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# Elliptic and parabolic problems

Pont-à-Mousson 1994



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# The support shrinking in solutions of parabolic equations with non-homogeneous absorption terms

## 1. Introduction

**1.1 STATEMENT OF THE PROBLEM.** This paper deals with the propagation and vanishing properties of local weak solutions of nonlinear parabolic equations. Let  $\Omega \subset \mathbf{R}^N$ ,  $N = 1, 2, \dots$ , be an open connected domain with the smooth boundary  $\partial\Omega$ , and  $T > 0$ . We consider the problem

$$\left. \begin{aligned} \frac{\partial}{\partial t} (|u|^{\alpha-1}u) &= \operatorname{div} (\vec{A}(x, t, u, \nabla u)) - B(x, t, u) + f(x, t) \\ &\text{in } Q = \Omega \times (0, T), \\ u(x, 0) &= u_0(x) \text{ in } \Omega \end{aligned} \right\} \quad (1)$$

assuming that the functions  $\vec{A}$  and  $B$  are subject to the following structural conditions: there exist constants  $\lambda > 0$  and  $p > 1$  such that

$$\forall (x, t, s, \rho) \in \Omega \times \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}^N \quad M_1 |\rho|^p \leq (\vec{A}(x, t, s, \rho), \rho) \leq M_2 |\rho|^p, \quad (2)$$

$$\forall (x, t, s) \in \Omega \times \mathbf{R}^+ \times \mathbf{R} \quad sB(x, t, s) \geq M_3 a(x, t) |s|^{\lambda+1}, \quad (3)$$

with  $a(x, t) \geq 0$  a given measurable bounded function satisfying

$$a^{-1} \in L^{(1+\lambda)/(\tilde{\lambda}-\lambda)}(Q), \quad 0 < \lambda < \tilde{\lambda}. \quad (4)$$

In (2)-(3)  $M_i$ ,  $i = 1, 2, 3$ , are positive constants. The additional (and crucial) assumption in all further consideration is:

$$\lambda < \alpha. \quad (5)$$

The right-hand side  $f(x, t)$  of equation (1) and the initial data  $u_0(x)$  are assumed to satisfy

$$u_0 \in L^{\alpha+1}(\Omega), \quad f \in L^{(1+\lambda)/\lambda}(Q), \quad f a^{-1/(1+\tilde{\lambda})}(Q). \quad (6)$$

We are interested in the qualitative properties of solutions of problem (1), understood in the following sense.

**Definition 1** A measurable in  $Q$  function  $u(x, t)$  is said to be a weak solution of problem (1) if

$$\text{a) } u \in L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(0, T; L^{\alpha+1}(\Omega));$$

$$\text{b) } \lim_{t \rightarrow 0} \|u(x, t) - u_0(x)\|_{L^{\alpha+1}(\Omega)} = 0;$$

c) for any test function  $\zeta(x, t) \in W^{1,\infty}(0, T; W_0^{1,p}(\Omega))$ , vanishing at  $t = T$ , the integral identity holds

$$\int_Q \{ |u|^{\alpha-1}u \zeta_t - (\vec{A}, \nabla \zeta) - B\zeta + f\zeta \} dxdt + \int_\Omega |u_0|^{\alpha-1}u_0 \zeta(x, 0) dx = 0. \quad (7)$$

So far, the theory of problems of the type (1) already accounts for a number of existence results. We refer the reader to papers [1, 9, 13, 18] and their references.

The class of equations of (1) includes, in particular, the following model equation

$$v_t = \Delta (|v|^{m-1}v) - M_3 |v|^{\gamma-1}v + f(x, t). \quad (8)$$

To pass to an equation of the form (1) with the parameters  $\alpha = 1/m$ ,  $\lambda = \gamma/m$ ,  $p = 2$  amounts to introduce the new unknown  $v := |u|^{\frac{1}{m}} \operatorname{sign} u$ . Assumption (5) holds if, for instance,

$$m \geq 1, \quad \gamma \in (0, 1).$$

In this choice of the exponents of nonlinearity the disturbances originated by data propagate with *finite speed*, (see [17] and references therein). Moreover, it is known [20, 21, 16] that in this range of the parameters the supports of nonnegative weak solutions to equation (8) may shrink as  $t$  grows. It is known also, [8, 12], that solutions of the Cauchy problem and the Cauchy-Dirichlet problem for equation (8) may even vanish on some subset of the problem domain  $Q$  despite of the fact that  $u_0$  and the boundary data are strictly positive. These properties were derived by means of comparison of solutions of (8) with suitable sub and supersolutions of these problems.

It is to be pointed out here that in our formulation the function  $\vec{A}(x, t, s, \rho)$  is not subject to any monotonicity assumptions neither in  $s$  nor in  $\rho$ . Next, we are not constrained by any special boundary conditions. Lastly, as follows from Definition 1.1 the solutions of problem (1) are not supposed to have a definite sign.

Our purpose is to generalize the referred results and to describe the dynamics of the supports of solutions of equation (1) without having recourse to any comparison method. We propose certain refinement of the energy methods in the literature [10, 24, 25, 5, 6, 7, 4, 3]. This techniques allows us to make certain conclusions about the properties of the supports of local weak solutions to problem (1) which rely only on some assumptions about the properties of initial data or even use only the information on the character of the nonlinearity of the equation in (1).

