APPROXIMATE CONTROLLABILITY FOR SOME NONLINEAR PARABOLIC PROBLEMS

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1 Introduction.

The main goal of this work is to present several results on the controllability of some nonlinear parabolic problems, mainly the semilinear problem

$$\text{(SL)} \begin{cases} y_t - \Delta y + f(y) = v\chi_\omega & \text{in } Q = \Omega \times (0,\mathrm{T}) \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0,\mathrm{T}) \\ y(\cdot,0) = y_0(\cdot) & \text{on } \Omega \end{cases}$$

where Ω is a bounded regular set of $\mathbb{R}^n$, ω is an open subset of Ω, χ_ω denotes the characteristic function of ω, $\mathrm{T} > 0$ is fixed and the initial datum y_0 is given in a functional space, *e.g.* $y_0 \in \mathrm{L}^2(\Omega)$.

The nonlinear term is given by the real function f and the control is represented by the function $v \in \mathrm{L}^2(\omega \times (0,\mathrm{T}))$. As usual, the study of the semilinear problem is carried out by considering previously some suitable linear problem

$$\text{(L)} \begin{cases} y_t - \Delta y + ay = v\chi_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(\cdot,0) = y_0(\cdot) & \text{on } \Omega, \end{cases}$$

where $a \in \mathrm{L}^\infty(Q)$ is given.

Due to the smoothing effect of parabolic equations the notion of controllability (*exact controllability*) must be relaxed: We say that the *approximate controllability property* holds for the problem (SL) (respectively (L)) if given $y_d \in \mathrm{L}^2(\Omega)$ and $\varepsilon > 0$ there exist $v \in \mathrm{L}^2(\omega \times (0,\mathrm{T}))$ and $y(\mathrm{T}:v)$ solution of (SL) (respectively (L)) satisfying

$$\|y(\mathrm{T}:v) - y_d\|_{\mathrm{L}^2(\Omega)} \le \varepsilon.$$

We start, in Section 2, by collecting some abstract and constructive proofs of the approximate controllability for the linear problem (L). The nonlinear case may yield different answers according to the behaviour of the function f near the infinity. This is presented in Section 3 jointly with some remarks about other nonlinear problems.

2 The approximate controllability for linear problems.

The study of the approximate controllability for linear parabolic problems has been developped on different levels of abstraction (a survey containing many references up to 1978 is due to D.Russell **[30]**). A very elegant proof of this property for the formulation (L) is due to J.L.Lions

Theorem 1 ([22])
The approximate controllability holds for problem (L).

Proof.
By linearity we can assume $y(\mathrm{T}:0) \equiv 0$. Let us show that if $g \in \mathrm{L}^2(\Omega)$ satisfies

$$(y(\mathrm{T}:v), g) = 0 \quad \forall v \in \mathrm{L}^2(\omega \times (0,\mathrm{T})), \tag{1}$$

then necessarily $g \equiv 0$. In that case the conclusion comes from a corollary of the Hahn-Banach Theorem. Here $(\cdot,\cdot)$ denotes the scalar product over $\mathrm{L}^2(\Omega)$. Define ψ as the unique solution of the time-reversed problem

$$\begin{cases} -\psi_t - \Delta\psi + a\psi = 0 & \text{in } Q \\ \psi = 0 & \text{on } \Sigma \\ \psi(\cdot,\mathrm{T}) = g(\cdot) & \text{on } \Omega. \end{cases}$$

Multiplying by y, integrating by parts and using (1) we get that

$$\int\int_{\Omega\times(0,\mathrm{T})} \psi v \chi_\omega dx dt = 0 \quad \forall v \in \mathrm{L}^2(\omega \times (0,\mathrm{T})).$$

In particular, $\psi \equiv 0$ on $\omega \times (0,\mathrm{T})$. Using the Unique Continuation Theorem (due to Mizohata **[28]** for $a \in \mathcal{C}^\infty(Q)$ and Saut-Scheurer **[31]** for $a \in \mathrm{L}^\infty(Q)$) we deduce that $\psi \equiv 0$ on $\Omega \times (0,\mathrm{T})$ and so $g \equiv 0$ on Ω. □

Other variants of the Hahn-Banach Theorem can be used to prove approximate controllability results under some contraints on the controls and/or the state. For instance, in many physical applications only nonnegative controls are admissible. The density of the range set $\{y(\mathrm{T}:v)\}$ when the control acts on Σ was first proved in Díaz [3]. A similar result for the formulation (L) is the following

Theorem 2 ([8])
Let $\mathcal{U}$ be a dense subset of

$$\mathrm{L}^2_+(\omega\times(0,\mathrm{T}))\left(:=\left\{v\in\mathrm{L}^2(\omega\times(0,\mathrm{T})):v\geq 0\ a.e.\right\}\right).$$

Then $\{y(\mathrm{T}:v):y$ solution of (L) $,v\in\mathcal{U}\}$ is a dense subset of $y(\mathrm{T}:0)+\mathrm{L}^2_+(\Omega)$.

