STABILIZATION OF SOLUTIONS TO A NON-LINEAR DIFFUSION EQUATION ON A MANIFOLD IN CLIMATOLOGY

J.I. Díaz and L. Tello
Depto. Matemática Aplicada
Universidad Complutense de Madrid
28.040 Madrid, SPAIN.

1. Introduction

The main goal of this communication is to present some of the results of Díaz and Tello [1994] concerning the stabilization of solutions to a non-linear model arising in Climatology. The model under consideration is based in a global energy balance of the atmosphere temperature over relatively long time scales. The so called *climate energy balance models* were introduced independently by M. Budyko [1969] and W. Sellers [1969]. The energy balance is stated in the following terms:

$$\text{Heat variation} = R_a - R_e + D,$$

where R_a represents the solar energy absorbed by the Earth, R_e is the energy emitted by the Earth to the outher space and D is the temperature diffusion. If we denote by u the temperature of the Earth surface then usually $R_a = QS(x)\beta(u)$ with Q the Solar Constant, $S(x)$ the insolation function and $\beta(u)$ the coalbedo (which is a nondecreasing function of u of the type $\beta(u) = 0,7$ if $u > -10$, $\beta(u) = 0,4$ if $u < -10$). The term R_e is also assumed to be a non decreasing function on u and can be taken as $R_e(t,x,u) = g(u) - f(t,x)$. Assuming (for simplicity) the heat capacity and the diffusion coefficient equal to one, we obtain an energy balance model of the type

$$(P)\begin{cases} u_t - \Delta_p u + g(u) \in QS(x)\beta(u) + f(t,x) & \text{in } (0,\infty)\times\mathcal{M} \\ u(0,x) = u_0(x) & \text{on } \mathcal{M} \end{cases}$$

where

- $(\mathcal{M}, g)$ is a compact bidimensional Riemannian manifold without boundary (as for instance $\mathcal{M} = S^2$ the unit sphere of $\mathbb{R}^3$).
- $\Delta_p u = \mathrm{div}_{\mathcal{M}}(|\mathrm{grad}_{\mathcal{M}} u|^{p-2}\mathrm{grad}_{\mathcal{M}} u)$, $p \geq 2$, where $\mathrm{grad}_{\mathcal{M}}$ is understood in the sense of the Riemann metric g. Budyko and Sellers considered $p = 2$. Later, Stone [1972] proposed the case $p = 3$ arguing that the diffusion coefficient must increase as the gradient of the temperature increases.
- $g : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function such that $|g(s)| \geq C|s|^r$ for some $r \geq 1$ (for instance $g(s) = Cs$ Budyko [1969], or $g(s) = C|s|^3 s$ Sellers [1969]).

- $S : \mathcal{M} \to I\!R$, $S(x) > s_0 > 0$, $S \in L^\infty(\mathcal{M})$.
- β is a bounded maximal monotone graph of $I\!R^2$, and $m \leq b \leq M$ for any $b \in \beta(s)$ for any $s \in I\!R$ (some times β is assumed *multivalued* at $u = -10$, Budyko [1969] or β *locally Lipschitz*, Sellers [1969]).
- $f \in L^\infty((0,\infty) \times \mathcal{M})$ (Budyko and Sellers proposed f constant).

In order to recall the expression of the diffusion operator $\Delta_p u$ on $\mathcal{M}$ we start by consider an atlas $\{W_\lambda, w_\lambda\}_{\lambda \in \Lambda}$ on $\mathcal{M}$. Let $(\theta_\lambda, \varphi_\lambda)$ be the coordinates framework in $w_\lambda(W_\lambda) \subset I\!R^2$ and let α_λ be a partition of unity subordinate to the covering W_λ. Then we assume $g = \sum \alpha_\lambda g^\lambda$ with g^λ given over each local chart. Given $p \in W_\lambda \subset \mathcal{M}$ the set $\{e_1 := \frac{\partial}{\partial \theta_\lambda}, e_2 := \frac{\partial}{\partial \varphi_\lambda}\}$ is a basis of the tangent space $T_p\mathcal{M}$. We consider the tangent bundle $T\mathcal{M}$ as $\cup_{p \in \mathcal{M}} T_p\mathcal{M}$ and for $u : \mathcal{M} \to I\!R$ we define

$$\mathrm{grad}_{\mathcal{M}} u = g^{ij} \frac{\partial u}{\partial y_j} e_i.$$

If $X : \mathcal{M} \to T\mathcal{M}$ we define

$$\mathrm{div}_{\mathcal{M}} X = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial y_i} \left(X^i \sqrt{\det g} \right).$$