The results we obtain below may be illustrated by the following simplified description. Let  $v(x, t)$  be a local weak solution of the model equation

$$v_t = \Delta_p (|v|^{m-1}v) - |v|^{\gamma-1}v,$$

where  $\Delta_p(\cdot)$  denotes the  $p$ -Laplace operator given by

$$\Delta_p v \equiv \operatorname{div} (|\nabla v|^{p-2} \nabla v), \quad p > 1.$$

Then:

- (i) if  $0 < \gamma < 1$ ,  $m(p-1) \geq 1$  and  $v(x, 0)$  is flat enough near the boundary of its support, the so-called "waiting time" of  $v$  is complete, i.e.,  $\forall t \in [0, T] \quad \operatorname{supp} v(\cdot, t) \subseteq \operatorname{supp} v(\cdot, 0)$  (we also may say that there is no dilatation of the initial support);
- (ii) if  $\gamma$ ,  $m$ , and  $p$  additionally satisfy the relation  $m + \gamma \leq p/(p-1)$ , we have shrinking of the initial support, i.e., the above inclusion is strict:  $\operatorname{supp} v(\cdot, t) \subset\subset \operatorname{supp} v(\cdot, 0)$  for  $t > 0$  small enough;
- (iii) under the assumptions of item (ii) on the exponents but without any assumption on the initial datum, a null-set with nonempty interior (or dead core) is formed, i.e.,  $\exists t^* > 0 : \forall t > t^* \quad \bar{\Omega} \setminus \{\operatorname{supp} v(\cdot, t)\} \neq \emptyset$ .

In order to compare these results and the theorems below, recall that  $\alpha = \frac{1}{m}$  and  $\lambda = \frac{\gamma}{m}$ . We also remark that the above results remain true when the diffusion is linear, i.e.,  $p = 2$  and  $m = 1$ , but in the presence of the non-homogeneous strong absorption term:  $\gamma \in (0, 1)$ ,  $a(x, t)$  is admitted to vanish at some set of zero measure.

**1.2. FORMULATION OF RESULTS.** Let us introduce the following notation: given  $T > 0$ ,  $t \in [0, T)$ ,  $x_0 \in \Omega$ ,  $\rho \geq 0$ , and nonnegative parameters  $\sigma$  and  $\mu$ ,

$$P(t, \rho) \equiv \{(x, s) \in Q : |x - x_0| < \rho(s) \equiv \rho + \sigma(s - t)^\mu, s \in (t, T)\} \equiv P(t, \rho; \sigma, \mu).$$

It is clear that the choice of the parameters  $\sigma, \mu, \rho, T$  determines the shape of the domains  $P(t, \rho)$ . We distinguish three cases.

- a)  $\sigma = 0, \mu = 0, \rho > 0$ ; in this case  $P(t, \rho)$  is a cylinder  $B_\rho(x_0) \times (t, T)$ ;
- b)  $\sigma > 0, \mu = 1, \rho > 0$ ;  $P(0, \rho)$  renders a truncated cone centered in the point  $x_0 \in \Omega$  and with the base  $B_\rho(x_0) := \{x \in \Omega : |x - x_0| < \rho\}$  on the plane  $t = 0$ ;
- c)  $\sigma > 0, 0 < \mu < 1, \rho = 0$ ; then  $P(t, 0)$  becomes a paraboloid.

To simplify the notation we will omit the arguments of  $P$  wherever possible. Treating separately cases a), b), c) we indicate specially which of the parameters are essential and which are not. The domains of the type  $P(t, \rho)$  will play the fundamental role in the definition of the local energy functions

$$E(P) := \int_{P(t, \rho)} |\nabla u(x, \tau)|^p dx d\tau, \quad C(P) := \int_{P(t, \rho)} |u(x, \tau)|^{\lambda+1} dx d\tau, \\ C_a(P) := \int_{P(t, \rho)} a(x, t) |u(x, \tau)|^{\lambda+1} dx d\tau,$$

$$b(T) := \operatorname{ess\,sup}_{s \in (t, T)} \int_{|x - x_0| < \rho + \sigma(s-t)^\mu} |u(x, s)|^{\alpha+1} dx,$$

associated to any of local weak solutions of problem (1).

Let us assume that

$$\left\| \frac{1}{a} \right\|_{L^{(1+\lambda)/(\lambda-\lambda)}(Q)} \leq K, \quad K = \operatorname{const}, \quad (9)$$

whence

$$C^{(1+\lambda)/(1+\lambda)} \leq C_a \left\| \frac{1}{a} \right\|_{L^{(1+\lambda)/(\lambda-\lambda)}(Q)} \leq K C_a. \quad (10)$$

We now pass to the precise statement of our results. The only global information we need will be formulated in terms of the global energy function

$$D(u(\cdot, \cdot)) := b(T, \Omega) + \int_Q (|\nabla u|^p + a |u|^{\lambda+1}) dx dt,$$

where

$$b(T, \Omega) := \operatorname{ess\,sup}_{t \in (0, T)} \int_\Omega |u(x, t)|^{\alpha+1} dx.$$