Proof.
We start by giving a proof for the case $\omega=\Omega$. Again, without loss of generality we can assume $y(\mathrm{T}:0)\equiv 0$. By linearity $\mathrm{F}:=\{y(\mathrm{T}:v):y$ solution of (L)$,v\in\mathcal{U}\}$ is such that $\overline{F}$ is a convex set. Then, assumed that there exists $y_1\in\mathrm{L}^2_+(\Omega)\setminus\overline{F}$, by the Hahn-Banach Theorem (in its geometrical form) we can separate y_1 from $\overline{F}$, *i.e.* there exists $\alpha\in\mathbb{R}$ and $g\in\mathrm{L}^2(\Omega)$ such that

$$(y(\mathrm{T}:v),g)<\alpha<(y_1,g)\quad\forall v\in\mathcal{U}.\tag{2}$$

Taking $v\equiv 0$ we deduce that $\alpha>0$. If ψ is given as in the previous proof, multiplying by y, integrating by parts and using (2) we conclude that

$$\int_Q\psi v\mathrm{d}x\mathrm{d}t\leq 0\quad\forall v\in\mathcal{U}.$$

Then $\psi\leq 0$ in Q which implies $g\leq 0$: A contradiction with (2).
A sketch of the proof for the general case $\omega\subset\Omega$ is as follows: assume that there exists a $g\in\mathrm{L}^2_+(\Omega)$ such that $g\notin\overline{\mathrm{F}}$. By the Projection Theorem there exists a unique $u\in\overline{\mathrm{F}}$ such that

$$(g-u,p-u)\leq 0\quad\forall p\in\overline{\mathrm{F}}.$$

Moreover, as $\overline{\mathrm{F}}$ is a convex closed cone we can take $p=e+u$ and $p=0$ respectively and obtain

$$(g-u,e)\leq 0\quad\forall e\in\overline{\mathrm{F}}\tag{3}$$
$$(g-u,u)=0.\tag{4}$$

Now, let $q\in\mathcal{C}([0,\mathrm{T}]:\mathrm{L}^2(\Omega))$ be the solution of the problem

$$\begin{cases}-q_t-\Delta q+aq=0 & \text{in } Q\\ q=0 & \text{on } \Sigma\\ q(\cdot,\mathrm{T})=g(\cdot)-u(\cdot) & \text{on } \Omega.\end{cases}\tag{5}$$

Multiplying (5) by z, with $z\in\mathrm{F}$ arbitrary, we obtain

$$0\geq\int_\Omega(g(x)-u(x))z(x,\mathrm{T})\mathrm{d}x=\int_{\omega\times(0,\mathrm{T})}qv\mathrm{d}x\mathrm{d}t$$

for any $v\in\mathcal{U}$. From the assumption on $\mathcal{U}$ we deduce that $q\leq 0$ on $\omega\times[0,\mathrm{T}]$. In particular,

$$0\leq g(x)\leq u(x)\quad a.e.\ x\in\omega$$

and

$$q\leq 0\quad\text{on } \partial\omega\times(0,\mathrm{T}).$$

Then by the Strong Maximum Principle (on the domain $\omega\times(0,\mathrm{T})$) we deduce that either $q\equiv 0$ on $\Omega\times(0,\mathrm{T})$ or $q<0$ on $\omega\times(0,\mathrm{T})$. But $q\equiv 0$ implies that $g=u$ which contradicts that $g\notin\overline{\mathrm{F}}$. Moreover, we have

$$0=(g-u,u)=\int_{\omega\times(0,\mathrm{T})}qv_0\mathrm{d}x\mathrm{d}t,$$

where we can assume, without loss of generality, that $u=y(\mathrm{T}:v_0)$, with $v_0\in\mathcal{U}$. Now, if $q<0$ on $\omega\times(0,\mathrm{T})$ we conclude that $v_0\equiv 0$ on $\omega\times(0,\mathrm{T})$. This implies that $u\equiv 0$ on Ω and from (3) we have that $g\leq 0$ on Ω, *i.e.* $g\equiv 0$ which is a contradiction. □

The rest oh this section will be devoted to the question of the construction of a sequence of control $\{v_k\}_{k\in\mathbb{N}}$ satisfying that $y(\mathrm{T}:v_k)\to y_d$ as $k\to\infty$. A first idea introduced in Lions [23] is to consider the auxiliary control problem

$$(\mathcal{P}_k)\ \begin{cases}\inf\{J_k(v):v\in\mathrm{L}^2(\omega\times(0,\mathrm{T}))\},\\ J_k(v)=\frac{1}{2}\|v\|^2_{\mathrm{L}^2(\omega\times(0,\mathrm{T}))}+\frac{k}{2}\|y(\mathrm{T}:v)-y_d\|^2_{\mathrm{L}^2(\Omega)}.\end{cases}$$