Finally, given $u : \mathcal{M} \to I\!R$ the diffusion operator is defined by

$$\Delta_p u = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial y_i} \left(\sqrt{\det g} \left| g^{kl} \frac{\partial u}{\partial y_l} e_k \right|^{p-2} g^{ij} \frac{\partial u}{\partial y_j} \right).$$

where $y_1 = \theta_\lambda$, $y_2 = \varphi_\lambda$, $|\cdot| = g(\cdot,\cdot)^{\frac{1}{2}}$, and g^{ij} are the coefficients of the inverse matrix of $g^\lambda = (g_{ij})$. For $p = 2$ the above expression coincides with the Laplace-Beltrami operator,

$$\Delta u = \frac{1}{\sqrt{detg}} \frac{\partial}{\partial y_i} \left(g^{ij} \sqrt{detg} \frac{\partial u}{\partial y_j} \right)$$

In the case $\mathcal{M} = S^2$ with the spherical coordinates atlas some authors assume u as a function only of φ (latitude). Introducing the change of variable $x = \cos\varphi$ we obtain the simpler expressions

$$\Delta_p u = \mathrm{div}\left((1-x^2)^{\frac{p}{2}} |u_x|^{p-2} u_x \right) \qquad \Delta u = \mathrm{div}\left((1-x^2) u_x \right)$$

and so, we obtain the so called *one-dimensional climate model* [notice the presence of a degenerated weight in the operator].

The general theory (existence and uniqueness of weak solutions) for this class of problems was carried out in Díaz [1993] for the one-dimensional model and then generalized in Díaz-Tello [1993] to the bidimensional case. The existence of solutions was obtained in the space $C([0,\infty); L^2(\mathcal{M})) \cap L^p_{loc}(0,\infty; V)$, where $V = \{u \in L^2(\mathcal{M}) : \mathrm{grad}_{\mathcal{M}} u \in L^p(T\mathcal{M})\}$. As usual (see e.g. Aubin

[19 and Chavel [1984]), given $p > 1$ we denote by $L^p(\mathcal{M})$ the set $\{u : \mathcal{M} \to I\!R$ measurable : $\int_{\mathcal{M}} |u|^p dS < \infty\}$ where $dS = \sum_{\lambda \in \Lambda} \alpha_\lambda \sqrt{\det g^\lambda} d\theta_\lambda d\varphi_\lambda$. This set is a Banach space with the norm

$$\int_{\mathcal{M}} |u|^p dS = \sum_{\lambda \in \Lambda} \int_{w_\lambda(W_\lambda)} \alpha_\lambda |u(w_\lambda^{-1}(\theta_\lambda, \varphi_\lambda))|^p \sqrt{\det g^\lambda} d\theta_\lambda d\varphi_\lambda.$$

Analogously $L^p(T\mathcal{M}) = \{X : \mathcal{M} \to T\mathcal{M} : \int_{\mathcal{M}} (g(X,X))^{\frac{p}{2}} dS < \infty\}$. Concerning the uniqueness of weak solutions the answers are of different nature: there is uniqueness of solutions to the *Sellers type model* (i.e. when β is a locally Lipschitz function) but this is not the case, in general, for the *Budyko type model* (i.e. when β is discontinuous or multivalued). Nevertheless, the uniqueness of solutions holds in the class of functions satisfying a suitable "nondegeneracy property" at the level $u = -10$ (see the mentioned references).