Our first result refers to the situation when the support of  $u$  (an arbitrary local weak solution of (1)) does not display the property of dilatation with respect to the initial support  $\operatorname{supp} u_0$  and the support of the forcing term  $\operatorname{supp} f(\cdot, t)$ . Assume that

$$u_0 \equiv 0 \quad \text{in} \quad B_{\rho_0}(x_0) \quad \text{for some } x_0 \in \Omega \text{ and } \rho_0 > 0 \quad (11)$$

$$f \equiv 0 \quad \text{in the cylinder} \quad P = P(0, \rho_0) = P(0, \rho_0; 0, 0) \quad (= B_{\rho_0}(x_0) \times (0, T)). \quad (12)$$

and claim the convergence (near  $\rho = \rho_0$ ) of the auxiliary integral

$$I := \int_{\rho_0+0} (\rho - \rho_0)^\beta \left[ \|u_0\|_{L^{\alpha+1}(B_{\rho_0}(x_0))}^{\alpha+1} + \|f a^{-1/(1+\lambda)}\|_{L^{(1+\lambda)/\lambda}(P(0, \rho))}^{(1+\lambda)/\lambda} \right]^{p/(p-1)} d\rho < \infty, \quad (13)$$

where

$$\beta = (1 - \delta\tilde{\theta})(1 + \kappa), \quad \delta = - \left( 1 + \frac{p-1-\lambda}{p(1+\lambda)} N \right), \quad \tilde{\theta} = \frac{pN - r(N-1)}{(N+1)p - Nr}, \quad (14)$$

with some

$$\kappa \in \left( 0, \frac{p(1+\alpha)}{(p-1-\lambda)(1-\tilde{\theta})} \right), \quad (15)$$

Note that condition (13) implies certain restrictions on the vanishing rates of the functions  $\|u_0\|_{L^{\alpha+1}(B_{\rho_0}(x_0))}$  and  $\|f(\cdot, t) a^{-1/(1+\lambda)}\|_{L^{(\lambda+1)/\lambda}(B_{\rho_0}(x_0))}$  as  $\rho \rightarrow \rho_0$ .

**Theorem 1** Assume (2), (3), (9) and

$$\lambda < \alpha \leq p - 1. \quad (16)$$

Let  $u_0$  and  $f$  satisfy (11), (12) and (13). Then there exists positive constants  $M$  (depending only on the constants in (2), (3),  $\rho_0$ ,  $\text{dist}(x_0, \partial\Omega)$  and the difference  $\lambda - \lambda$ ) such that any weak solution of (1) with bounded global energy,  $D(u) \leq M$ , possesses the property

$$u(x, t) \equiv 0 \quad \text{in } B_{\rho_0}(x_0) \times (0, T).$$

Under some additional assumptions on the structural exponents  $\alpha, \lambda, p$  and the function  $f$  one may get a stronger result which means that the support of  $u(\cdot, t)$  shrinks strictly with respect to the initial support.

**Theorem 2** Assume (2) – (5), (16), (9) and let

$$1 + \lambda \leq \alpha \frac{p}{p-1}. \quad (17)$$

Let  $u_0$  satisfy (11). Assume

$$f \equiv 0 \quad \text{in the truncated cone } P \equiv P(0, \rho_0 : \sigma, 1) \text{ for some } \sigma > 0 \quad (18)$$

and let (13) be true. Then there exist positive constants  $M, \bar{\lambda} > \lambda$  and  $t^*$  such that each weak solution of problem (1) with global energy satisfying the inequality  $D(u) \leq M$ , possesses the property

$$u(x, t) \equiv 0 \quad \text{in } P(0, \rho_0 : \sigma, 1) \cap \{t \leq t^*\}.$$

**Remark 1** It is curious to observe that the assertion of Theorem 2 has a local character in the sense that different parts of the boundary of  $\text{supp } u_0$  may originate pieces of the boundary of the null-set of  $u(x, t)$ , which display different shrinking properties. Having a possibility to control the rate of vanishing of  $u_0$  and  $f(x, t)$ , one may design solutions of problem (1) which have prescribed shapes of supports. For the model equation (8) this phenomenon is already known as “the heat crystal” [23, Ch.3, Sec.3].

The last of our main results refers to the case when the initial datum need not vanish, that is, the parameter  $\rho_0$  in the conditions of Theorems 1 and 2 is assumed to be zero. Assuming  $f \equiv 0$  we show how the strong absorption term causes the formation of the null-set of the solution.

**Theorem 3** Assume (2) – (5), (16) – (17), (9). Let  $f \equiv 0$ . Then there exist positive constants  $M, t^*$ , and  $\mu \in (0, 1)$  such that any weak solution of problem (1) satisfying the inequality  $D(u) \leq M$  possesses the property

$$u(x, t) \equiv 0 \quad \text{in } P(t^*, 0 : 1, \mu).$$

## 2. Differential Inequalities

**2.1 FORMULA OF INTEGRATION BY PARTS.** It follows from results of [] that for local weak solutions of equation (1) the following formula of integration by parts holds:

$$\begin{aligned} i_1 + i_2 + i_3 &=: \frac{\alpha}{\alpha + 1} \int_{P \cap \{t=T\}} |u|^{\alpha+1} dx + \int_P (\vec{A}, \nabla u) dx d\theta + \int_P u B dx d\theta \\ &= \int_{\partial_t P} (\vec{n}_x, \vec{A}) u d\Gamma d\theta + \frac{\alpha}{\alpha + 1} \int_{\partial_t P} n_\tau |u|^{\alpha+1} d\Gamma d\theta \\ &+ \frac{\alpha}{\alpha + 1} \int_{P \cap \{t=0\}} |u_0|^{\alpha+1} dx + \int_P u f dx d\theta \\ &:= j_1 + j_2 + j_3 + j_4. \end{aligned} \quad (1)$$

Here  $d\Gamma$  is the differential form on the hypersurface  $\partial_t P \cap \{t = \text{const}\}$ ,  $\vec{n}_x$  and  $n_\tau$  are the components of the unit normal vector to  $\partial_t P$ ,  $|\vec{n}_x|^2 + |n_\tau|^2 = 1$ .

