y \in \Lambda(z)$.

Let us indicate the proof of the rest. We first recall that Λ is *upper hemicontinuous* at $z_0 \in L^2((0,T) \times \Omega)$ if

$$
\limsup_{z_n \to z_0} \sigma(\Lambda(z_n), k) \le \sigma(\Lambda(z_0), k) \quad \text{for any } k \in L^2((0,T) \times \Omega),
$$

where

$$
\sigma(\Lambda(z), k) := \sup_{y \in \Lambda(z)} \int_{(0,T)\times\Omega} k(t,x)y(t,x)\,dx\,dt.
$$

From the compactness of $\Lambda(z)$, for any $n \in \mathbb{N}$ there exists $y_n \in \Lambda(z_n)$ such that

$$
\sigma(\Lambda(z_n), k) = \int_{(0,T)\times\Omega} k(t,x)y_n(t,x)\,dx\,dt.
$$

As $\Lambda(z) \subset \mathcal{X}$, the compactness of $\mathcal{X}$ implies the existence of $y \in \mathcal{X}$ such that $y_n \to y$ in $L^2((0,T) \times \Omega)$. If we denote $\hat{\varrho}_n = \hat{\varrho}_n(z_n)$, we know the existence of $v_n \in \text{sign}(\hat{\varrho}_n)$ and $b_n \in \beta(z_n)$ such that

$$
\left.
\begin{aligned}
y_{n,t} - \Delta y_n + g(z_n)y_n &= \hat{F}_n + |\hat{\varrho}_n|_{L^2((0,T) \times \Omega)} v_n \chi_\omega & &\text{in } (0,T) \times \Omega \\
\frac{\partial y_n}{\partial \vec{n}} &= 0 & &\text{on } (0,T) \times \partial\Omega \\
y_n(0, \cdot) &= y_0(\cdot) & &\text{on } \Omega \\
\|y_n(T, \cdot) - y_d(\cdot)\|_{L^2(\Omega)} &\leq \varepsilon.
\end{aligned}
\right\}
$$

with

$$\hat{F}_n = -f(s_0) + g(z_n) + Qb_n.$$

By Lemma 3.1 of [18] we have $z_n \to z$ (and so $\hat{\varrho}_n \to \hat{\varrho}$) strongly in $L^2((0,T) \times \Omega)$. Using again that $\text{sign}(\cdot)$ and β are bounded strongly-weakly closed graphs we deduce that y satisfies (17), with $z = z_0$, for some $v \in \text{sign}(\hat{\varrho})$ and some $b \in \beta(z_0)$. This proves that $y \in \Lambda(z_0)$. □

Remark 1

Theorem 1 can be improved in several directions. First of all, as in [17], [18] it is possible to obtain the approximate controllability for the model $\mathcal{P}(f, \beta)$ on the spaces $L^p(\Omega)$, $1 \leq p < \infty$. Second, the boundedness assumption (8) on β seems to be generalizable to a linear growth condition

$$
\left.
\begin{aligned}
&\text{there exists } C_1 \text{ and } C_2 \text{ two positive constants such that} \\
&|b| \leq C_1 + C_2|r|, \quad \forall b \in \beta(r), \quad \forall r \in \mathbb{R}.
\end{aligned}
\right\} \tag{18}
$$

Finally, the Laplace operator can be replaced by a self-adjoint second order uniformly elliptic operator (for instance modelling the case of a nonisotropic conductivity coefficient $k(x)$ in (1)). The case in which $f = f(t, x, u)$ and $\beta = \beta(t, x, u)$ can be also treated analogously if the time-dependence is smooth enough. □

Remark 2

It is easy to see that the proof of Theorem 1 can be also applied to the case in which the multivalued maximal monotone graph appear in the left-hand side of the equation, *i.e.*

$$y_t - \Delta y + f(t, x, y) + Q\beta(y) \ni v(t,x)\chi_\omega \tag{19}$$

and β satisfies (8) (or more generally (18)). This class of problems appears in other different contexts. It seems interesting to point out that, again, the behaviour of $\beta(y)$ when it becomes infinity is a crucial point. Indeed, in Díaz [9] it was shown that the *obstacle problem*, which corresponds to the equation (19) with $f(t, x, u) \equiv f(t, x)$ and

$$
\beta(r) = \begin{cases}
\{0\}, & \text{if } r \geq 0 \\
(-\infty, 0], & \text{if } r = 0 \\
\emptyset, & \text{if } r < 0
\end{cases}
$$

is not always approximately controllable. □

3　The Sellers type model.

As it was pointed out in the Introduction, Sellers assume a nonlinear emmision radiation flux bassed on the Stefan-Boltzmann law (see (6)). In that case, assumption (9) is not satisfied and which turns out to be relevant (for the study of the approximate controllability property) is the superlinear character of $R_e(\cdot, \cdot, u)$ as $|u| \to \infty$. A representative class of functions f is given now by the condition

$$
\left.
\begin{aligned}
&f \text{ satisfies (7) } f(0) = 0, \ f'(s) > 0 \text{ for any } s \neq 0 \text{ and} \\
&\text{there exists } p > 1 \text{ and some constants } C_1, C_2, M > 0 \text{ such that} \\
&|f(s)| \geq C_1 + C_2|s|^p \text{ for any } s \in \mathbb{R} \text{ with } |s| > M.
\end{aligned}
\right\} \tag{20}
$$