Theorem 3 ([23])
Assume (for simplicity) $y_0\equiv 0$. Then a) *Problem $(\mathcal{P}_k)$ has a unique solution v_k and $y(\mathrm{T}:v_k)\to y_d$ as $k\to\infty$.* b) *We have the characterization $v_k=-kp_k\chi_\omega$ where (y_k,p_k) satisfies the optimality system*

$$(\mathcal{P}_k^*)\ \begin{cases}y_t-\Delta y+ay+kp\chi_\omega=0 & in\ Q\\ -p_t-\Delta p+ap=0 & in\ Q\\ y=p=0 & on\ \Sigma\\ y(\cdot,0)=0,\ p(\cdot,\mathrm{T})=y(\cdot,\mathrm{T})-y_d(\cdot) & on\ \Omega.\end{cases}$$

Idea of the proof.

a) The existence and uniqueness of v_k solution of $(\mathcal{P}_k)$ follow from well-known results ([22]). By Theorem 1 given $\varepsilon > 0$ there exists $v_\varepsilon \in L^2(\omega \times (0,T))$ such that

$$\|y(T:v) - y_d\|_{L^2(\Omega)} \le \frac{\varepsilon}{2}.$$

Then, as $J_k(v_k) \le J_k(v_\varepsilon)$ we have

$$k\|y(T:v_k) - y_d\|^2_{L^2(\Omega)} \le \|v_\varepsilon\|^2_{L^2(\omega\times(0,T))} + \frac{k\varepsilon^2}{4}$$

and so $y(T:v_k) \to y_d$ in $L^2(\Omega)$ as $k \to \infty$.
b) Next, it is enough to remark that the Euler equation associated to $(\mathcal{P}_k)$ is

$$\int_{\omega\times(0,T)} v_k v \,dx dt + k\int_\Omega (y(T:v_k) - y_d)\, y(T:v) dx = 0 \quad \forall v \in L^2(\omega \times (0,T))$$

and that this is satisfied for the function $-kp_k\chi_\omega$ assuming that (y_k, p_k) satisfies $(\mathcal{P}^*_k)$. □

Remark 1
System (P^*_k) can be treated directly *i.e.* without using the fact that (P^*_k) is the optimality system of the problem (P_k). So, in Lions **[23]** the existence and uniqueness of a solution (y_k, p_k) of (P^*_k) are shown, as well as that $y_k(T) \to y_d$ in $L^2(\Omega)$ as $k \to \infty$. We also remark that the system (P^*_k) remains still the optimality system of the problem

$$(\mathcal{P}^+_k) \qquad \inf\{J_k(v) : v \in L^2_+(\omega \times (0,T))\}.$$

This is shown in Díaz-Henry-Ramos [8]. The statement of Theorem 3 remains the same. The proof of b) uses the fact that the Euler equation of $(\mathcal{P}^+_k)$ becomes now the variational inequality

$$\int_{\omega\times(0,T)} v_k(v - v_k) dx dt + k\int_\Omega (y(T:v_k) - y_d)\,(y(T:v) - y(T:v_k))\, dx \ge 0$$

$\forall v \in L^2_+(\omega \times (0,T))$. □

A second constructive method use some *duality arguments* which are inspired on the HUM (Hilbert Uniqueness Method), introduced by J.L.Lions for the study of the exact controllability. We start by formulating the approximate controllability property in the following terms: *Given $\varepsilon > 0$ and $y_d \in L^2(\Omega)$ find a control $v \in L^2(\omega \times (0,T))$ such that $y(T:v) \in y_d + \varepsilon B$, where B denotes the unit ball in $L^2(\Omega)$.*

As pointed out in Lions **[25]**, it is easy to see that, as a matter of fact, there are *infinitely many controls v* driving the system from the initial datum y_0 to the ball $y_d + \varepsilon B$

at time T. Indeed, let $\delta \in (0,T)$ arbitrary. We take $v = \overline{v}_\delta$ arbitrary in $L^2(\omega \times (0,\delta))$. Let $y_\delta = y(\delta : v)$. Then, according Theorem 1 there exists a control $\hat{v}_\delta \in L^2(\omega \times (\delta, T))$ driving the system from y_δ to $y_d + \varepsilon B$. Then

$$v_\delta(x,t) = \begin{cases} \overline{v}_\delta(x,t) & \text{for } x \in \omega \text{ and } 0 < t < \delta, \\ \hat{v}_\delta(x,t) & \text{for } x \in \omega \text{ and } \delta \le t < T \end{cases}$$

satisfies the required property (it leads to the system for y_0 to a state $y(T:v_\delta)$ in $y_d + \varepsilon B$). In consequence, it is then natural to ask for the optimal control driving the system from y_0 to the ball $y_d + \varepsilon B$. The problem possed in Lions **[25]** is the following: *Given $\varepsilon > 0$ and $y_d \in L^2(\Omega)$ find*

$$(\mathcal{P}_\varepsilon) \qquad \inf\{\|v\|_{L^2(\omega\times(0,T))} : y(T:v) \in y_d + \varepsilon B\}.$$

