CONTROLLABILITY AND OBSTRUCTION FOR SOME NON LINEAR PARABOLIC PROBLEMS IN CLIMATOLOGY

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1 Introduction.

In 1955, John von Neumann wrote: *Probably intervention in atmospheric and climate matters will come in a few decades, and will unfold on a scale difficult to imagine at present* ([26]). Today one phase of this programme is almost a dream come true: the "rain making" research initiated by I. Langmuir and coworkers have originated already sucessful experiences (see Dennis [4]). While is not easy to evaluate the significance of the efforts made thus far, the evidence seems to indicate that the aim is an attainable one.

The main goal of this work is to carry out a theoretical study on the remainig part of the von Neumann programme: the control of the climate. Our modest goal is to study such a *general philosophy* by considering simple climate models which introduced in 1969 by M.I. Budyko and W. D. Sellers are today well-accepted in the Climatology literature.

In a first part we recall some of the important facts on the modelling the Earth climate by the so called Energy Balance Models and some of the results on their mathematical treatment.

In a second part we consider the question of the controllability of those models. Continuing our previous research (see Díaz [8]) in wh ich it was shown how the *obstruction phenomenon* leads to the general uncontrollability of the Sellers model, we show here that a chance still remains: *the restricted (approximate) controllability*. We will show that a very large class of desired climate states are attainable (in a weak sense) by introducing suitable spatially localized controls on the climate system.

2 Energy Balance Models.

Climate models have different characteristics than weather prediction models: the time scale is completely different (centuries versus days or weaks) and their

main goal is also complementary (prognostic in the weather prediction and diagnostic in the case of climate models). Climate models were introduced in order to understand past and future climates and their sensitivity on a few of relevant features (which a quantitative analysis reduces to some parameters).

Two of the most important ingredients of the models concern the Solar radiation R_a (short-wave energy from Sun) and the Earth radiation R_e (long-wave radiation escaping into space). The consideration of other features under different degree of accuracy introduces a *hierarchy* in the class of climate models. So, according to the time variable the models are classified into equilibrium and dynamical models. With respect to the space variable the models are called as 0-D zero-dimensional (if only the mean Earth temperature is analized), 1-D latitudinal or vertical models, 2-D horizontal or meridional plane models and so up the most sophisticated 3-D General Circulation model. More complex models have been also considered in the literature by coupling the study of the Earth temperature with different phenomena from the Glaceology, Celestial Mechanics, Geophysics, etc. (see the general expositions of Ghill and Childress [16], North [27] and Henderson-Sellers and Mc Guffie [17]).

In this work we shall pay attention to a general 2-D horizontal energy balance model which takes into consideration the main structure of the 1-D models introduced (independently) by Budyko [3] and Sellers [30].

If we represent the Earth by a compact two-dimensional manifold without boundary $\mathcal{M}$ and we denote by $u(t,x)$ the annually (or seasonally) averaged Earth surface temperature, our model is formulated as the reaction-diffusion equation

$$c(t,x)u_t(t,x) - div(k(t,x)grad\ u(t,x)) = R_a(t,x,u(t,x)) - R_e(t,x,u(t,x)) \quad (1)$$

where the heat capacity $c(t,x)$ is a positive function largely determined by oceans (recall that the 70 per cent of the Earth's surface is covered by oceans). After averaging $c \sim 1.05 \times 10^{23} Jm^{-2}K^{-1}$. The diffusion operator in (1) has a double justification:

$$div(k\ grad\ u) = div(F_c + F_a)$$

with $F_c = k_c\ grad\ u$ the conduction heat flux and F_a the advection heat flux. In Meteorology and Oceanography it is usually assumed $F_a = -\mathbf{v}T$ where $\mathbf{v}$ and T are the velocity and temperature of the fluid. In planetary scales $O(10^4Km)$ the velocity is eliminated using the *eddy diffusive approximation*

$$divF_a \simeq div(k_e grad\ u) \quad (2)$$

where the eddy diffusion coefficient is again a positive number (and more generally a positive function). Obviously the differential operators div and $grad$ must be suitably understood with respect to the Riemannian metric. An important variant is due to P.H. Stone [31] who pointed out that in the case of rotating atmospheres the eddy diffusive approximation really leads to a nonlinear diffusion operator of the form

$$div(k_e^* |grad\ u| grad\ u) \quad (3)$$

for some $k_e^* > 0$ (see [31] formula 2.24). In terms of equation (2) the nonlinear operator (3) means that the eddy diffusion coeficient k_e must increase as the gradient of the averaged temperature increases.