**2.2. THE ENERGY DIFFERENTIAL INEQUALITIES. DOMAINS OF THE TYPE C).** Now we derive differential inequalities for the energy function  $E + C$  which later on will be utilized for the proofs of Theorems 1-3. We begin with the most complicated case c) where the domain  $P$  is a paraboloid determined by the parameters  $\mu \in (0, 1)$ ,  $\sigma > 0$ , and  $t$ :

$$P = P(t) = \{(x, \tau) : |x - x_0| \equiv \rho(\tau) \leq \sigma(\tau - t)^\mu, \tau \in (t, T)\}, \quad t \in (0, T).$$

We assume that  $f \equiv 0$  and that  $P$  does not touch the initial plane  $\{t = 0\}$ . These assumptions simplify the basic energy equality (1)  $i_1 + i_2 + i_3 = j_1 + j_2$ .

Let us estimate the first term  $j_1$ . It is easy to see that

$$\vec{n} \equiv (\vec{n}_x, n_\tau) = \frac{1}{(\sigma^2 \mu^2 + (\theta - t)^{2(1-\mu)})^{1/2}} \left( (\theta - t)^{1-\mu} \vec{e}_x - \mu \sigma \vec{e}_\tau \right)$$

where  $\vec{e}_\tau$  and  $\vec{e}_x$  are unit vectors orthogonal to the hyperplane  $t = 0$  and the axis  $t$  respectively.

Let  $(\rho, \omega)$ ,  $\rho > 0$ ,  $\omega \in \partial B_1$ , be the polar coordinate system in  $\mathbf{R}^N$ . Given an arbitrary function  $F(x, t)$ , we use the notation  $x = (\rho, \omega)$  and  $F(x, t) = \Phi(\rho, \vec{\omega}, t)$ . There holds the equality

$$I(t) := \int_P F(x, \theta) dx d\theta \equiv \int_t^T d\theta \int_0^{\rho(\theta, t)} \rho^{N-1} d\rho \int_{\partial B_1} \Phi(\rho, \vec{\omega}, \theta) |J| d\omega,$$

where  $J$  is the Jacobi matrix and, due to the definition of  $P$ ,  $\rho(\theta, t) = \sigma(\theta - t)^\mu$ . It is easy to check that:

$$\begin{aligned}
\frac{dI(t)}{dt} &= - \int_0^{\rho(\theta,t)} \rho^{N-1} d\rho \int_{\partial B_1} \Phi(\rho, \bar{\omega}, \theta) |J| d\omega \Big|_{\theta=t} \\
&+ \int_t^T \rho_t(\theta, t) \rho^{N-1}(\theta, t) d\theta \int_{\partial B_1} \Phi(\rho, \bar{\omega}, t) |J| d\omega \\
&= \int_{\partial_t P} \rho_t F(x, \theta) d\Gamma d\theta. \tag{2}
\end{aligned}$$

Treating the energy function  $E$  as a function of  $t$ , with the use of (2), (2), and the Hölder inequality, we have now:

$$\begin{aligned}
|j_1| &\leq \left| \int_{\partial_t P} (\bar{n}_x, \bar{A}) u d\Gamma d\theta \right| \leq M_2 \int_{\partial_t P} |\bar{n}_x| |\nabla u|^{p-1} |u| d\Gamma d\theta \\
&= M_2 \left( -\frac{dE}{dt} \right)^{(p-1)/p} \left( \int_t^T \frac{|\bar{n}_x|^p}{|\rho_t|^{p-1}} \left( \int_{\partial B_{\rho(\theta,t)}} |u|^p d\Gamma \right) d\theta \right)^{1/p}. \tag{3}
\end{aligned}$$

To estimate the right-hand side of (3) we use the following interpolation inequality: given  $v \in W^{1,p}(B_\rho)$  and  $\lambda \leq p-1$ ,

$$\|v\|_{p,S_\rho} \leq L_0 \left( \|\nabla v\|_{p,B_\rho} + \rho^\delta \|v\|_{\lambda+1,B_\rho} \right)^\delta \cdot \left( \|v\|_{r,B_\rho} \right)^{1-\delta} \tag{4}$$

with a universal constant  $L_0 > 0$  not depending on  $v(x)$  and the exponents

$$r \in [1, Np/(N-1)], \quad \tilde{\theta} = \frac{pN - r(N-1)}{(N+r)p - Nr} \in (0, 1), \quad \delta = - \left( 1 + \frac{p-1-\lambda}{p(1+\lambda)} N \right)$$

(see, e.g., Diaz-Veron [14]). Let us introduce the notation

$$E_*(t, \rho) := \int_{B_\rho} |\nabla u|^p dx, \quad C_*(t, \rho) := \int_{B_\rho} |u|^{\lambda+1} dx,$$

so that

$$E = \int_t^T E_*(\theta, \rho(\theta, t)) d\theta, \quad C = \int_t^T C_*(\theta, \rho(\theta, t)) d\theta,$$

and make use of the Hölder inequality

$$\left( \int_{B_\rho} |u|^r dx \right)^{1/r} \leq \left( \int_{B_\rho} |u|^{1+\lambda} dx \right)^{(1+\lambda)/qr} \cdot \left( \int_{B_\rho} |u|^{\alpha+1} dx \right)^{(1+\alpha)(q-1)/qr},$$

where

$$q = \frac{\alpha - \lambda}{\alpha - r + 1}, \quad r \in [1 + \lambda, 1 + \alpha].$$

To estimate the second factor in the right-hand side of (3), we choose  $r$  satisfying the inequalities

$$1 < 1 + \lambda < \frac{p(1+\alpha)}{\alpha - \lambda + p} < r < 1 + \alpha < \frac{pN}{N-1}.$$