If $\|y_d\|_{L^2(\Omega)} \le \varepsilon$ this problem has the trivial solution $v = 0$ (since $y(T:0) = 0 \in y_d + \varepsilon B$). So, in what follows we assume that

$$\|y_d\|_{L^2(\Omega)} > \varepsilon.$$

Theorem 4 ([25])
Problem $(\mathcal{P}_\varepsilon)$ has a unique solution $v_\varepsilon \in L^2(\omega \times (0,T))$. Moreover, $v_\varepsilon = \hat{\varrho}\chi_\omega$, where $\hat{\varrho}$ is the unique solution of the auxilary problem

$$\left.\begin{array}{ll} -\varrho_t - \Delta\varrho + a\varrho = 0 & \text{in } Q \\ \varrho = 0 & \text{on } \Sigma \\ \varrho(\cdot, T) = \varrho_0(\cdot) & \text{on } \Omega \end{array}\right\} \tag{6}$$

and $\varrho_0 = \hat{\varrho}_0$ is given by the minimization problem

$$\begin{cases} \inf\{I(\varrho_0, y_d, \varepsilon) : \varrho_0 \in L^2(\Omega)\}, \\ I(\varrho_0, y_d, \varepsilon) = \frac{1}{2}\int_{\omega\times(0,T)} \varrho^2 dx dt + \varepsilon\|\varrho\|_{L^2(\Omega)} - \int_\Omega y_d \varrho_0 dx. \end{cases} \tag{7}$$

(Here ϱ is the solution of the problem (6)).

Idea of the proof.
Define the functionals

$$F(v) = \frac{1}{2}\int_{\omega\times(0,T)} v^2 dx dt, \quad G(v) = \begin{cases} 0 & \text{if } f \in y^1 + \varepsilon B \\ +\infty & \text{otherwise} \end{cases}$$

$$L \in \mathcal{L}\left(L^2(\omega \times (0,T)) : L^2(\Omega)\right), \qquad Lv = y(T:v).$$

Then problem $(\mathcal{P}_\varepsilon)$ is equivalent to

$$\inf\{F(v)+G(Lv) : v \in L^2(\omega \times (0,T))\}.$$

Using the Fenchel-Rockafellar Duality Theorem (see *e.g.* Ekeland-Temam [18]) we have that

$$\inf\{F(v)+G(Lv) : v \in L^2(\omega \times (0,T))\} = -\inf\{F^*(L^*\varrho_0)+G^*(-\varrho_0) : \varrho_0 \in L^2(\Omega)\}.$$

where in general ϕ^* denotes the convex dual of a proper function $\phi : H \to]-\infty,+\infty]$ on a Hilbert space H. It is not dificult to see that

$$L^*\varrho_0 = -\varrho\chi_\omega, \quad G^*\varrho_0 = \int_\Omega y_d \varrho_0 dx + \varepsilon\|\varrho_0\|_{L^2(\Omega)}, \quad F^* = F,$$

and the conclusion holds. □

This second constructive method was sistematically developed in Fabré-Puel-Zuazua [12],[13]. By introducing suitable variants of the functional $I(\varrho_0, y_d, \varepsilon)$ and studying the associated minimization property they obtain the approximate controllability in $L^p(\Omega)$ for $1 \le p < \infty$ and $C_0(\Omega)$. Moreover they show that the wanted controls are of the type 'quasi bang-bang' (*i.e.* they take only the values $-k$ and k, for some suitable $k > 0$, except a set of points which at least has empty interior).

Theorem 5 ([12],[13])
Let $a \in L^\infty(Q)$ and denote by $\mathcal{X} = L^p(\Omega)$ with $1 \le p < \infty$ either $\mathcal{X} = C_0(\Omega)$ (the space of uniformly continuous functions in Ω that vanish on $\partial\Omega$ endowed with the norm of supremum) and $\mathcal{X}' = L^{p'}(\Omega)$ ($\frac{1}{p}+\frac{1}{p'} = 1$ if $1 < p < \infty$, $p' = \infty$ if $p = 1$) either $\mathcal{X}' = \mathcal{M}(\Omega)$ (the space of bounded measures on Ω) respectively. Let $y_d \in \mathcal{X}$ such that $\|y_d\| > \varepsilon$. Given $\varrho_0 \in \mathcal{X}'$ consider the auxiliar problem

$$\left.\begin{array}{ll} -\varrho_t - \Delta\varrho + a\varrho = 0 & \text{in } Q \\ \varrho = 0 & \text{on } \Sigma \\ \varrho(\cdot,T) = \varrho_0(\cdot) & \text{on } \Omega \end{array}\right\} \tag{8}$$

and define the functional

$$J(\varrho_0, y_d, \varepsilon) = \frac{1}{2}\left(\int_{\omega\times(0,T)} |\varrho| dx dt\right)^2 + \varepsilon\|\varrho_0\|_{\mathcal{X}'} - \langle y_d, \varrho_0 \rangle_{\mathcal{X}\mathcal{X}'}.$$

Then:
1. $J(\cdot, y_d, \varepsilon)$ is a real strictly convex continuous and coercive function. In particular, it achieves its minimum at a unique point $\hat{\varrho}_0 \in \mathcal{X}'$.

