Inequalities for the Capillary Problem with Volume Constraint

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Dedicated to our friend and colleague Ivar Stakgold

Abstract
We use **rearrangements methods** to estimate the minimal height of a liquid in a tube. We also use comparison techniques in order to give upper and lower bounds for the critical volume which gives rise to a free boundary. The main motivation comes from a conjecture posed by R. Finn in 1986.

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1 Introduction

In this paper we discuss the equilibrium surface of a liquid in a cylindrical tube of cross–section D and horizontal basis. Its height is denoted by $u(x, y)$ and is determined by means of the principle of virtual work. (see e.g. [5]). If $\kappa > 0$ is the capillarity constant, $\gamma \in [0, \pi/2)$ the wetting angle and V the prescribed volume of the liquid, the total energy is

$$J[u] = \int_D \sqrt{1 + |\nabla u|^2}\, dx\, dy + \frac{1}{2}\kappa \int_D u^2\, dx\, dy - \cos\gamma \oint_{\partial D} u\, ds\,.$$

The height $u(x, y)$ is the solution of the variational problem

$$(1) \qquad J[v] \to \min\,, \quad v \in K := \{w \in BV(D) : w \geq 0\,, \int_D w\, dx\, dy = V\}\,,$$

where $BV(D)$ is the space of bounded variations functions.

It was shown in [6] that (1) has a unique solution $u \in C^1(\bar{D})$ which is locally analytic in $D^+ := \{(x, y) : u(x, y) > 0\}$, has the property that $\operatorname{div} Tu \in L^\infty(D^+)$, $Tu := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$, and satisfies the Laplace equation and the capillary boundary condition

$$(2) \qquad \begin{cases} \operatorname{div} Tu = \kappa u + \lambda & \text{in } D^+\,, \quad u \geq 0 \quad \text{in } D\,, \\ (Tu, n) = \cos\gamma & \text{on } \partial D\,. \end{cases}$$

Here n is the outer normal to ∂D and $(\ ,\ )$ denotes the scalar product. The constant λ is a Lagrange multiplier which is easily obtained from (2) by integration by parts. More precisely we have

$$(3) \qquad \lambda = \frac{1}{A^+}(L\cos\gamma - \kappa V)\,,$$

where

$$A^+ = \text{ meas } D^+ \quad \text{and} \quad L = \oint_{\partial D} ds\,.$$

If no volume constraint is imposed then the height u_0 is a solution of (1) with K replaced by $K_0 = \{w \in BV(D)\}$. The minimizer is positive and satisfies $\operatorname{div} Tu = \kappa u$ in D. The volume V_0 of the liquid is then

$$V_0 = L\cos\gamma/\kappa\,. \tag{4}$$

The maximum principle applies and implies that u_0 takes its maximum at the boundary and its minimum, say α, at an inner point. It is well–known [6] that there exists a critical volume V^c such that $D_0 = D - D^+ = \{(x,y) \in D : u(x,y) = 0\}$ is void if $V > V^c$. It is readily seen that $D_0 \neq \phi$ for $V \leq V^c$. By shifting the solution $u_0(x,y)$ of the problem without volume constraint, it follows that

$$V^c = V_0 - \alpha A\,, \quad A := \text{ meas } D\,. \tag{5}$$

The case $\kappa = 0$ is peculiar. It corresponds to the capillary problem without gravity. A solution exists only if $\lambda A^+ = \cos\gamma L$ [5].

Equation (2) can also be used to describe a fluid in a tube with a pressure different than the one of the surrounding media. In this case λ is determined by the difference of the pressures. The aim of this paper is to estimate u and in particular α, by means of rearrangement methods [1]. It was motivated by the following conjecture of Finn [5]: *Among all cross–sections D of given area, the disk has the minimal α.* For references on this and related problems we refer to [2], [3], [4], [5], [7], [8]. We also construct an upper solution to localize D^+.

2 Rearrangement Inequalities

Let us first introduce some notation connected with the solution $u(x,y)$ of problem (2). Given $\tilde{u} \in \mathbb{R}^+$ let

$$D(\tilde{u}) := \{(x,y) \in D : u(x,y) < \tilde{u}\}$$

$a(\tilde{u}) :=$ meas $D(\tilde{u})$ is the distribution function. The increasing rearrangement of u is defined by $\tilde{u}(s) := \inf\{t : a(t) \geq s\}$. The boundary of $D(\tilde{u})$ consists of the sets

$$\Gamma(\tilde{u}) := \{(x,y) \in D : u(x,y) = \tilde{u}\} \quad \text{and} \quad \Gamma_0(\tilde{u}) := \partial D \cap \partial D(\tilde{u})\,.$$

For $\tilde{u} \in (\alpha, \min_{\partial D} u)$, $\Gamma_0(\tilde{u})$ is empty.