It is easy to check then

$$p < qr = \frac{(\alpha - \lambda)r}{\alpha + 1 - r}, \quad \gamma = 1 - \frac{\tilde{\theta}}{p} - \frac{1 - \tilde{\theta}}{qr} \in \left( 1 - \frac{1}{qr}, 1 - \frac{1}{p} \right),$$

$$\frac{\gamma p}{p-1} \in \left( \frac{qr-1}{qr} \frac{p}{p-1}, 1 \right), \quad (1-\gamma)p \in (0, 1). \tag{5}$$

Then, by virtue of (4),

$$\begin{aligned}
\int_{\partial B_\rho} |u|^p d\Gamma &\leq L_0 \left( \int_{B_\rho} |\nabla u|^p dx + \rho^{\delta p} \left( \int_{B_\rho} |u|^{\lambda+1} dx \right)^{p/(\lambda+1)} \right)^{\tilde{\theta}} \left( \int_{B_\rho} |u|^r dx \right)^{p(1-\tilde{\theta})/r} \\
&\leq K \rho^{\delta \tilde{\theta} p} \left( \int_{B_\rho} |\nabla u|^p dx + \int_{B_\rho} |u|^{\lambda+1} dx \right)^{\tilde{\theta}} \\
&\quad \times \left( \int_{B_\rho} |u|^{\lambda+1} dx \right)^{p(1-\tilde{\theta})/qr} \left( \int_{B_\rho} |u|^{\alpha+1} dx \right)^{p(q-1)(1-\tilde{\theta})/qr} \\
&\leq K \rho^{\delta \tilde{\theta} p} (E_* + C_*)^{\tilde{\theta} + (1-\tilde{\theta})p/qr} b^{(q-1)(1-\tilde{\theta})p/qr}, \tag{6}
\end{aligned}$$

where

$$\begin{aligned}
K &= L_0 \max \left( \rho_0^{-\delta p}, \left( \text{ess sup}_{(t,T)} \int_{B_{\rho(\theta)}} |u|^{\lambda+1} dx \right)^{\frac{p}{\lambda+1}-1} \right)^{\tilde{\theta}} \\
&\leq L_0 \max \left( \rho_0^{-\delta p}, \left( \text{meas } B_{\rho(T)} \right)^{\frac{\alpha-\lambda}{\alpha+1} \left( \frac{p}{\lambda+1}-1 \right)} (b(T))^{\frac{\lambda+1}{\alpha+1} \left( \frac{p}{\lambda+1}-1 \right)} \right)^{\tilde{\theta}}, \quad \rho \leq \rho_0.
\end{aligned}$$

Returning to (3) and applying once again the Hölder inequality, we have from (6)

$$\begin{aligned}
|j_1| &\leq L \left( -\frac{dE}{dt} \right)^{(p-1)/p} \left( \int_t^T \frac{|\bar{n}_x|^p}{|\rho_t|^{p-1}} K \rho^{\delta \tilde{\theta} p} (E_* + C_*)^{p(1-\gamma)} b^{(q-1)(1-\tilde{\theta})p/qr} d\tau \right)^{1/p} \\
&\leq L \Lambda(t) \left( -\frac{d(E+C)}{dt} \right)^{(p-1)/p} \cdot b^{(q-1)(1-\tilde{\theta})p/qr} (E+C)^{1-\gamma} \tag{7}
\end{aligned}$$

for a suitable positive constant  $L$  and the exponent  $\mu = 1/[1 - p(1-\gamma)]$ .

$$\Lambda(t) := \left( \int_t^T \left( \frac{1}{|\rho_t|^{p-1}} \rho^{\delta \tilde{\theta} p}(\tau) \right)^\mu d\tau \right)^{1/\mu p} < \infty. \tag{8}$$

To satisfy (8) one has to take  $\mu$  small enough, since the condition of convergence of the integral  $\Lambda(t)$  is:

$$(1 - \mu)(2p - 1) + \mu\delta\bar{\theta}p > -(1 - \bar{\theta}) \left( 1 - \frac{p(\alpha - r + 1)}{(\alpha - \lambda)r} \right).$$

So, we have obtained the estimate of the following type:

$$|j_1| \leq L_1 \Lambda(t) D(u)^{(q-1)(1-\bar{\theta})/qr} (E + C)^{1-\gamma} \left( -\frac{d(E + C)}{dt} \right)^{(p-1)/p} \quad (9)$$

where  $L_1$  is a universal positive constant,  $D(u)$  is the total energy of the solution under investigation.

Let us estimate  $j_2$ . For this purpose we use the interpolation inequality

$$\|v\|_{\alpha+1, \partial B_\rho} \leq L_0 \left( \|\nabla v\|_{p, B_\rho} + \rho^\delta \|v\|_{\lambda+1, B_\rho} \right)^s \cdot \|v\|_{r, B_\rho}^{1-s} \quad (10)$$

with a universal positive constant  $L_0 > 0$ , the exponent

$$s = \frac{(\alpha + 1)N - r(N - 1)}{(N + r)p - Nr} \cdot \frac{p}{\alpha + 1},$$

and  $\delta$  from (4), which holds for each  $v \in W^{1,p}(B_\rho)$ . Now we choose the exponent  $r$  in (4) as follows

$$1 + \lambda < r < \frac{\alpha p}{p - 1},$$

whence,

$$\kappa = (1 + \alpha) \left( \frac{s}{p} + \frac{1-s}{qr} \right) \in (0, 1), \quad \eta = \kappa + \frac{(q-1)(1-s)}{qr} (1 + \alpha) = \kappa + \xi > 1.$$