2. If $\hat{\varrho}$ denotes the solution of (8) for $\varrho_0 = \hat{\varrho}_0$ there exists $w \in \operatorname{sign}(\hat{\varrho})\chi_\omega$ such that the solution of

$$\left.\begin{array}{ll} y_t - \Delta y + ay = |\hat{\varphi}|_{L^1(\omega\times(0,T))} w\chi_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(\cdot,0) = 0 & \text{on } \Omega \end{array}\right\}$$

satisfies that $\|y(T) - y_d\|_{\mathcal{X}} \le \varepsilon$. □

Remark 2
In Fabré-Puel-Zuazua [14] the optimal control problem $(\mathcal{P}_\varepsilon)$ associated to the L^2-approximate controllability is studied but assuming that $v \in L^r(\omega \times (0,T))$, $2 \le r \le +\infty$ and replacing $\|v\|_{L^2(\omega\times(0,T))}$ by $\|v\|^2_{L^r(\omega\times(0,T))}$. □

Remark 3
Many of the above results remain valid for other linear parabolic problems. This is the case of the Stokes problem

$$\left.\begin{array}{ll} \vec{y}_t - \Delta\vec{y} = -\nabla\vec{y} + \vec{v}\chi_\omega & \text{in } Q \\ \operatorname{div} \vec{y} = 0 & \text{in } Q \\ \vec{y} = \vec{0} & \text{on } \Sigma \\ \vec{y}(\cdot,0) = \vec{y}_0(\cdot) & \text{in } \Omega \end{array}\right\}$$

where $\vec{v} \in (L^2(\omega \times (0,T)))^n$. The approximate controllability is now formulated as the density of the set $\left\{\vec{y}(T : \vec{v}) : \vec{v} \in (L^2(\omega \times (0,T)))^n\right\}$ in $H = \left\{\vec{w} \in (L^2(\Omega))^n : \operatorname{div} \vec{w} = 0\right\}$. We point out that the property holds even for controls $\vec{v}$ of the type $\vec{v} = (v_1, v_2, 0)$ (see Lions [27] and Fursikov-Imanuvilov [17]). The very special case of the controls $\vec{v} = (v_1, 0, 0)$ leads also to a positive answer for suitable domains Ω of $\mathbb{R}^3$ (Díaz-Fursikov [7]). □

3 The approximate controllability for nonlinear problems.

In order to fix ideas we shall study the approximate controllability for the semilinear problem (SL) of the Introduction. Results for other nonlinear problems will be mentioned at the end of this Section.

As we shall see below, the results are of different nature according to whether the domain of controllability ω satisfies $\omega = \Omega$ or $\omega \subset\subset \Omega$.

3.1 The special case $\omega \equiv \Omega$.

The most favorable situation for which the approximate controllability holds correspond to when we can introduce arbitrary actions (controls) at any point of the domain. In that case it is possible to give a positive answer even in the case for which the existence and uniqueness of the solutions are not assured by the general theory. As pointed out at the Introduction, given y_d and $\varepsilon > 0$ the approximate controllability holds if we find a control v_ε and a function y_ε such that i) y_ε satisfies (SL) and ii) $\|y_\varepsilon(T) - y_d\| \leq \varepsilon$. Thus we merely need to justify the existence of a solution y_ε corresponding to the control v_ε but not the existence and uniqueness for an arbitrary control $v \in L^2(0,T : L^2(\Omega))$.

Theorem 6 ([6])
Let $y_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ and assume $f : \mathbb{R} \to \mathbb{R}$ to be continuous. Then the approximate controllability property holds for the problem (SL).

Proof.
Given $y_d \in L^2(\Omega)$ there exist two regular functions u_ε and z_ε such that $u_\varepsilon \in L^2(Q)$ z_ε satisfies (L) with $a \equiv 0$ $v = u_\varepsilon$ and verifies $\|z_\varepsilon(T,\cdot) - y_d\|_{L^2(\Omega)} \leq \varepsilon$. From standard regularity results we know that $z_\varepsilon \in L^2(0,T : H^2(\Omega)) \cap H^1(0,T : L^2(\Omega)) \cap L^\infty(Q)$. Then definying $v_\varepsilon = (z_\varepsilon)_t - \Delta z_\varepsilon + f(z_\varepsilon)$ and $y_\varepsilon = z_\varepsilon$ we have that $v_\varepsilon \in L^2(Q)$ and y_ε satisfies the required condition. □

Remark 4
Theorem 6 admits an arbitrary version ([6]) which is of special interest when the general theory does not assure the global existence of solutions (as, for instance, is the case if $f(s) = -|s|^{p-1}s$ with $p > 1$) or the uniqueness of solutions (case of $f(s) = -|s|^{p-1}s$ with $0 < p < 1$ or the three-dimensional Navier-Stokes problem). A pionneering result (assumming some additional conditions) can be found in Henry [19]. □

A more complicated situation arises when the controls (even actuating in the whole domain) are subject to some constraints. Here we adapt the so-called *cancellation method*, introduced in Henry [19], to the case of nonnegative controls.