One of the main interest of this kind of models is its simplicity for the study of the effect of variations on the data (mainly on the solar constant Q). This study was firstly obtained of $p = 2$ heuristically (see e.g. North [1993] and its references) and more rigurously by Hetzer [1990] for the Sellers model. In this communication we extend some of the above results to the general formulation (i.e. $p \geq 2$ and β not necessarily Lipschitz continuous). The stabilization of solutions to the solutions to the stationary problem is proved firstly by using some abstract results (see Section 2) and after by more *ad hoc* arguments (Section 3). Finally, some partial results on the structure of the set of stationary solutions are presented in Section 4.

2. Some abstract results for the Semigroup associated to the Sellers type Climate Model.

In this section, we shall always assume that β is a locally Lipschitz continuous function and that $f(t,x) \equiv f(x)$ with $f \in L^\infty(\mathcal{M})$. We define the nonlinear operator $Au = -\Delta_p u + g(u) - QS(\cdot)\beta(u) - f(x)$ with $D(A) = \{w \in V : Aw \in L^2(\mathcal{M})\}$. Then we have

Lemma 1.
i) A generates a semigroup $\{S(t)\}_{t \geq 0}$ *on* $L^2(\mathcal{M})$.
ii) $\forall \lambda > 0$ $J_\lambda = (I + \lambda A)^{-1}$ *is a compact map from* $L^2(\mathcal{M})$ *into* $L^2(\mathcal{M})$.
iii) $S(t)$ *is equicontinuous on any compact interval of* $[0,\infty)$.
iv) $S(t)$ *is a compact semigroup for any* $t \in (0,\infty)$.

Proof. Parts i), ii) and iv) are today more or less standard (see Vrabie [1987] and Díaz [1993]). Part iii) is obtained by proving that if $u_0 \in V \cap L^\infty(\mathcal{M})$ then $u_t \in L^2_{loc}((0,\infty) \times \mathcal{M})$ and $u \in L^\infty(0,\infty; V)$ in a similar way to Theorem 5 of Díaz-Thelin [1994]. ∎

Following Temam [1988] we introduce some notions:

Definition 1 *A set $\mathcal{A} \subset L^2(\mathcal{M})$ is called an **attractor** for $S(t)$ if $S(t)\mathcal{A} = \mathcal{A}$ $\forall t \geq 0$ and $\exists\, U$ an open neighborhood of $\mathcal{A}$ such that $\forall u_0 \in U$, $dist_{L^2}(S(t)u_0, \mathcal{A})$ tends to 0 as $t \to +\infty$. Moreover, $\mathcal{A}$ is a **global attractor** of $S(t)$ if $\mathcal{A}$ is a compact attractor and uniformly attracts any bounded set B of $L^2(\mathcal{M})$.*

Definition 2 *$\mathcal{B} \subset U \subset L^2(\mathcal{M})$ is **absorbing** in U if $\forall B_0 \subset U$ bounded, $\exists t_1(B_0)$ such that $S(t)B_0 \subset \mathcal{B}$ $\forall t \geq t_1(B_0)$.*

Proposition 1. ***Assume $p \geq 2$. Then there exists a compact global attractor for the semigroup $\{S(t)\}_{t\geq 0}$ associated to the Sellers type model.***

Proof. Multiplying by u and using Hölder and Young inequalities, we obtain

$$\frac{d}{dt}\|u(t,.)\|^2_{L^2(\mathcal{M})} \leq -C_1\|u(t,.)\|^2_{L^2(\mathcal{M})} + C_2, \quad C_1, C_2 > 0.$$

By Gronwall Lemma,

$$\|u(t,.)\|^2_{L^2(\mathcal{M})} \leq \|u_0\|e^{-C_1 t} + \frac{C_1}{C_2}(1 - e^{-C_1 t}) \to \frac{C_1}{C_2} \text{ as } t \to +\infty,$$

which shows that the set $\mathcal{B} = B_{L^2(\mathcal{M})}(0, \frac{C_1}{C_2} + \varepsilon)$ is a bounded absorbing set in $L^2(\mathcal{M})$. Then, using the compactness of the semigroup and an abstract result (see Temam [1988], Theorem 1.1) we obtain that the $\omega-$limit set of $\mathcal{B}$ satisfies $\omega(\mathcal{B}) = \mathcal{A}$.