Similarly to the previous estimate, using (10) we have:

$$\begin{aligned} \int_{\partial B_\rho} |u|^{\alpha+1} dx &\leq L \left( \int_{B_\rho} |\nabla u|^p dx + \int_{B_\rho} |u|^{\lambda+1} dx \right)^{s(\alpha+1)/p} \\ &\times \left[ \left( \int_{B_\rho} |u|^{\lambda+1} dx \right)^{1/qr} \left( \int_{B_\rho} |u|^{\alpha+1} dx \right)^{(q-1)/qr} \right]^{(1-s)(\alpha+1)} K^{s(\alpha+1)/\bar{\theta}p}. \end{aligned}$$

Here  $K$  is defined in (6). Using the Hölder inequality and reminding that always  $|n_\tau| \leq 1$ , we arrive to the inequality

$$\begin{aligned} |j_2| &= \left| \int_t^T |n_\tau| d\tau \int_{\partial B_{\rho(\tau)}} |u|^{\alpha+1} d\Gamma \right| \\ &\leq L (E + C + b(T, \Omega)) (b(T, \Omega))^{\eta-1} \left( \int_t^T (K^{s(\alpha+1)/\bar{\theta}p})^{1/(1-\kappa)} d\tau \right)^{1-\kappa}. \quad (11) \end{aligned}$$

We now turn to estimating the left-hand side of (1). By (2)-(3) we have at once that

$$i_1 + i_2 + i_3 \geq i_1 + M_1 E + M_3 C_a \geq M_4 D^{1-m} (E + C + i_1)^m, \quad m = \frac{1 + \bar{\lambda}}{1 + \lambda} > 1, \quad (12)$$

$$M_4 = M_4(M_1, M_3, m).$$

Since the right-hand side of (1) is an increasing function of  $T$ , we may always replace  $i_1$  by  $\frac{\alpha}{\alpha+1}b(T)$  in the left-hand side of (1). Gathering now (1) with  $j_3 = j_4 = 0$  and (11), (12), we get:

$$\begin{aligned} M_4 D^{1-m} \left( E + C + \frac{\alpha}{1 + \alpha} b \right)^m &\leq L (E + C + b) b^{\eta-1} \left( \int_t^T K^{s(1+\alpha)/(\bar{\theta}p(1-\kappa))} d\tau \right)^{1-\kappa} \\ &+ L_1 \Lambda(t) D^{(q-1)(1-\bar{\theta})/qr} (E + C) \left( -\frac{d(E + C)}{dt} \right)^{(p-1)/p}. \end{aligned}$$

Let us now choose  $\bar{\lambda}$  satisfying the inequality

$$1 < m = \frac{1 + \bar{\lambda}}{1 + \lambda} \leq \eta(\alpha, \lambda, p), \quad (13)$$

and assume  $T - t$  and  $D(u)$  be so small that

$$L (b(T, \Omega))^\kappa \int_t^T (K^{s(\alpha+1)/\bar{\theta}p})^\epsilon d\tau < \frac{M_4}{2}.$$

The we arrive at the inequality

$$(E + C)^m \leq (E + C + b(T, \Omega))^m \leq L_2 \Lambda(t) D(u)^{(m+1)(q-1)(1-\bar{\theta})/qr} (E + C)^{1-\gamma} \left( -\frac{d(E + C)}{dt} \right)^{(p-1)/p}$$

whence we get the desired differential inequality for the energy function  $Y(t) := E + C$ :

$$Y^\nu(t) \leq c(t) (-Y(t))', \quad \nu = \frac{(m-1+\gamma)}{p-1}, \quad (14)$$

with

$$c(t) = \left( L_1 (M_*)^{(q-1)(1-\bar{\theta})/qr} \Lambda(t) \right)^{p/(p-1)}. \quad L_1 = \text{const} > 0$$

for  $M_* := D(u)$ . Note that  $c(t) \rightarrow 0$  as  $t \rightarrow T$ . According to (5)  $p\gamma/(p-1) < 1$ ; thus we may take, additionally to (13),

$$\nu \left( \frac{p(m-1)}{p-1} + \frac{\gamma p}{p-1} \right) < 1.$$

2.3. THE ENERGY DIFFERENTIAL INEQUALITIES. DOMAINS OF THE TYPES A)-B). In these cases the differential inequality for the energy function  $E + C$  is derived in same way that in the case c) but with certain simplifications due to the choice of the domain  $P$ .

Let us begin with the case b). Let

$$P = \{(x, t) : |x - x_0| < \rho + \sigma\theta, \sigma \in (0, T)\}, \quad \rho \geq \rho_0 > 0.$$

The unit outer normal to  $\partial_t P$  has the form

$$\vec{n} = \frac{1}{\sqrt{1 + \sigma^2}}(1, -\sigma)$$

and if we treat now the energy function  $Y := E + C$  as a function of  $\rho$ , we have:

$$\begin{aligned} \frac{dY(\rho)}{d\rho} &= \frac{d}{d\rho} \left\{ \int_0^T d\theta \int_0^{\rho+\sigma\theta} \tau^{N-1} d\tau \int_{\partial B_1} |J| (|\nabla u|^p + |u|^{\lambda+1}) \Big|_{x=(\tau, \omega)} d\omega \right\} \\ &= \int_0^T d\theta \int_{\partial B_1} \left\{ (\rho + \sigma\theta)^{N-1} |J| (|\nabla u|^p + |u|^{\lambda+1}) \Big|_{x=(\rho+\sigma\theta, \omega)} \right\} d\omega \\ &= \int_{\partial_t P} (|\nabla u|^p + |u|^{\lambda+1}) d\Gamma d\theta. \end{aligned} \quad (15)$$