The solar energy absorbed by the Earth R_a is assumed to be of the form

$$R_a = QS(x)\beta(u) \quad (4)$$

where Q is the *Solar constant* (i.e. the annual average amount of radiation energy per unit time passing through a unit area perpendicular to the Sun's rays at the Earth orbit). Averaging $Q \sim 1.370\ W/m^2$. $S(x)$ is the distribution of solar radiation over the Earth and $\beta(u)$ is the *planetary coalbedo* representing the fraction absorbed according the average temperature. Usually $\beta(u)$ is assumed to be a non-decreasing function of u taking constant values a_i and a_f (both positive and less than one) for small and respectively large values of u. It is not completely clear how is produced the transition: Budyko [3] propose a discontinuity at $u = -10°C$

$$\beta(u) = \begin{cases} a_f & \text{over ice-free} & \{x \in \mathcal{M} : u(t,x) > -10\} \\ a_i & \text{over ice-covered} & \{x \in \mathcal{M} : u(t,x) < -10\}. \end{cases} \quad (5)$$

In contrast to that, Sellers [30] propose a continuous linear piecewise function with a very large increasing rate near -10. We remark that in seasonally averaged models the terms $QS(x)$ are replaced by a more general function $S(t,x)$ "almost" periodic in time. This is of relevance in the study of *ice ages* since snowcover over the summer is a necessary condition for the growth of continental glaciers as, for instance, the ones of Antarctica and Greenland (see the work by North Mengel and Short [28] and its references). We also point out that the modeling of clouds is one of the most important open problems in the study of the solar energy absortion.

The mean emmitted energy flux $R_e(t,x,u)$ is determined empirically and depends on the amount of greenhouse gases, clouds and water vapor in the atmosphere. It seems natural to assume that R_e increases with u but the increasing rate is controversial: Sellers [30] propose a Stefan-Boltzman radiation law

$$R_e = \sigma u^4(1 - m\ tanh(\frac{19u^6}{10^6})) \quad (6)$$

where u is represented in Kelvin degrees (here $\sigma > 0$ is the emmisivity and $m > 0$ the atmospheric opacity). Budyko [3] replaces it by a Newton linear type radiation ansatz

$$R_e = A + Bu \quad (7)$$

which is a linear approximation of (6) near $u = 15°C$ (the actual mean temperature). Here $A = 210\ W/m^2$ and $B = 1.9\ W/°Cm^2$. We point out that the term R_e takes also in account the anthropogenerated changes.

For our mathematical study we can simplify the model assuming $c = k = 1$. On the other hand, it is useful to treat the possible discontinuous function β (as, for instance, the one given in (5)) in the class of maximal monotone graphs of $\mathbb{R}^2$ (see e.g. Brezis [2]). So our problem can be formulated in the following terms

$$(P) \begin{cases} u_t - \Delta_{\mathcal{M}} u + g(u) \in QS(x)\beta(u) + f(t,x) & \text{in } (0,\infty) \times \mathcal{M} \\ u(0,x) = u_0(x) & \text{on } \mathcal{M} \end{cases}$$

where g is a continuous and nondecreasing function. Here we are assuming $R_e(t,x,u) = g(u) - f(t,x)$, $S \in C(\mathcal{M})$ with $S(x) > 0$ for any $x \in \mathcal{M}$ and β is a maximal monotone graph of $\mathbb{R}^2$ such that

$$\left. \begin{array}{l} D(\beta) = \mathbb{R} \text{ and there exist } m \text{ and } M \text{ satisfying} \\ 0 < m \le b \le M \quad \forall b \in \beta(r), \ \forall r \in \mathbb{R}. \end{array} \right\} \qquad (8)$$

We recall that $\Delta_{\mathcal{M}}$ represents the Laplacian operator on $\mathcal{M}$. If $\mathcal{M} = S^2$ (the unit sphere of $\mathbb{R}^3$) and we denote by ϕ and λ to the colatitude and the longitude then

$$\Delta_{\mathcal{M}} u = \frac{1}{\sin\phi}\left(\frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin\phi}\frac{\partial^2 u}{\partial\lambda^2}\right). \qquad (9)$$

The mentioned one-dimensional models arise when u is assumed to be constant on the same latitude circles. Introducing the variable $x \in (0,1)$ by $x = \cos\phi$ then (9) leads to the degenerated operator

$$\frac{\partial}{\partial x}\left((1 - x^2)\frac{\partial u}{\partial x}\right). \qquad (10)$$

We point out that from the climatological view point one of the main subjects of research is the bifurcation diagram (with respect to Q) of the number of solutions of the associated stationary problem and their stability (see the surveys mentioned at the begining of this section).

For a mathematical approach we send the reader to the works by Hetzer [19], Hetzer and Schmidt [20], Xu [32], Díaz [6] and Díaz and Tello [12], [13].

3   Controllability and Obstruction.

The main goal of this section is to study if possible antropogenerated actions on the climate system allows to carry the average temperature from a given distribution $y(0,x)$ to a desired distribution $y_d(x)$ after a given period of time T. Such type of questions was already considered by J. Fourier [15] and some of the most relevant applied mathematicians of this century (J. von Neumann [26] and J.L. Lions [22] [24] among them). The connection between this question and the study of the irreversibility of the antropogenetic changes already introduced in the atmosphere since the beginings of the Industrial Society is obvious. It is also clear that many of the actual world decission on greenhouse gases emmision norms follow also this philosofy.