The main goal of this section is to show that (20), and more precisely the condition $p > 1$, leads to an *obstruction phenomenon* on the solutions of $\mathcal{P}(f, \beta)$: there is a *universal bound* function $\mathcal{Y}_\infty(t, x)$ (independently of the control v) which plays the role of an obstacle on $|y(t, x; v)|$ for any control v and any solution y of $\mathcal{P}(f, \beta)$. Such a property was already observed in Díaz [9], [12] and Diaz and Ramos [14] for other semilinear problems: see in Henry [21] a related result showing an *universal energy estimate* due to A.Bamberger. In consequence, it is clear that the Sellers type model is not, in general, approximately controllable.

We shall need a technical assumption on $\partial\omega$:

$$\partial\omega \text{ satisfies an interior and exterior sphere condition.} \tag{21}$$

Theorem 3 (Obstruction for the Sellers type model)

Assume (7), (8), (20) *and* (21). *Then there exists a function* $\mathcal{Y}_\infty \in \mathcal{C}([0,T] \times (\Omega \setminus \overline{\omega}))$ *such that for any* $v \in L^2((0,T) \times \omega)$ *and any solution* $y(t, x; v)$ *of* $\mathcal{P}(f, \beta)$ *we have*

$$|y(t, x; v)| \leq \mathcal{Y}_\infty(t, x), \quad \text{for } (t, x) \in (0,T] \times (\Omega \setminus \overline{\omega}). \tag{22}$$

Proof.

Let us construct an auxiliar function $\mathcal{Y}_\infty(t,x)$ by showing that the problem

$$\left.\begin{array}{ll} \mathcal{Y}_t - \Delta \mathcal{Y} + f(\mathcal{Y}) = Q, & \text{in } (0,T) \times (\Omega \setminus \varpi) \\ \dfrac{\partial \mathcal{Y}}{\partial \vec{n}} = 0, & \text{on } (0,T) \times \partial\Omega \\ \mathcal{Y} = \infty, & \text{on } (0,T) \times \partial\omega \\ \mathcal{Y}(0,\cdot) = y_{0,+}(\cdot) := \max\{0, y_0(\cdot)\}, & \text{on } \Omega \setminus \varpi \end{array}\right\} \quad (23)$$

has a weak solution. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz continuous function strictly increasing, convex on $[0,\infty)$ and concave on $(-\infty, 0]$ satisfying

$$|g(s)| \leq |f(s)| \quad \text{for } s \in \mathbb{R} \quad (24)$$

and

$$\left.\begin{array}{l} \text{there exists } p > 1 \text{ and some constants } k_1, \ k_2, \ L > 0 \\ \text{such that } |g(s)| \geq k_1 + k_2|s|^p \text{ for any } |s| > L \end{array}\right\} \quad (25)$$

(such a function g can be easily constructed due to assumption (20)).

Let us prove that problem

$$\left.\begin{array}{ll} z_t - \Delta z + g(z) = Q, & \text{in } (0,T) \times (\Omega \setminus \varpi) \\ \dfrac{\partial z}{\partial \vec{n}} = 0, & \text{on } (0,T) \times \partial\Omega \\ z = \infty, & \text{on } (0,T) \times \partial\omega \\ z(0,\cdot) = y_{0,+}(\cdot), & \text{on } \Omega \setminus \varpi \end{array}\right\} \quad (26)$$

has a solution $\mathcal{Z}_\infty$. We introduce the family of truncated problems

$$\left.\begin{array}{ll} z_{m,t} - \Delta z_m + g(z_m) = Q, & \text{in } (0,T) \times (\Omega \setminus \varpi) \\ \dfrac{\partial z_m}{\partial \vec{n}} = 0, & \text{on } (0,T) \times \partial\Omega \\ z_m = m, & \text{on } (0,T) \times \partial\omega \\ z_m(0,\cdot) = \min\{m, y_{0,+}(\cdot)\}, & \text{on } \Omega \setminus \varpi \end{array}\right\} \quad (27)$$

where $m \in \mathbb{N}$ is fixed. Problem (27) has a unique regular solution. Moreover, by the comparison principle we know that $0 \leq z_m \leq z_{m+1}$ in $[0,T] \times (\Omega \setminus \varpi)$. In order to to show that $z_m \rightarrow \mathcal{Z}_\infty$ as $m \rightarrow \infty$ we only had to construct a supersolution. Thanks to the conditions $p > 1$ and (21) we can apply the results of Bandle and Marcus [4] assuring the existence of the minimal solution of the stationary problem

$$\left.\begin{array}{ll} -\Delta V + \frac{1}{3}g(V) = 0, & \text{in } \Omega \setminus \varpi \\ \dfrac{\partial V}{\partial \vec{n}} = 0, & \text{on } \partial\Omega \\ V = +\infty, & \text{on } \partial\omega \end{array}\right\} \quad (28)$$