■

A better information can be obtained by using another abstract result (Temam [1988], Theorem VII.4.1).

Proposition 2. *The functional*

$$J(w) = \frac{1}{p}\int_{\mathcal{M}} |\nabla w|^p + \int_{\mathcal{M}} G(w) - Q\int_{\mathcal{M}} S(x)j(w) - \int_{\mathcal{M}} fw \qquad (1)$$

where

$$\partial j = \beta \quad \text{and} \quad G(r) = \int_0^r g(s)ds$$

is a Lyapunov function for the semigroup $S(t)$.

Corollary 1. *Let $\mathcal{E}$ be the set of fixed points of the semigroup $S(t)$, then $\mathcal{A} = M_+(\mathcal{E})$ where M_+ denotes the unstable manifold at $\mathcal{E}$. Moreover, if $\mathcal{E}$ is a discrete set, then $\mathcal{A}$ is the union of $\mathcal{E}$ and the heteroclinic curves.*

3. St lization of Solutions to a more general Climate Model.

In this section we shall use "ad hoc" techniques valid for β a general bounded maximal monotone graph (remember the case of the Budyko type model) and f also dependent on time. We assume that there exists $f_\infty \in V'$ such that

$$\int_{t-1}^{t+1} \|f(\tau,\cdot) - f_\infty(\cdot)\|_{V'} d\tau \to 0 \quad \text{as } t \to \infty.$$

As usual, given u be a bounded weak solution of (P), we define the ω-limit set of u by

$$\omega(u) = \{u_\infty \in V \cap L^\infty(\mathcal{M}) : \exists t_n \to +\infty \text{ such that } u(t_n,\cdot) \to u_\infty \text{ in } L^2(\mathcal{M})\}.$$

Theorem 1. ***Let $u_0 \in L^\infty(\mathcal{M}) \cap V$. Then***
(i) $w(u) \neq \emptyset$.
(ii) If $u_\infty \in \omega(u)$ then $\exists t_n \to +\infty$ such that $u(t_n+s,\cdot) \to u_\infty$ in $L^2(-1,1;L^2(\mathcal{M}))$ and u_∞ is a weak solution of the stationary problem,

$$(P^Q_\infty) \qquad -\Delta_p u_\infty + g(u_\infty) \in QS\beta(u_\infty) + f_\infty \text{ in } \mathcal{M}.$$

(iii) In fact, if $u_\infty \in w(u)$ then $\exists\{\hat{t}_n\} \to +\infty$ such that $u(\hat{t}_n,\cdot) \to u_\infty$ strongly in V.

Proof. i) We first prove that $u \in L^\infty(0,\infty;V)$. The conclusion is then obvious.
ii) The first part comes from the integrability of u_t. To prove the second part we consider the test functions $v(t,x) = \xi(x)\varphi(t-t_n)$ with $\xi \in V \cap L^\infty(\mathcal{M})$ and $\varphi \in \mathcal{D}(-1,1)$, $\varphi \geq 0$, $\int_{-1}^1 \varphi = 1$. Then

$$\int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} u_t\xi\varphi(t-t_n) + \int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} |\nabla u|^{p-2}\nabla u\nabla\xi\varphi(t-t_n) + \int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} g(u)\xi\varphi(t-t_n)$$
$$= \int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} Qz\xi\varphi(t-t_n) - \int_{t_n-1}^{t_n+1}\int_{\mathcal{M}} f(t,x)\xi\varphi(t-t_n) \quad z \in \beta(u(t,x)).$$