A mathematical statement of the question under consideration can be the following: given ω an open submanifold of $\mathcal{M}$, $T > 0$, an initial distribution of temperatures $u_0 : \mathcal{M} \to \mathbb{R}$ and a desired temperature $y_d : \mathcal{M} \to \mathbb{R}$, we want to find a control $v : (0,T) \times \omega \to \mathbb{R}$ such that $y(T : v) = y_d$ where $y(\cdot : v)$ denotes the solution of problem (P) replacing $f(t,x)$ by $f(t,x) + v(t,x)\chi_\omega$ with χ_ω the characteristic function of ω. When the answer is positive we say that (P) *is controllable*. Nevertheless, the parabolic character of the equation of (P) implies some regularizing effects making impossible our goal except for a very limited class of desired states y_d. A relaxed statement comes in a natural way: *the approximate controllability*. Given $\varepsilon > 0$ we seek now a control v_ε (defined again on $(0,T) \times \omega$) such that $d(y(T,v_\varepsilon), y_d) \le \varepsilon$. In the above expression $d(\cdot,\cdot)$ represents the distance measured in some space of functions defined on $\mathcal{M}$ (usually $L^2(\mathcal{M})$, or, more generally, $L^p(\mathcal{M})$ with $1 \le p \le \infty$).

The nature of our spatial domain $\mathcal{M}$ leads to some additional (and technical) difficulties in our study. A simpler formulation which still gives a representative idea of the treatement in more complex situations corresponds to the case in which we replace $\mathcal{M}$ by an open regular bounded set Ω of $\mathbb{R}^2$ (here $\mathbb{R}^2$ can be also substitued by $\mathbb{R}^N$ with $N \in \mathbb{N}$). As boundary condition on $(0,T) \times \partial\Omega$ we can chose the one of Neumann type since it leads to a set of test functions for the weak formulation very similar to the one corresponding to the case of a Riemannian manifold without boundary. Another unrelevant simplification is to assuming $f \equiv 0$. Thus the new formulation is the following: given ω an open bounded subset of $\Omega, y_0, \ y_d : \Omega \to \mathbb{R}$ and $\varepsilon > 0$ find $v_\varepsilon : (0,T) \times \omega \to \mathbb{R}$ such that $d(y(T : v_\varepsilon), y_d) \le \varepsilon$ where, in general, $y(T : v)$ represents the solution of problem

$$(\mathcal{P}_\omega) \begin{cases} y_t - \Delta y + g(y) \in QS(x)\beta(y) + v\chi_\omega & \text{in } (0,T) \times \Omega \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0,\cdot) = y_0(\cdot) & \text{on } \Omega, \end{cases}$$

where n is the outer unit vector to $\partial\Omega$.

In a previous work (Díaz [8]) it was shown that the answer to the approximate controllability property depends on the asymptotic behaviour of the nonlinearities of the equation (and not on its regularity). So, a positive answer is collected in the following result

Theorem 1 ([8])

Assume $y_0, \ y_d \in L^2(\Omega)$, β satisfying (8) and g a nondecreasing function such that

$$|g(s)| \le C_1 + C_2|s| \quad \forall s \in \mathbb{R}, \ |s| > \overline{M} \qquad (11)$$

for some nonnegative constants C_1, C_2 and $\overline{M}$. Then problem (P_ω) is approximate controllable in $L^2(\Omega)$, i.e. there exist $v_\epsilon \in L^2((0,T) \times \omega)$ such that

$$\| y(T : v_\epsilon) - y_d \|_{L^2(\Omega)} \leq \epsilon.$$

The above theorem can be easily extended to the case in which we replace $L^2(\Omega)$ by $L^p(\Omega)$ with $1 \leq p < \infty$ or $C(\overline{\Omega})$. The main idea of the proof is the application of the Kakutani fixed point theorem similarly to the work Fabré, Puel and Zuazua [14] (see also Henry [18], Lions [21] [23], Díaz [7] and Díaz and Ramos [10] [11] for other related works).