Define $w = w(t)$ as the unique solution of the one-dimensional problem

$$\left.\begin{array}{ll} \dfrac{dw}{dt}(t) + \dfrac{1}{3}g(w(t)) = 0, & \text{in } (0,T) \\ w(0) = \|y_{0,+}\|_{L^\infty(\Omega \setminus \varpi)}, & \end{array}\right\} \quad (29)$$

Since $p > 1$, the existence (and the uniqueness) of w is assured even in the case in which $\|y_{0,+}\|_{L^\infty(\Omega \setminus \varpi)} = +\infty$. Denote by $\mathcal{U}_Q$ the unique solution of the problem

$$\left.\begin{array}{ll} \mathcal{U}_t - \Delta \mathcal{U} + \frac{1}{3}g(\mathcal{U}) = Q, & \text{in } (0,T) \times (\Omega \setminus \varpi) \\ \dfrac{\partial \mathcal{U}}{\partial \vec{n}} = 0, & \text{on } (0,T) \times \partial\Omega \\ \mathcal{U} = 0, & \text{on } (0,T) \times \partial\omega \\ \mathcal{U}(0,\cdot) = 0 & \text{on } \Omega \setminus \varpi. \end{array}\right\} \quad (30)$$

Finally, we define

$$\overline{z}(t,x) = V(x) + w(t) + \mathcal{U}_Q(t,x). \quad (31)$$

Using the convexity of g we have that $\overline{z}$ is a supersolution of (27) for any $m \in \mathbb{N}$ and therefore $z_m \leq \overline{z}$ in $[0,T] \times (\Omega \setminus \varpi)$. Thus, there exists $\mathcal{Z}_\infty(t,x) = \lim_{z \to \infty} z_m(t,x)$. It is a routine matter to check that $\mathcal{Z}_\infty$ is the *minimal solution* of (26) and that $z_m \leq \mathcal{Z}_\infty$.

The construction of $\mathcal{Y}_\infty^+(t,x)$, *minimal solution* of (23), is completly similar, since by (24) we have $0 \leq \mathcal{Y}_m \leq z_m \leq \mathcal{Z}_\infty$ if $\mathcal{Y}_m$ is the solution of the truncated problem (23)$_m$ formulated in a similar manner to (27). This proves that $\mathcal{Z}_\infty$ is a supersolution for (23)$_m$, for any $m \in \mathbb{N}$, and so $0 \leq \mathcal{Y}_m \leq \mathcal{Y}_\infty^+ \leq \mathcal{Z}_\infty$.

Finally, let us prove (22). Let $v \in L^2((0,T) \times \omega)$ and let $y(t,x;v)$ be any solution of $\mathcal{P}(f,\beta)$. Then by (8)

$$y_t - \Delta y + f(y) \leq Q \quad \text{in } (0,T) \times (\Omega \setminus \varpi).$$

Moreover, we have

$$\left.\begin{array}{ll} \dfrac{\partial y}{\partial \vec{n}} = \dfrac{\partial \mathcal{Y}_\infty^+}{\partial \vec{n}}, & \text{on } (0,T) \times \partial\Omega \\ y \leq \mathcal{Y}_\infty^+, & \text{on } (0,T) \times \partial\omega \\ y(0,\cdot) \leq \mathcal{Y}_\infty^+(0,\cdot), & \text{on } \Omega \setminus \varpi. \end{array}\right\}$$

Then we conclude that $y \leq \mathcal{Y}_\infty^+$ in $(0,T) \times (\Omega \setminus \varpi)$. In an analogous way we can construct $\mathcal{Y}_\infty^-$ as the *maximal solution* of the problem

$$\left.\begin{array}{ll} \mathcal{Y}_t - \Delta \mathcal{Y} + f(\mathcal{Y}) = 0, & \text{in } (0,T) \times (\Omega \setminus \varpi) \\ \dfrac{\partial \mathcal{Y}}{\partial \vec{n}} = 0, & \text{on } (0,T) \times \partial\Omega \\ \mathcal{Y} = -\infty, & \text{on } (0,T) \times \partial\omega \\ \mathcal{Y}(0,\cdot) = -y_{0,-}(\cdot) := \min\{0, y_0(\cdot)\}, & \text{on } \Omega \setminus \varpi \end{array}\right\} \quad (32)$$

and then $\mathcal{Y}_\infty^- \leq y$ in $(0, T) \times (\Omega \setminus \varpi)$. Taking $\mathcal{Y}_\infty(t, x) = \min\{\mathcal{Y}_\infty^+(t, x), -\mathcal{Y}_\infty^-(t, x)\}$ the proof of (22) is complete. □