Theorem 7 ([8])
Let f be a continuous nondecreasing (or Lipschitz continuous) function such that $f(0) = 0$. Let $\mathcal{U}$ be a dense subset of $L_+^2(\Omega)$. Then the set $\{y(T : v) : y$ solution of (SL) *and $v \in \mathcal{U}\}$ is a subset dense of $y(T : 0) + L_+^2(\Omega)$.*

Proof.
Without loss of generality we can assume $y_0 \equiv 0$ and thus $y(T : 0) = 0$. By Theorem 2, given $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{U}$ such that the solution y of (L), with $a \equiv 0$ and $y_0 \equiv 0$, satisfies $\|y(T : u_\varepsilon) - y_d\|_{L^2(\Omega)} \leq \varepsilon$. Now, let $\hat{u}_\varepsilon \in L_+^\infty(Q)$ with $\|u_\varepsilon - \hat{u}_\varepsilon\|_{L^2(Q)}$ small enough so that $\|y(T : u_\varepsilon) - y(T : \hat{u}_\varepsilon)\|_{L^2(\Omega)} \leq \varepsilon$. From the assumptions on f we know that $f(y(\cdot : \hat{u}_\varepsilon)) \in L^\infty(Q)$: indeed, if f is not increasing we introduce the change of unknown $\hat{y} = e^{\lambda t}y$ for a suitable $\lambda \in \mathbb{R}$. Now let $v_\varepsilon \in \mathcal{U}$ such that

$$\|v_\varepsilon - f(y(\cdot : \hat{u}_\varepsilon))\|_{L^2(\Omega)} \leq \varepsilon.$$

Finally, we consider $\tilde{y}$ solution of the auxiliar nonlinear problem

$$\begin{cases} \tilde{y}_t - \Delta\tilde{y} + f(y(\cdot : \hat{u}_\varepsilon) + \tilde{y}) = v_\varepsilon + u_\varepsilon - \hat{u}_\varepsilon & \text{in } Q \\ \tilde{y} = 0 & \text{on } \Sigma \\ \tilde{y}(\cdot, 0) = 0 & \text{on } \Omega. \end{cases}$$

Then the function $y(\cdot) := y(\cdot : \hat{u}_\varepsilon) + \tilde{y}$ satisfies (SL) and

$$\|y(T) - y_d\| \leq \|y(T : \hat{u}_\varepsilon) - y(T : u_\varepsilon)\| + \|y(T : u_\varepsilon) - y_d\| + \|\tilde{y}\| \leq 3\varepsilon.$$

The control v_ε is find, through the control of the linear problem, by the *cancellation* (at least approximately) of the nonlinear term. □

3.2 The case $\omega \subset\subset \Omega$ and f sublinear near the infinity.

When $\omega \subset\subset \Omega$ the answers to the approximate controllability question have different nature according to whether the nonlinear term $f(y)$ is sublinear or superlinear at the infinity. The following result concerns the sublinear case. It was obtained in [12], [13] under a global Lipschitz condition on f and later extended in [10] to the present statement.

Theorem 8 ([13],[10])
Let f be a continous function such that

$$|f(s)| \leq C_1 + C_2 s \quad \textit{if} \quad |s| > M, \quad \textit{for some } M, C_1, C_2 > 0 \tag{9}$$

and there exists $s_0 \in \mathbb{R}$, $C_3, \delta > 0$ such that

$$|f(s) - f(s_0)| \leq C_3|s - s_0| \quad \textit{for any } s \in (s - \delta, s + \delta). \tag{10}$$

Then the approximate controllability property holds for (SL).

Proof.
Define the function

$$g(s) = \begin{cases} \dfrac{f(s) - f(s_0)}{s - s_0} & \text{if } s \neq s_0 \\ 0 & \text{if } s = s_0. \end{cases}$$

By (10) there exists $K > 0$ such that $|g(s)| \leq K$ for any $s \in \mathbb{R}$. Given $z \in L^2(Q)$ and $v \in L^2(Q)$ we define the auxiliary functions $e(\cdot : z)$ and $y(\cdot : z)$ as the solutions of the *linear* problems

$$\left.\begin{array}{ll} e_t - \Delta e + g(z)e = -f(s_0) + g(z)s_0 & \text{in } Q \\ e = 0 & \text{on } \Sigma \\ e(\cdot, 0) = y_0(\cdot) & \text{on } \Omega, \end{array}\right\}$$

and

$$\left.\begin{array}{ll} y_t - \Delta y + g(z)y = v\chi_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(0, \cdot) = 0 & \text{on } \Omega. \end{array}\right\}$$