We point out that Theorem 1 applies to the special case of the Budyko model since there $g(y) = By$ and (11) fails for the Sellers model (assume $m = 0$ in (6) and also $u > 0$ in order to reduce the study to a nondecreasing function g). In fact, it was shown in Díaz [8] (see also [5]) that if we assume

$$g(y) = \lambda |y|^{p-1} y \text{ for } y \in \mathbb{R} \text{ and some } \lambda > 0 \text{ and } p > 1 \tag{12}$$

then an *obstruction phenomenon* appears

Theorem 2

Assume (12) and that $\partial \omega$ satisfies the interior and exterior sphere condition. Let $y_0 \in L^\infty(\Omega)$. Then, there exists a function $Y_\infty \in C([0,T] \times (\Omega - \overline{\omega}))$ such that for any $v \in L^2((0,T) \times \omega)$ and any solution $y(t,x : v)$ of (P_ω) we have

$$|y(t,x : v)| \leq Y_\infty(t,x) \text{ for } (t,x) \in (0,T] \times (\Omega - \overline{\omega}). \tag{13}$$

The obstruction function Y_∞ in (13) was constructed in [8] such that

$$\begin{aligned} Y_\infty(t,x) &= +\infty && \text{on } (0,T) \times \partial \omega \\ \tfrac{\partial Y_\infty}{\partial n}(t,x) &= 0 && \text{on } (0,T) \times \partial \Omega. \end{aligned}$$

In consequence, condition (12) implies that problem (P_ω) is not (in general) approximate controllable since if $|y_d(x)| > Y_\infty(T,x)$ a.e. x in a positively measured subset D of $\Omega - \overline{\omega}$ then for any $v \in L^2((0,T) \times \omega)$

$$\|y(T;v) - y_d\|_{L^2(\Omega)} \geq \|Y_\infty(T,\cdot) - y_d\|_{L^2(\Omega)}$$

and so, if $\epsilon > 0$ is small enough, it is imposible to choose v satisfying the required properties. We remark that a previous uniform estimate (independetly of the control) for superlinear equations but when the control acts on the boundary was due to A. Bamberger (see Henry [18]). Due to the relevance of the Sellers model, a natural question arises: is problem P_ω approximate controllable in a smaller class of desired states y_d?

The main contribution of this work is to give a positive answer to the above question. For the sake of the exposition we shall simplify, even more, the model under consideration to

$$(P_p) \begin{cases} y_t - \Delta y + \lambda |y|^{p-2} y = v \chi_\omega & \text{in } (0,T) \times \Omega \\ \tfrac{\partial y}{\partial n} = 0 & \text{on } (0,T) \times \partial \Omega \\ y(0,\cdot) = y_0(\cdot) & \text{on } \Omega. \end{cases}$$

The extension of the following results to the case of problem (P_ω), assumed (12), is merely a technical matter and can be carried out as in [8].

The starting point of our approach consists in improving the estimate (13) by obtaining some *sharp obstruction functions*. This is the objective of the next result

Proposition 1

Given $y_0 \in L^1(\Omega)$ there exist $\underline{Y}_\infty$, $\overline{Y}_\infty \in C((0,T] \times \Omega - \overline{\omega})$ such that $\underline{Y}_\infty$ is a weak solution to the problem

$$\begin{cases} \underline{Y}_t - \Delta \underline{Y} + \lambda |\underline{Y}|^{p-2} \underline{Y} = 0 & \text{in } (0,T) \times (\Omega - \overline{\omega}) \\ \underline{Y}_\infty = -\infty & \text{on } (0,T) \times \partial \omega \\ \tfrac{\underline{Y}_\infty}{\partial n} = 0 & \text{on } (0,T) \times \partial \Omega \\ \underline{Y}_\infty(0,\cdot) = y_0(\cdot) & \text{on } \Omega \end{cases}$$

and $\overline{Y}_\infty$ satisfies the same conditions except that $\overline{Y}_\infty = +\infty$ on $(0,T) \times \partial \omega$.

Idea of the proof. As in Bandle, G. Díaz and J.I. Díaz [1], given $N \in \mathbb{N}$ we define $\underline{Y}_N$ as the (unique) solution of the problem

$$\begin{aligned} Y_t - \Delta Y + \lambda |Y|^{p-2} Y &= 0 && \text{in} && (0,T) \times (\Omega - \overline{\omega}) \\ Y &= -N && \text{on} && (0,T) \times \partial \omega \\ \tfrac{\partial Y}{\partial n} &= 0 && \text{on} && (0,T) \times \partial \Omega \\ Y(0,\cdot) &= \text{Sup}\{(y_0)_-(\cdot), -N\} && \text{on} && \Omega. \end{aligned}$$

where $(y_0)_-(x) = \inf\{y_0(x), 0\}$. Using the maximum principle and the assumption $p > 1$ it is easy to see that there exists $\underline{Z}_\infty \in C((0,T) \times (\Omega - \overline{\omega}))$ such that $\underline{Z}_\infty \leq .... \leq \underline{Y}_2 \leq \underline{Y}_1 \leq 0$. Then we can define $\underline{Y}_\infty(t,x) = \lim_{N \to \infty} \underline{Y}_N(t,x)$ and by duality arguments it is proven that $\underline{Y}_\infty$ satisfies the requiered conditions. The arguments for $\underline{Y}_\infty$ are completely similar. ∎

We point out that if we assume, formally, $Q = 0$ in P_ω then the obstruction functions of Proposition 1 is sharper than the ones given in Theorem 2, i.e.