Corollary 1

Assume the same conditions of Theorem 3. Let $y_d \in L^2(\Omega)$ be such that there exists a positively measured set D in $\Omega \setminus \varpi$ such that

$$|y_d(x)| > \mathcal{Y}_\infty(T, x) \quad a.e. \ x \in D. \tag{33}$$

Then, for any $v \in L^2((0, T) \times \omega)$ we have

$$\|y(T, \cdot; v) - y_d(\cdot)\|_{L^2(\Omega)} \geq \|\mathcal{Y}(T, \cdot; v) - y_d(\cdot)\|_{L^2(D)}. \tag{34}$$

In particular, problem $\mathcal{P}(f, \beta)$ is not approximatelly controllable. □

Remark 3

Using the functions $\mathcal{Y}_\infty^-$ and $\mathcal{Y}_\infty^+$ and monotonicity properties it is possible to show the existence of $(\mathcal{Y}_\infty)_*$ and $(\mathcal{Y}_\infty)^*$, minimal and maximal solutions of the multivalued problem

$$\left.\begin{array}{ll} \mathcal{Y}_t - \Delta\mathcal{Y} + f(\mathcal{Y}) \in Q\beta(\mathcal{Y}), & \text{in } (0, T) \times (\Omega \setminus \varpi) \\ \dfrac{\partial\mathcal{Y}}{\partial\vec{n}} = 0, & \text{on } (0, T) \times \partial\Omega \\ \mathcal{Y} = \pm\infty, & \text{on } (0, T) \times \partial\omega \\ \mathcal{Y}(0, \cdot) = y_0(\cdot), & \text{on } \Omega \setminus \varpi \end{array}\right\} \tag{35}$$

i.e. such that any other possible solution $\mathcal{Y}$ of (35) (*i.e.* taking any finite or infinite value on $(0, T) \times \partial\omega$) satisfies

$$\mathcal{Y}_\infty^- \leq (\mathcal{Y}_\infty)_* \leq \mathcal{Y} \leq (\mathcal{Y}_\infty)^* \leq \mathcal{Y}_\infty^+ \quad \text{on } [0, T] \times (\Omega \setminus \varpi).$$

We also remark that the behaviour of $\mathcal{Y}_\infty^+$ near $\partial\omega$ can be estimated from above and below in term of $C d(x, \partial\omega)^{\frac{2}{1-p}}$ (see Bandle and Marcus [3], G.Díaz and Letelier [8], Bandle, G.Díaz and J.I.Díaz [2], [9] and Díaz and Ramos [14]. □

The negative result given in Corollary 1 leads to the following question: let $y_d \in L^2(\Omega)$ such that

$$(\mathcal{Y}_\infty)_*(T, x) \leq y_d(x) \leq (\mathcal{Y}_\infty)^*(T, x) \quad \text{for a.e. } x \in \Omega \setminus \varpi. \tag{36}$$

Is it possible to find $v_\varepsilon \in L^2((0, T) \times \omega)$ such that

$$\|y(T, \cdot; v_\varepsilon) - y_d(\cdot)\|_{L^2(\Omega)} \leq \varepsilon$$

for a given $\varepsilon > 0$?. Such a question was already raised in Díaz [12]. A partial positive answer (relative to the simplified case $\beta \equiv 0$) is the main objective of the work Díaz and Ramos [15].

4   Conclusion.

The results presented in the above sections show that the sole difference between the Budyko and Sellers type models which becomes relevant when we study the approximate controllability of those simple climate models is their distinct behaviour, for large value of the temperature, of the mean emmitted energy flux $R_e(t, x, u)$ (i.e. function f in the simplified formulation of $\mathcal{P}(f, \beta)$). In the case of Budyko type models, Theorem 1 shows the approximate controllability for any arbitrary temperature distribution y_d. For Sellers type models the answer is negative for some temperature distributions y_d (see Corollary 1) but it seems possible to adapt the techniques of Díaz and Ramos [15] in order to give a positive answer in a *reasonable* class of desired states y_d (see (36)).

We end this work by remarking that at present the mean Earth surface temperature is about 15°C. This implies that the Budyko choice for the mean emmitted energy flux $R_e(t, x, u)$ can be understood as a linearized version of a, perhaps, more exactly choice (the one by Sellers) but of a simplier treatment and leading to a negligeable error at least in a moderate range of temperatures.

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