By Theorem 1 we can chose $v = v(z) \in L^2(\omega \times (0,T))$ such that

$$\|y(T) - y_d + e(T)\|_{L^2(\Omega)} \leq \varepsilon.$$

Moreover, the function $y := e + y$ satisfies

$$\left.\begin{array}{ll} y_t - \Delta y + g(z)y = -f(s_0) + g(z)s_0 + v\chi_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(\cdot, 0) = y_0(\cdot) & \text{on } \Omega, \end{array}\right\} \qquad (11)$$

and

$$\|y(T) - y_d\|_{L^2(\Omega)} \leq \varepsilon. \qquad (12)$$

Consider now the multivalued mapping $\Lambda : L^2(Q) \to \mathcal{P}(L^2(Q))$ given by

$$\Lambda z = \{y : \text{ satisfying (11) and (12)}\}.$$

It can be shown that Λ verifies the assumptions of the Kakutany Fixed Point Theorem and so there exists y solution of (11) with $z = y$ and satisfies (12) which ends the proof by using the definition of g. □

By applying Theorem 2 (instead of Theorem 1) in the above proof we can improve the conclusion of Theorem 8 relative to nonnegative controls.

Corollary 1 ([8])
Let f satisfying (9) and (10). Let $\mathcal{U}$ be a dense subset of $L^2_+(\omega \times (0,T))$. Then $\{y(T : v) : y$ solution of (SL), $v \in \mathcal{U}\}$ is a dense subset of $y(T : 0) + L^2_+(\Omega)$. □

Remark 5
The Kakutany Fixed Point Theorem was also used in Henry [19]. The applicability of other fixed point theorems can be found in Carmichael and Quinn [2]. We also remark that the programme of the above proof can be successfully applied to show the approximate

controllability in $L^p(\Omega)$ with $1 \leq p < \infty$ and in $\mathcal{C}_0(\Omega)$ (see [13]). Finally, we mention the work [5] where Theorem 8 was extended to a multivalued semilinear equation arising in Climatology. □

Remark 6
Assumptions (9) and (10) holds if, for instance, $f(s) = \lambda|s|^{p-2}s$ with $0 < p < 1$ and $\lambda \in \mathbb{R}$ (notice that this function is not globally Lipschitz). Abstract results on the approximate controllability for some nonlinear parabolic problems, using also some sublinear assumptions on the nonlinear terms, are due to Seidman [32] and Naito and Seidman [29]. □

3.3 The case $\omega \subset\subset \Omega$ and f superlinear near the infinity.

When the nonlinear term $f(y)$ is superlinear near the infinity there appears an *obstruction* over the solutions of the equation and the approximate controllability fails. This fact was first pointed out by A.Bamberger in [19] when

$$f(s) = \lambda|s|^{p-2}s, \qquad p > 1, \quad \lambda > 0 \qquad (13)$$

$\Omega = (0,1)$ and the internal control in (SL) is replaced by the homogeneous equation and the boundary control $y_x(0,t) = v(t)$. He uses an energy method to prove that $\|y(T : v)\|_{L^2(\Omega_\varepsilon)} \leq C$ with $\Omega_\varepsilon = (\varepsilon, 1)$, $0 < \varepsilon < 1$ and C independent of v. A different technique was used in Díaz [3] for f given by (13), Ω arbitrary and the boundary controls $y(t,x) = v(t,x)$, $(t,x) \in \Sigma$. This thechnique can be easily adapted to the case of internal controls

Theorem 9 ([10])
Let f given by (13) and let $\omega \subset\subset \Omega$. Then for any $v \in L^2(\omega \times (0,T))$ arbitrary we have the estimate

$$|y(x,t : v)| \leq C(p,n)\left(\frac{1}{d(x)^\theta} + \frac{1}{t^{\frac{\theta}{2}}}\right) \quad \textit{a.e. } (x,t) \in (\Omega \setminus \overline{\omega}) \times (0,T)$$

where $\theta = \frac{2}{p-1}$, $d(x) = dist(x, \partial\omega)$ and $C(p,n)$ is a positive constant independent on v.

Proof.
We introduce the function

$$y(t,x) = C(p,n)\left(\frac{1}{d(x)^\theta} + \frac{1}{t^{\frac{\theta}{2}}}\right) \quad \text{for } x \in \Omega \setminus \overline{\omega} \text{ and } t > 0.$$

A careful choice of the constant $C(\mathrm{p},\mathrm{n})$ (see *e.g.* **[21]**) allows to check that y satisfies

$$\begin{array}{ll} \mathrm{y}_t - \Delta \mathrm{y} + \lambda|\mathrm{y}|^{p-1}\mathrm{y} \geq 0 & \text{in } (\Omega \setminus \overline{\omega}) \times (0,\mathrm{T}), \\ \mathrm{y} \geq 0 & \text{on } \Sigma \\ \mathrm{y} \to +\infty & \text{on } \partial\omega \times (0,\mathrm{T}) \\ \mathrm{y} \to +\infty & \text{on } \Omega \times \{0\}. \end{array}$$