$$\underline{Y}_\infty(t,x) \leq -Y_\infty(t,x) \leq \underline{Y}_\infty(t,x) \leq \overline{Y}_\infty(t,x).$$

Now we are in a condition to state our *restricted approximate controllability criterion*:

Theorem 3

Let $y_0 \in C(\overline{\Omega})$ and consider $y_d \in C(\overline{\Omega})$ such that

$$\underline{Y}_\infty(T,x) < y_d(x) < \overline{Y}_\infty(T,x) \quad \forall x \in \Omega - \overline{\omega}. \tag{14}$$

Then for any $\epsilon > 0$ there exists $v_\epsilon \in C([0,T] \times \overline{\omega})$ such that if $y(t:v)$ is the corresponding solution of (P_p) we have

$$\|y(T:v_\epsilon) - y_d\|_{C(\overline{\Omega})} \leq \epsilon. \tag{15}$$

The above statement is an obvious consequence of the following more general result:

Theorem 4

Let $y_0 \in C(\overline{\Omega})$ and let $\epsilon > 0$ fixed. Consider $y_d \in C(\overline{\Omega})$ such that

$$\underline{Y}_\infty(T,x) - \epsilon < y_d(x) < \overline{Y}_\infty(T,x) + \epsilon \quad \forall x \in \Omega - \overline{\omega} \tag{16}$$

Then there exists $v_\epsilon \in C([0,T] \times \overline{\omega})$ satisfying (15).

Remark. The assumption (16) is optimal. Indeed, assume v_ϵ such that (15) holds. Then by the comparison principle

$$\underline{Y}_\infty(t,x) < y(t,x:v_\epsilon) < \overline{Y}_\infty(t,x) \quad \forall(t,x) \in [0,T] \times (\Omega - \overline{\omega})$$

and so

$$\underline{Y}_\infty(T,x) - \epsilon < y(T,x:v_\epsilon) - \epsilon \leq y_d(x) \leq y(T,x:v_\epsilon) + \epsilon < \overline{Y}_\infty(T,x) + \epsilon$$

which proves (14).

The proof of Theorem 4 consists of several steps. We start by proving the *restricted approximate controllability* for an auxiliary control problem with controls acting on the boundary

Theorem 5

Let $y_0 \in C(\overline{\Omega} - \omega)$, $\epsilon > 0$ fixed and let $y_d \in C(\overline{\Omega} - \omega)$ satisfying (16). Then there exists $u_\epsilon \in C([0,T] \times \partial\omega)$ such that if $\tilde{y}(t,x:u_\epsilon)$ denotes the solution of the problem

$$(\mathcal{P}_{\Omega-\omega}) \begin{cases} \tilde{y}_t - \Delta\tilde{y} + \lambda|\tilde{y}|^{p-2}\tilde{y} = 0 & in \ (0,T) \times (\Omega - \omega) \\ \tilde{y} = u_\epsilon & on \ (0,T) \times \partial\omega \\ \frac{\partial\tilde{y}}{\partial n} = 0 & on \ (0,T) \times \partial\Omega \\ \tilde{y}(0,x) = y_0(x) & on \ \Omega - \overline{\omega}, \end{cases}$$

we have

$$\|\tilde{y}(T,\cdot:u_\epsilon) - y^d(\cdot)\|_{C(\Omega-\omega)} \leq \epsilon.$$

The proof of Theorem 5 uses another auxiliary result:

Lemma 1

Let G be an open regular bounded set of $\mathbb{R}^N$. For $a \in L^\infty((0,T) \times G)$ and $y_0 \in C(\overline{G})$ given we denote by $y(t,x:u)$ the solution of the linear control problem

$$(PL) \begin{cases} y_t - \Delta y + ay = 0 & in \ (0,T) \times G \\ y = u & on \ (0,T) \times \partial_D G \\ \frac{\partial y}{\partial n} = 0 & on \ (0,T) \times \partial_N G \\ y(0,\cdot) = y_0(\cdot) & on \ G. \end{cases}$$

where $\partial G = \partial_D G \cup \partial_N G$. Let $\epsilon > 0$ and $y_d \in C(\overline{G})$. Then (i) There exists $u_\epsilon \in C^0([0,T] \times \partial_D G)$ such that

$$\|y(T,\cdot:u_\epsilon) - y_d(\cdot)\|_{C(\overline{G})} \leq \epsilon. \tag{17}$$

Moreover, there exists $q = q(N) > 0$ and two functions $\hat{\varphi}$ and h such that

$$u_\epsilon(t,x) = h(t,x)\|(T-t)^q \frac{\partial\hat{\varphi}}{\partial n}(t,\cdot)\|_{L^1((0,T)\times\partial_D G)} \tag{18}$$

with

$$h(t,x) \in sign(\frac{\partial\hat{\varphi}}{\partial n}(t,x)) \quad \forall(t,x) \in (0,T) \times \partial_D G. \tag{19}$$

(ii) If $a \geq 0$ a.e. on $(0,T) \times G$, the function u_ϵ given in (18) satisfies that

$$\|u_\epsilon\|_{C([0,T]\times\partial_D G)} \leq C \tag{20}$$

for some $C > 0$ independent of a.