Following the above scheme for estimating the term  $j_1$  in (1) and applying (15), we arrive at the following inequality

$$|j_1| \leq \frac{K}{\sqrt{1 + \sigma^2}} \left( \frac{dE}{d\rho} \right)^{(p-1)/p} \rho^{\delta\bar{\theta}} (b(T))^{(q-1)(1-\bar{\theta})/qr} \left( \int_0^T (E_*) + C_* \right)^{\bar{\theta} + p(1-\bar{\theta})/qr} d\theta$$

Let  $r$  be such that  $\bar{\theta} + (1 - \bar{\theta})p/qr = 1$ . Such a choice is always possible, since

$$\bar{\theta} + \frac{(1 - \bar{\theta})p}{qr} = 1 \quad \Leftrightarrow \quad r = \frac{p(1 + \alpha)}{p + \alpha - \lambda},$$

and the last equality is compatible with the conditions  $p > 1 + \lambda$ ,  $\alpha > \lambda$ , and the starting choice of  $r$ :  $r \in [1 + \lambda, 1 + \alpha]$ . The estimate for  $j_1$  then takes the form

$$|j_1| \leq \frac{K\rho^{\delta\bar{\theta}}}{\sqrt{1 + \sigma^2}} \left( \frac{dE}{d\rho} \right)^{(p-1)/p} (b(T))^{(q-1)(1-\bar{\theta})/qr-\varepsilon} (E + C)^{\varepsilon+1/p}$$

with an arbitrary  $\varepsilon \in (0, (q-1)(1-\bar{\theta})/qr)$ .

The estimate for  $j_2$  is the same that of the case c). The only difference is that now we need not claim that  $T$  is small. The value of the coefficient in the estimate for  $j_2$  is controlled now by the choice of  $\sigma$ , since  $n_\tau = -\sigma/\sqrt{1 + \sigma^2}$ . Due to (11) we have  $j = 0$ . At last, we estimate  $j_4$  with the help of the Hölder and Young inequality

$$j_4 \leq \tau C + L(\tau) \int_P a^{-1/\lambda} |f|^{(\lambda+1)/\lambda} dx d\theta.$$

Gathering these estimates with (1), (12), we arrive to the inequality

$$Y(\rho) \leq c(\rho)Y^{\varepsilon+1/p}(\rho) (Y'(\rho))^{(p-1)/p} + F(\rho), \quad \rho > \rho_0$$

with the coefficient  $c(\rho) = \rho^{\delta\bar{\theta}} K (D(u))^{(q-1)(1-\bar{\theta})/qr-\varepsilon}$  and the right-hand side term

$$F(\rho) = \frac{\alpha}{\alpha + 1} \int_{R_\rho(x_0)} |u_0|^{\alpha+1} dx + L(\tau) \int_P a^{-1/\lambda} |f|^{(\lambda+1)/\lambda} dx d\theta.$$

It is easy to see now that the function

$$Z := Y^{p(1+\varepsilon)/(p-1)}(\rho)$$

satisfies the inequality

$$Z^\gamma(\rho) \leq \frac{p-1}{p(1+\varepsilon)} c^{p/(p-1)} Z'(\rho) + F^{p/(p-1)}(\rho), \quad \rho > \rho_0, \quad \gamma = \frac{m}{1+\varepsilon} < 1. \quad (16)$$

In the case a), the desired inequality (16) for the energy function  $Z(\rho) := (E + C)^{p(1+\varepsilon)/(p-1)}$  defined on the cylinders  $P = \{(x, t) : |x - x_0| < \rho, t > 0\}$  is a by-product of the previous consideration, since the term  $j_2$  of the right-hand side of (1) vanishes.

### 3. Analysis of the Differential Inequalities

#### 3.1 THE MAIN LEMMA.

**Lemma 1** Let a function  $U(\rho)$  be defined for  $\rho \in (\rho_0, R)$ ,  $\rho_0 \geq 0$  and possesses the properties:  $0 \leq U(\rho) \leq M = \text{const.}$ ,  $U'(\rho) \geq 0$  and

$$AU^s(\rho) \leq G\rho^{-\delta}U'(\rho) + \varphi(\rho) \quad \text{as } \rho \in (\rho_0, R) \quad (1)$$

where  $R < \infty$ ,  $s \in (0, 1)$ ,  $A, G, \delta$  are finite positive constants, and  $\varphi(\rho)$  is a given function. If the integral

$$i(\rho) := \int_{\rho_0}^{\rho} \sigma^\delta (\sigma - \rho_0)^{-(1+\delta)/(1-s)} \varphi(\sigma) d\sigma$$

converges and the equation

$$(\rho - \rho_0)^{(1+\delta)/(1-s)} \left\{ \left( \frac{A(1-s)}{G(1+\delta)} \right)^{1/(1-s)} - \frac{1}{G} i(\rho) \right\} = M \quad (2)$$

has a root  $\rho_* \in (\rho_0, R)$ , then  $U(\rho_0) = 0$ .