The goal of this section is to derive a differential inequality for the rearrangement $\tilde{u}(a)$. The divergence theorem yields

$$\int_{D(\tilde{u})} \operatorname{div} Tu\, dx\, dy = \int_{\Gamma(\tilde{u})} \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} ds + \cos\gamma \int_{\Gamma_0(\tilde{u})} ds \tag{6}$$

on the other hand we obtain from (2) and Cavalieri's principle

$$\begin{aligned} \int_{D(\tilde{u})} \operatorname{div} Tu\, dx\, dy &= \kappa \int_{D(\tilde{u})} u\, dx\, dy + \lambda a \\ &= \kappa \int_0^a \tilde{u}(s)\, ds + \lambda a =: \nu(a) \end{aligned} \tag{7}$$

Define $q(t) = \frac{t^2}{\sqrt{1+t^2}}$. It is an increasing function in $\mathbb{R}^+$ which is convex for $0 \le t \le \sqrt{2}$ and concave for $t > \sqrt{2}$. Denote by $q_0(t)$ any function which for positive t is increasing, convex and satisfies

$$(i) \quad q_0(t) \le q(t) \quad \text{for} \quad t > 0, \quad q_0(0) = 0$$
$$(ii) \quad q_0(t)/t \to \cos\gamma \quad \text{as} \quad t \to \infty.$$

Examples

(1) $q_0(t) = \cos\gamma t^2/(1+t)$ (This function was proposed by Talenti [8]).

$$(2) \qquad q_0(t) = \begin{cases} q(t) & t \in (0, t_0) \\ \cos\gamma t + q(t_0) - \cos\gamma t_0 & t \ge t_0 \end{cases}$$

where t_0 is the positive root of $q'(t_0) = \cos\gamma$. Put

$$t_m = \begin{cases} |\nabla u| & \text{on } \Gamma(\tilde u) \\ m & \text{on } \Gamma_0(\tilde u) \end{cases}, \quad dp_m = \frac{ds}{t_m}$$

and $P_m = \oint_{\partial D(\tilde u)} dp_m$. The latter definition makes sense if $|\nabla u| \ne 0$ on $\Gamma(\tilde u)$. According to Sard's lemma this is true for almost all $\tilde u > 0$. The right-hand side of (6) can now be estimated as follows

$$(8) \qquad \nu(a) \ge \int_{\partial D(\tilde u)} q_0(t_m)\, dp_m$$

Since q_0 is convex, Jensen's inequality applies and yields

$$(9) \qquad \nu(a) \ge P_m\, q_0(L(\partial D(\tilde u))/P_m)$$

This is true for all m and also if we let m tend to infinity.

$$P_m \to P = \int_{\Gamma(\tilde u)} \frac{ds}{|\nabla u|} \quad \text{as} \quad m \to \infty\,.$$

From the coarea formula it follows that

$$(10) \qquad P = \frac{da}{d\tilde u}\,.$$

By the isoperimetric inequality we have $L^2(\partial D(\tilde u)) \ge 4\pi a$. Since q_0 is monotone,

$$(11) \qquad P q_0(L(\partial D(\tilde u))/P) \ge q_0(\sqrt{4\pi a}\tilde u'(a))/\tilde u'(a).$$

Finally we obtain the <u>rearrangement inequality</u>

$$(\mathrm{I}) \qquad \kappa V(a) + \lambda a \ge \frac{q_0(\sqrt{4\pi a}\tilde u'(a))}{\tilde u'(a)}, \quad V(a) := \int_{D(\tilde u)} u\, dx dy \quad \text{in} \quad (0, A)\,.$$

Let us now discuss the case of equality. Consider a solution $\tilde u^*$ of

$$(12) \qquad \kappa V^*(a) + \lambda^* a = \frac{q_0(\sqrt{4\pi a}\tilde u^{*\prime}(a))}{\tilde u^{*\prime}(a)}, \quad V^*(a) := \int_0^a \tilde u^*(s)\, ds$$

After the change of variable $\pi r^2 = a$ and $u^*(r) = \tilde{u}^*(a)$, $a = \pi r^2$, (12) becomes

$$\kappa u^*(r) + \lambda^* u^*(r) = \frac{1}{r}\left(\frac{q_0(u^{*'})r}{u^{*'}}\right)' \quad \text{in} \quad (0, R). \tag{13}$$

If we impose the boundary conditions $u^{*'}(0) = 0$ and $u^{*'}(R) = \cos\gamma$, $\pi R^2 = A$, then u^* can be interpreted as the radial solution of

$$\operatorname{div}(g(|\nabla u^*|)\nabla u^*) = \kappa u^* + \lambda^* \quad \text{in} \quad D^*, g(t) := \frac{q_0(t)}{t^2} \tag{14}$$

where D^* is the circle of the same area as D. The existence of such a solution is guaranteed for the examples given before. $u^{*'}$ is positive regardless of the sign of λ^*.