Changing variables, namely $s = t - t_n$ and defining $U_n(s,x) = u(t_n+s,x)$, we obtain the a priori estimates

$$\| U_n \|_{L^\infty(-1,1;V)} \leq C_1, \quad \| \nabla U_n \|_{L^\infty(-1,1;L^p(T\mathcal{M}))} \leq C_2, \quad \| z_n \|_{L^\infty(-1,1;L^\infty(\mathcal{M}))} \leq C_3$$

and thus the following convergences

$$\begin{array}{llll} U_n \rightharpoonup u_\infty & \text{in} & L^s((-1,1);V) & \forall s > 1 \\ |\nabla U_n|^{p-2}\nabla U_n \rightharpoonup Y & \text{in} & L^s((-1,1);L^p(T\mathcal{M})) & \forall s > 1 \\ z_n \rightharpoonup z_\infty \in \beta(u_\infty) & \text{in} & L^s((-1,1)\times\mathcal{M}) & \forall s > 1. \end{array}$$

Passing to the limit we arrive to

$$\int_{-1}^1\int_{\mathcal{M}} Y\nabla\xi\varphi + \int_{\mathcal{M}} g(u_\infty)\xi = \int_{\mathcal{M}} QSz_\infty\xi + \int_{\mathcal{M}} f_\infty\xi \quad \forall\xi \in V \cap L^\infty(\mathcal{M}).$$

The main difficulty is to prove that $\int_{-1}^{1} Y(s,\cdot)\varphi(s) = |\nabla u_\infty|^{p-2}\nabla u_\infty$. For this, we use a Minty type argument reducing the problem to the inequality

$$\lim_{n\to\infty}\int_{-1}^{1}\int_{\mathcal{M}}(|\nabla U_n|^{p-2}\nabla U_n - |\nabla\chi|^{p-2}\nabla\chi)\cdot(\nabla u_\infty - \chi)\varphi(s) \geq 0$$

for any $\chi = u_\infty + \lambda\xi$ (this holds due, essentially, to the coercitivity of the diffusion operator).

iii) This part also uses the coercivity of the operator and the fact that

$$\int_{-1}^{1}\int_{\mathcal{M}}|\nabla U_n|^{p-2}\nabla U_n - |\nabla u_\infty|^{p-2}\nabla u_\infty)\cdot(\nabla U_n - \nabla u_\infty)\varphi(s) \to 0.$$

(We send the reader to Díaz-Tello [1994] for details). ∎

4. Existence of Stable Stationary Solutions.

In this last section we examine the sensitivity of the two-dimensional equilibrium climate model to changes in the solar constant Q. It is well-known that the 0-dimensional model presents a S-shaped bifurcation curve (see e.g. North [1993]). Some of those aspects remain true for the 2-dimensional model:

Theorem 2. *For any $Q > 0$ there exists at least a stable solution of the stationary problem (P_∞^Q) associated to (P).*

Proof. Applying the Weierstrass theorem we show that the function J defined by (1) has a global minimum w_Q on V. Moreover, as J is Gateaux-differentiable w_Q is a (stable) solution of (P_∞^Q). ∎

Under some suitable assumptions we can prove uniqueness of solutions for Q large (resp. Q small) enough.

Proposition 3. *Assume*
(H_β) $\exists\ r_1 \leq r_2$ *such that* $\beta(r) = m \quad \forall r \leq r_1$ *and* $\beta(r) = M \quad \forall r \geq r_2$,
$(H_{f,g})$ $\exists\ Q_1 > 0$ *such that the solution* $\underline{u}^{Q_1}$ *of* $-\Delta_p\underline{u} + g(\underline{u}) = QS(x)m + f$ *in* $\mathcal{M}$ *satisfies that* $\underline{u}^{Q_1}(x) < r_1$ *a.e.* $x \in \mathcal{M}$. *Then*
i) $\forall Q \in (0, Q_1]$ *the problem* (P_∞^Q) *has a unique solution.*
ii) $\exists\ Q_2 > Q_1$ *such that* $\forall Q \in [Q_2, +\infty)$ *problem* (P_∞^Q) *has a unique solution.*

The proof is based on the comparison principle applied to the operator $\Delta_p + g$.

Remark. The bifurcation diagram for the Sellers model (β Lipschitz) and $p = 2$ was studied in Hetzer [1992]. The extension to $p \neq 2$ and β a maximal monotone graph satisfying (H_β) is under current research.

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