**Proof.** Let us consider the function

$$z(\rho) = \left( \frac{A(1-s)}{G(1+\delta)} \right)^{1/(1-s)} (\rho - \rho_0)^{(1+\delta)/(1-s)},$$

satisfying the conditions

$$Az^s = G\rho^{-\delta}z' \quad \text{as } \rho \in (\rho_0, R), \quad z(\rho_0) = 0. \quad (3)$$

Introduce the function

$$\Phi(\rho) := \exp \left( -\frac{sA}{G} \int_{\rho_0}^{\rho} \sigma^{\delta} d\sigma \int_0^1 (\theta U + (1-\theta z))^{s-1} d\theta \right)$$

and observe that always

$$U^s - z^s \equiv \int_0^1 (\theta U + (1-\theta z))^{s-1} d\theta (U - z).$$

Subtracting now termwise equality (3) from inequality (1) and multiplying the result by the function  $\rho^{-\delta}G^{-1}\Phi(\rho)$ , we get:

$$\frac{d}{d\rho} \{(u-z)\Phi\} \geq -\frac{\rho^{\delta}}{B}\Phi\varphi. \quad (4)$$

Integrate inequality (4) over the interval  $(\rho_0, \rho)$ :

$$U(\rho) \geq z(\rho) + \frac{1}{\Phi(\rho)}U(\rho_0) - \frac{1}{G\Phi(\rho)} \int_{\rho_0}^{\rho} \sigma^{\delta}\Phi(\sigma)\varphi(\sigma)d\sigma. \quad (5)$$

Let us relax (5), having rewritten it in the form

$$\begin{aligned} M &\geq U(\rho_0) - \frac{1}{G} \int_{\rho_0}^{\rho} \sigma^{\delta}\varphi(\sigma) \\ &\times \exp \left( \frac{sA}{G} \int_{\sigma}^{\rho} \tau^{\delta}d\tau \int_0^1 (\theta U(\tau) + (1-\theta)z(\tau))^{s-1} d\theta \right) d\sigma \end{aligned}$$

and then make use of the following relations:

$$\begin{aligned} &\exp \left( \frac{sA}{G} \int_{\sigma}^{\rho} \tau^{\delta}d\tau \int_0^1 (\theta U(\tau) + (1-\theta)z(\tau))^{s-1} d\theta \right) \\ &\leq \exp \left( \frac{sA}{G} \int_0^1 (1-\theta)^{s-1} d\theta \int_{\sigma}^{\rho} \tau^{\delta}z^{s-1}(\tau)d\tau \right) \\ &= \exp \left( \int_{\sigma}^{\rho} \frac{d(z(\tau))}{z(\tau)} \right) = \exp \left( \ln \left( \frac{z(\rho)}{z(\sigma)} \right) \right) = \frac{z(\rho)}{z(\sigma)}. \end{aligned}$$

In the result we have

$$\begin{aligned} 0 \leq U(\rho_0) &\leq M - z(\rho) \left\{ 1 - \frac{1}{G} \int_{\rho_0}^{\rho} \sigma^{\delta} \frac{\varphi(\sigma)}{z(\sigma)} d\sigma \right\} \\ &\equiv M - (\rho - \rho_0)^{(1+\delta)/(1-s)} \left\{ \left( \frac{A(1-s)}{G(1+\delta)} \right)^{1/(1-s)} - \frac{z(\rho)}{G} \right\} := F(\rho). \quad (6) \end{aligned}$$

Assuming existence of some  $\rho_* \in (\rho_0, R)$  such that  $F(\rho_*) = 0$ , we get:  $U(\rho_0) = 0$ .

□

### 3.2. PROOFS OF THEOREMS 1-3.

We begin with the proof of Theorem 1. One has just to verify that the conditions of Lemma 1 are fulfilled. Assume  $u_0(x) \equiv 0$  in a ball  $B_{\rho_0}(x_0)$  and  $f \equiv 0$  in the cylinder  $P(0, \rho)$ , having this ball as the down-base. Let  $R > 0$  be such that  $P(0, R) \subset Q$ , and the integral  $I$  defined in the conditions of Theorem 1 is convergent. Assuming the restrictions on the structural constants listed in the conditions of Theorem 1, we derive for the corresponding energy function inequality (16). By Lemma 1, we see that it is sufficient to point out a threshold value of the total energy  $M_*^{p(1+\epsilon)/(p-1)}$  such that equation (2) would have a solution  $\rho_*$ . Recall that in the case of inequality (16) the coefficient  $G$  of inequality (1) depends only on structural constants and the energy  $M_*^{p(1+\epsilon)/(p-1)}$ , but does not depend on  $t$ . So, for the function  $F(\rho)$  defined in (6) satisfies  $F(\rho) \rightarrow -\infty$  as  $M \rightarrow 0$  for each  $\rho \in (\rho_0, R)$  fixed. Further,  $G$  is a linear function of the argument  $M_*^{p(1+\epsilon)/(p-1)}$  so that  $G \rightarrow \infty$  as  $M \rightarrow \infty$ ,  $G \rightarrow 0$  as  $M \rightarrow 0$ . Then from (6), having just compared the orders of  $M$  of positive and negative terms of  $F(\rho)$ , that  $F(\rho) > 0$  for large  $M$ . This means that  $F(R)$ , being viewed as a function of  $M$ , is always nonnegative for small  $M$ , which proves the theorem. □

The proof of Theorem 2 literally repeats the arguments just presented. The only difference is that now one has to add condition (11), needed for the derivation of (16).

For the proof of Theorem 3 we assume that the value of  $T$  is taken so as to satisfy  $P \subset Q$ . Remind that the coefficient  $c(t)$  in inequality (14) may be estimated from above by  $l := c(0)$ . Introduce the function  $z(T-t) := Y(t)$ . Since it satisfies the inequality

$$z^{p/(p-1)}(t) \leq lz'(t) \quad \text{as } t \in (0, T), \quad z(0) = 0, \quad z(t) \in [0, d(u)],$$

there remains to apply Lemma 1 with  $i(\rho) \equiv 0$  to complete the proof of Theorem 3.

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