REMARK 2.1. *It can be shown that for smooth boundaries the function $\tilde{u}(a)$ is absolutely continuous.*

3 Comparison of the minimum height

Let $u(x, y)$ be the solution of (1) and let $u_0(x, y)$ be the solution of (1) without volume constraint. Besides of (1) we shall consider the following "comparison" problem

$$\int_{D^*} G(|\nabla v|)\, dx\, dy + \frac{\kappa^*}{2}\int_{D^*} v^2\, dx\, dy - \cos\gamma \oint_{\partial D^*} v\, ds \to \min, \tag{15}$$

where $v \in K^* = \{w \in BV(D^*),\ w \geq 0,\ \int_{D^*} w^*\, dx\, dy = V^*\}$.

Here $G' = tg(t) = q_0(t)/t$. The corresponding Euler equation is (14) with the boundary condition

$$g(|\nabla u^*|)\partial u^*/\partial n = \cos\gamma .$$

The solution of (15) will be denoted by u^*. Accordingly we write u_0^* for the solution of (15) with K^* replaced by $K_0^* = BV(D^*)$. In the sequel the $*$ refers to quantities related to (15).

THEOREM 3.1. *The solutions of problems (1) and (15) without volume constraints satisfy $\alpha \geq \alpha^*$.*

Proof. From (I) and (12) we infer

$$\kappa[V(a) - V^*(a)] \geq \frac{q_0(\sqrt{4\pi a}\tilde{u}_0'(a))}{\tilde{u}_0'(a)} - \frac{q_0(\sqrt{4\pi a}\tilde{u}_0^{*'}(a))}{\tilde{u}_0^{*'}(a)} . \tag{16}$$

Integrating (14) over D^* and taking into account the boundary conditions satisfied by the minimizer of (15) we find $V_0^* = L^*\cos\gamma/\kappa$. By (4) and the isoperimetric inequality $V_0 \geq V_0^*$. Put $\delta(a) = V(a) - V^*(a)$. By the previous observation $\delta(A) \geq 0$. From (16) it then follows that $\delta(a) \geq 0$ for all a. In fact, suppose that $\delta(a)$ takes a negative minimum at a_1. Then $0 \leq \delta''(a_1) = \tilde{u}_0'(a_1) - \tilde{u}_0^{*'}(a_1)$. By (16) and the monotonicity of $q_0(t)/t$ (which follows immediately from the definition)

$$\kappa(V(a_1) - V^*(a_1)) \geq 0$$

which is a contradiction to our assumption. Since $V(a) \geq V^*(a)$ and $V(0) = V^*(0) = 0$ the assertion is now obvious.

REMARK 3.1. *The solution of (15) without volume constraint never coincides with the corresponding one of (1) even if we choose for q_0 the function given in the second example. This follows immediately from the fact that t_0 is always smaller than the value for which $q(t)/t = \cos\gamma$. The inequality in Theorem 3.1 is therefore never isoperimetric.*

COROLLARY 3.1. *If $V^* - V_0^* \leq V - V_0$, then the solutions of (1) and (15) satisfy $\alpha \geq \alpha^*$.*

Proof. Let $V = V_0 + hA$ and $V^* = V_0^* + h^*A$. By our assumption, $h \geq h^*$. If $h \geq -\alpha_0 := \min u_0$, then $u = u_0 + h$ is the desired solution. If $h \leq -\alpha_0$, $\bar{u} = u_0 - \alpha_0$ is a super solution and by the comparison principle [5], $\min u = 0$. Consequently $\alpha = \min(\alpha_0 + h, 0)$ and $\alpha^* = \min(\alpha_0^* + h^*, 0)$. Since $\alpha_0^* + h^* \leq \alpha_0 + h$, the proof is completed.