Applying the maximum principle we deduce that $y(x,t:v) \leq \mathrm{y}(x,t)$ for any $t \in (0,\mathrm{T}]$ and *a.e.* $x \in \Omega\setminus\overline{\omega}$. In a similar way we prove that $-\mathrm{y}(t,x) \leq y(t,x:v)$ and the conclusion holds. □

Remark 7
As a matter of fact the above estimate can be improved by introducing the function $\mathrm{U}_\infty(x,t)$ solution of the problem

$$\begin{array}{ll} \mathrm{U}_t - \Delta \mathrm{U} + \lambda|\mathrm{U}|^{p-1}\mathrm{U} = 0 & \text{in } (\Omega \setminus \overline{\omega}) \times (0,\mathrm{T}), \\ \mathrm{U} = 0 & \text{on } \Sigma \\ \mathrm{U} \to +\infty & \text{on } \partial\omega \times (0,\mathrm{T}) \\ \mathrm{U}(\cdot,0) = y_0(\cdot) & \text{on } \Omega. \end{array}$$

Thanks to the assumption $\mathrm{p} > 1$ it is possible to show (see **[1]**) the existence of a minimal solution U_∞. In fact, we have $\mathrm{U}_\infty > 0$ in $(\Omega \setminus \overline{\omega}) \times (0,\mathrm{T})$, assumed $y_0 \geq 0$. As in Theorem 9 we conclude the estimate

$$y(x,t:v) \leq \mathrm{U}_\infty(x,t) \quad \text{for } a.e.\ x \in \Omega \setminus \overline{\omega} \text{ and } t \in [0,\mathrm{T}]$$

where v is again arbitrary in $\mathrm{L}^2(\omega \times (0,\mathrm{T}))$. We conjecture that (even in this superlinear case) the approximate controllability property holds if we assume the desired state $y_d \in \mathrm{L}^2(\Omega)$ such that

$$\mathrm{U}_{-\infty}(x,\mathrm{T}) < y_d(x) < \mathrm{U}_\infty(x,\mathrm{T}) \quad \text{for } a.e.\ x \in \Omega \setminus \overline{\omega}$$

(here $\mathrm{U}_{-\infty}$ denotes the solution of the above problem replacing $+\infty$ by $-\infty$). □

Remark 8
We also conjecture that if f represents a superlinear source near the infinity (*e.g.* f given by (13) but with $\lambda < 0$ instead $\lambda > 0$) there is not any obstruction and the approximate controllability holds. □

Remark 9
Theorems 8 and 9 show that the approximate controllability property holds or not for (SL) according to whether the function f is sublinear or superlinear near the infinity. This fact contrasts with the occurrence of a free boundary for which the answers are of different nature according to whether f is sublinear (the positive case) or superlinear (the negative case) near the origin (see *e.g.* **[20]** and **[9]**). □

Remark 10
The existence of ***universal solutions*** taking values $+\infty$ or $-\infty$ over $\partial\omega \times (0,\mathrm{T})$ can also be obtained for many other nonlinear equations such as the nonlinear diffusion equation

$$y_t - \Delta\left(|y|^{m-1}y\right) = v\chi_\omega,$$

and the quasilinear equation associated to the $(\mathrm{m}+1)$-Laplacian operator

$$y_t - \Delta_{m+1}y = v\chi_\omega, \qquad \Delta_{m+1}y := \operatorname{div}\left(|\nabla y|^{m-1}\nabla y\right),$$

always under the condition $\mathrm{m} > 1$. Other kind of ***universal solution*** can also be obtained (see **[4]**) for the Burger equation

$$y_t - y_{xx} + yy_x = v\chi_\omega.$$

The uncontrollability for this equation was also shown in Fursikov-Imanuvilov **[16]** by using an energy method. We also mention that it is possible to show the exact controllability for the Burger equation over very special functional spaces (see El Badia-Ain Seba **[11]**). Finally we point out that other nonlinear problems also leads to positive or negative answers to the question of the approximate controllability according the behaviour of the data (see in Díaz **[3]**,**[4]** a study of the parabolic obstacle problem). □

Remark 11
The approximate controllability for the Navier-Stokes equation is, at the present, an open problem. The interest of this question was already raised in Lions **[24]** establishing some connections with the study of the turbulence. A partial result is due to Fernández-Cara and Real **[15]** and shows that the subspace spanned by $\vec{y}(\mathrm{T}:v)$ is dense in a suitable Hilbert space. We point out that if the conjecture of the Remark 7 is true then the subspace spanned by $\{y(\mathrm{T}:v) : v \in \mathrm{L}^2(\omega \times (0,\mathrm{T}))\}$ is dense in $\mathrm{L}^2(\Omega)$ (even for the superlinear case). □

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