Sketch of the proof. Part (i) is an adaptation of the duality method introduced in [23] and [14]. We start by defining the space $V = C(\overline{G})$ and let V' its dual (i.e. the set of Baire measures of bounded variation: Yosida [33] p. 119). Given $\varphi_0 \in V'$ we consider the retrograde problem

$$(PLR) \begin{cases} -\varphi_t - \Delta\varphi + a\varphi = 0 & in \ (0,T) \times G \\ \varphi = 0 & on \ (0,T) \times \partial_D G \\ \frac{\partial\varphi}{\partial n} = 0 & on \ (0,T) \times \partial_N G \\ \varphi(T,x) = \varphi_0(x) & on \ G. \end{cases}$$

As in Proposition 5.5 of [14] it can be shown that there exists a positive number q (depending on the dimension N) such that the solution φ of (PLR) satisfies that

$$L(\varphi_0,a) := (T-t)^q \frac{\partial\varphi}{\partial n} \in L^1((0,T) \times \partial_D G). \tag{21}$$

We introduce the functional

$$J(\varphi_0:a,y_d) := \frac{1}{2}\left(\int_0^T \int_{\partial_D G} |(T-t)^q \frac{\partial\varphi}{\partial n}(\sigma,t)|d\sigma dt\right)^2 + \epsilon\|\varphi_0\|_{V'} - <y_d, \varphi_0>_{V\times V'}$$

It is clear that J is a strictly convex and continuous function on V'. Moreover using the unique continuation theorem (see [25] and [29]) J is a coercive functional

$$\liminf_{|\varphi_0|_{V'} \to \infty} \frac{J(\varphi_0 : a, y_d)}{|\varphi_0|_{V'}} \geq \epsilon \tag{22}$$

(see Proposition 2.1 of [14]) and so J achieves its minimum at a unique point $\hat{\varphi}_0$ in V'. The associated Euler-Lagrange equation implies the existence of h satisfying

$$h \in sign(L(\hat{\varphi}_0 : a))\chi_{[0,T] \times \partial_D G}$$

and

$$0 = \int_0^T \int_{\partial_D G} (T-t)^q \frac{\partial \theta}{\partial n} h d\sigma dt + \epsilon[|\hat{\varphi}_0 + \theta_0|_{V'} - |\hat{\varphi}_0|_{V'}] - < y_d, \theta_0 >_{VV'} \tag{23}$$

for any $\theta_0 \in V'$ and where θ denotes the solution of (PLR) replacing φ_0 by θ_0. On the other hand, multiplying by θ the equation of (PL) (with u given by (18))

$$\int_G y(T,x)\theta_0(x)dx = -\int_0^T \int_{\partial_D G} u_\epsilon(\sigma,t)\frac{\partial \theta}{\partial n}(\sigma,t)d\sigma dt. \tag{24}$$

From (23) and (24) we get

$$< y_d - y(T,\cdot), \theta_0 >_{V'V} \leq \epsilon(|\hat{\varphi}_0 + \theta_0|_{V'} - |\hat{\varphi}_0|_{V'}) \leq \epsilon|\theta_0|_{V'}$$

and in consequence

$$\|y(T,\cdot) - y_d\|_{C(\overline{G})} \leq \sup_{\theta_0 \in V'} \frac{< y_d - y(T,\cdot), \theta_0 >_{V'V}}{\|\theta_0\|_{V'}} \leq \epsilon.$$

In order to prove part (ii) we denote by $\hat{\varphi}_{0,+}$ and $\hat{\varphi}_{0,+}$ the positive and negative parts of $\hat{\varphi}_0$. Let $\hat{\varphi}$ the solution of (PLR) corresponding to the initial datum $\hat{\varphi}_0$ and let ψ_+ and let ψ_- the solutions of (PLR) assuming $a = 0$ in the equation and corresponding to initial data $\hat{\varphi}_{0,+}$ $\hat{\varphi}_{0,-}$ respectively. Then, by the comparison principle we have $\psi_- \leq \hat{\varphi} \leq \psi_+$, $\psi_- \leq 0$ and $\psi_+ \geq 0$ in $(0,T) \times G$. Besides

$$\frac{\partial \psi_-}{\partial n} \geq \frac{\partial \hat{\varphi}}{\partial n} \geq \frac{\partial \psi_+}{\partial n} \quad \text{on } (0,T) \times \partial_D G. \tag{25}$$

Then, for any $\varphi_0 \in V'$ we have

$$J(\varphi_0 : a, y_d) \leq I(\varphi_0 : y_d) \tag{26}$$

where I is the functional (independent of a) given by

$$I(\varphi_0 : y_d) := \frac{1}{2}\left(\int_0^T \int_{\partial_D G}(T-t)^q \max\{|\frac{\partial \psi_-}{\partial n}(\sigma,t)|, |\frac{\partial \psi_+}{\partial n}(\sigma,t)|\}d\sigma dt\right)^2$$

$$+ \epsilon \|\varphi_0\|_{V'} - < y_d, \varphi_0 >_{V \times V'}.$$

From (22) we deduce that

$$\liminf_{|\varphi_0|_{V'} \to \infty} \frac{I(\varphi_0 : y_d)}{|\varphi_0|_{V'}} \geq \epsilon.$$

So, there exists $M > 0$ (independent of a) such that

$$I(\varphi_0 : y_d) \geq \frac{\epsilon}{2} \| \varphi_0 \|_{V'} \text{ assumed } \| \varphi_0 \|_{V'} \geq M.$$

This implies that if $\hat{\varphi}_0$ is the minimum of J in V' then there exists $\hat{M} > 0$ independently of a such that

$$\| \hat{\varphi}_0 \|_{V'} \leq \hat{M}. \tag{27}$$

Using (25), (27) and (18) we get (20). ∎

Proof of Theorem 5. From assumption (16) and the construction of $\underline{Y}_\infty$ and $\overline{Y}_\infty$ we deduce that there exists $N_0 \in I\!N$ such that

$$\underline{Y}_N(T,x) - 2\varepsilon \leq y_d(x) \leq \overline{Y}_N(T,x) + 2\varepsilon \quad \forall x \in \Omega - \overline{\omega}.$$

Let $N \geq N_0$ large enough and define

$$f_N(s) = \begin{cases} -\lambda N^p & \text{if} & s \leq -N \\ \lambda|s|^{p-1}s & \text{if} & -N \leq s \leq N \\ \lambda N^p & \text{if} & s \geq N. \end{cases}$$

Since f_N is a (globally) Lipschitz function and bounded, as in Theorem 1.2 of [14], there exists $u_\epsilon^N \in \mathcal{C}([0,T] \times \partial\omega)$ such that if $y^*(t,x : u_\epsilon^N)$ denotes the solution of

$$\begin{cases} y_t^* - \Delta y^* + f_N(y^*) = 0 & \text{in } (0,T) \times (\Omega - \overline{\omega}) \\ y^* = u_\epsilon^N & \text{on } (0,T) \times \partial\omega \\ \frac{\partial y^*}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega \\ y^*(0,x) = y_0(x) & \text{on } \Omega - \overline{\omega} \end{cases}$$

the we have

$$\| y^*(T,\cdot : u_\epsilon^N) - y^d(\cdot) \|_{C(\overline{\Omega}\backslash\omega)} \leq \varepsilon.$$

Moreover such a control u_ϵ^N can be found as a fixed point of the application $\Lambda : \mathcal{C}([0,T] \times (\overline{\Omega} - \omega)) \to \mathcal{P}(\mathcal{C}([0,T] \times (\overline{\Omega} - \omega)))$ defined by

$$\Lambda(z) = \{y(\cdot,\cdot : u) \text{ solution of (PL) with } a = \frac{f_N(z)}{z} \text{ and } u \text{ satisfying (17), (18)}\}.$$

From estimate (20) of Lemma 1 we deduce that if u_ϵ^N is a fixed point of Λ it must satisfy

$$\| u_\epsilon^N \|_{C([0,T] \times \partial_D G)} \leq C$$

with C (independent of N) given in (20). Then, by the maximum priciple we conclude that if $N \geq N_0$ is large enough then the function $y^*(t, x : u_\varepsilon^N)$ satisfies

$$|y^*(t, x : u_\varepsilon^N)| \leq N \quad \forall\, (t, x) \in [0, T] \times (\overline{\Omega} - \omega)$$

and so, in fact, $y^*(0, \cdot : u_\varepsilon^N)$ satisfies the requirements of the statement of Theorem 5. ▨

In order to complete the proof of Theorem 2 we need to use some other auxiliary results.

Lemma 2 *(Díaz and Fursikov [9])*

Let $u_\varepsilon \in \mathcal{C}([0, T] \times \partial\omega)$ fixed. There exists $\hat{v}_\varepsilon \in \mathcal{C}([0, T] \times \overline{\omega})$ such that the solution $\hat{y}(t : \hat{v}_\varepsilon)$ of

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + \lambda|\hat{y}|^{p-1}\hat{y} = \hat{v}_\varepsilon & in\ (0, T) \times \omega \\ \hat{y} = u_\varepsilon & on\ (0, T) \times \partial\omega \\ \hat{y}(0, \cdot) = y_0(\cdot) & on\ \omega \end{cases}$$

satisfies

$$\|\hat{y}(T : \hat{v}_\varepsilon) - y_d\|_{\mathcal{C}(\overline{\omega})} \leq \varepsilon.$$

We would need to regularize the matching between the functions $\tilde{y}$ and $\hat{y}$ given in Theorem 5 and Lemma 2 respectively.

Lemma 3

Let ω_ε be an open regular subset of ω such that $d(\omega_\varepsilon, \partial\omega) \leq \varepsilon$. Then there exists $y^* \in \mathcal{C}([0, T] \times \overline{\Omega}) \cap \mathcal{C}^2((0, T) \times \Omega)$ such that $y^* = \hat{y}$ on $[0, T] \times \overline{\omega}_\varepsilon$ and $y^* = \tilde{y}$ on $[0, T] \times (\overline{\Omega} - \omega)$.

The proof of this result uses standard regularization techniques and the details are left to the reader. The last technical result is consequence of the continuous dependence of the solutions of problem (P_p) with respect to different initial data.

Lemma 4

Let y^* be the function given in Lemma 3. Define

$$v_\varepsilon := y_t^* - \Delta y^* + \lambda|y^*|^{p-1}y^* \quad in\ (0, T) \times \Omega.$$

Then $v_\varepsilon = 0$ on $(0, T) \times (\Omega - \overline{\omega})$ and if $y(t, \cdot : v_\varepsilon)$ is the corresponding solution of (P_p) we have

$$\|y^*(t, \cdot) - y(t, \cdot : v_\varepsilon)\|_{\mathcal{C}(\overline{\Omega})} \leq \varepsilon \quad \forall\, t \in [0, T].$$

Proof of Theorem 2. Let v_ε be the function defined in Lemma 4. Then using Theorem 5 and lemmas 2, 3 and 4 we have that

$$\begin{aligned} \|y(T : v_\varepsilon) - y_d\|_{\mathcal{C}(\overline{\Omega})} &\leq \|y^*(T, \cdot) - y(T, \cdot : v_\varepsilon)\|_{\mathcal{C}(\overline{\Omega})} + \|y^*(T, \cdot) - y_d\|_{\mathcal{C}(\overline{\Omega})} \\ &\leq \|y^*(T, \cdot) - y(T, \cdot : v_\varepsilon)\|_{\mathcal{C}(\overline{\Omega})} + \|\tilde{y}(T, \cdot) - y_d\|_{\mathcal{C}(\overline{\Omega}-\omega)} \\ &\quad + \|y^*(T, \cdot) - y_d\|_{\mathcal{C}(\overline{\omega})} + \varepsilon \leq 4\varepsilon \end{aligned}$$

and the conclusion holds. ▨

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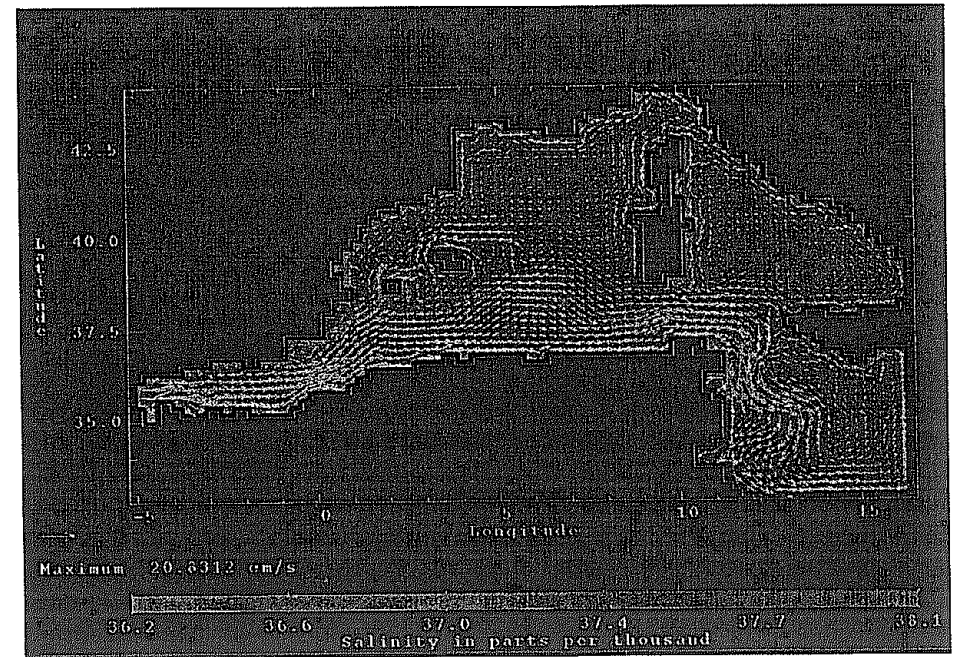
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Grupo de Análisis Matemático Aplicado de la Universidad de Málaga
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MODELADO DE SISTEMAS EN OCEANOGRAFIA, CLIMATOLOGIA Y CIENCIAS MEDIO-AMBIENTALES: ASPECTOS MATEMATICOS Y NUMERICOS