COROLLARY 3.2. *Assume $V \leq V^c(D)$ and $V^* - V_0^* \leq V - V_0$. Then $A_+ \geq A_+^*$.*

Proof. Suppose that the $A_+ \leq A_+^*$. $\tilde{u}$ satisfies the inequality (I) with (cf. (3)) $\lambda = \frac{\kappa}{A^+ \cos\gamma}(V_0 - V)$ and $\tilde{u}^*$ satisfies (12) with $\lambda^* = \frac{\kappa}{A_+^* \cos\gamma}(V_0^* - V^*)$. By (4) and the isoperimetric inequality $V_0 \geq V_0^*$. Thus because of our assumption $V \geq V^*$.The same arguments as for the Theorem 3.1 apply because $\lambda < \lambda^*$. Hence $V(a) \geq V^*(a)$ in $(0, A)$. By assumption $0 = \tilde{u}(a) = V'(a) < {V^*}'(a) = \tilde{u}^*(a)$ in $(A - A_+^*, A - A_+)$. Since $V(a) = V^*(a) = 0$ in $(0, A - A_+^*)$ this is impossible.

OBSERVATION 3.1. *As already noted the level lines $\Gamma(\tilde{u})$ are closed for $\tilde{u} \in (\alpha, \min_{\partial D} u)$. According to a result of Payne and Philippin [7] the function*

$$P = 2(1 - (1 + |\nabla u|^2))^{-1/2} \frac{\kappa u^2}{2}$$

takes its maximum at the points P on ∂D where $u(P) = \min_{\partial D} u(x, y)$. This together with Bernstein's result that $|\nabla u|^2$ takes its maximum on the boundary implies that

$$|\nabla u|^2 \leq ctg^2\gamma \quad in \quad D(\beta)\,, \quad \beta = \min_{\partial D} u\,.$$

If $ctg\gamma \leq \sqrt{2}$, then (I) with $q_0(t)$ holds in (α, β). If this suffices to establish Finn's conjecture at least for angles close to $\pi/2$ is still an open question.

4 Additional remarks

As we have seen before the estimate $\alpha \geq \alpha^*$ provides an upper bound for V^c in terms of κ, γ, L and A. We shall construct a lower bound by means of an upper solution.

The crucial tool is the following comparison lemma due to Concus & Finn (see [5]). If $\partial D = \Gamma_0 + \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\bar{u}$, $\underline{u}$ are two functions satisfying

$$\operatorname{div} T\underline{u} - \kappa\underline{u} \geq \operatorname{div} T\bar{u} - \kappa\bar{u} \quad \text{in} \quad D\,,$$

$\bar{u} \geq \underline{u}$ in Γ_0, $(n, T\bar{u}) \geq (n, T\underline{u})$ in Γ_1, then $\bar{u} \geq \underline{u}$ in D.

A suitable candidate for $\bar{u}$, already used by Finn [5], p. 113, and Gerhardt [6] is $\bar{u} = c - (R^2 - r^2)^{1/2}$ which satisfies

$$\operatorname{div} T\bar{u} = \frac{2}{R} \quad \text{for} \quad r < R\,, \quad (T\bar{u}, n) = 1 \quad \text{for} \quad r = R\,. \tag{17}$$

If $c = \frac{2}{\kappa R}$, then $\bar{u}$ satisfies the comparison principle for u_0 [5]. Consequently, if r_0 is the inradius of D,

$$\alpha < \frac{2}{\kappa r_0}\,. \tag{18}$$

This together with (5) implies

$$V^c \geq \frac{L\cos\gamma}{\kappa} - \frac{2A}{\kappa r_0}. \tag{19}$$

$\bar{u}$ can also be used to localize the set D^+ in case of a small volume V. This idea has already been applied in [4] in the case of fixed λ and will be repeated for the sake of completeness.

Let $P \in D$ be a point such that $\operatorname{dist}(P, \partial D) = \rho$ and let B_ρ be the disk of radius ρ centered at P. Then the function $\bar{u} = \rho - (\rho^2 - r^2)^{1/2}$ satisfies

$$\operatorname{div} T\bar{u} = \frac{2}{\rho} \leq \kappa\bar{u} + \frac{2}{\rho} \quad \text{in } B_\rho\,, \ (T\bar{u}, n) = 1 \quad \text{on } \partial B_\rho\,.$$

It is therefore an upper solution for the solutions u of (2) with $\lambda \geq \frac{2}{\rho}$. This construction is independent of the particular position of P. Hence $u = 0$ on $\{P \in D : \operatorname{dist}(P, \partial D) = \rho\}$. By the comparison lemma, $u \equiv 0$ in $\{P \in D : \operatorname{dist}(P, \partial D) > \rho\}$. Hence

$$D^+ \subset D_\rho := \{P \in D : \operatorname{dist}(P, \partial D) < \rho\}\,. \tag{20}$$

In view of (3) we must have

$$\lambda = \frac{1}{A^+}(L\cos\gamma - \kappa V) \geq \frac{2}{\rho}\,. \tag{21}$$

Since $A^+ \leq A_\rho := \operatorname{meas} D_\rho$, (21) holds for all V such that $\kappa V \leq L\cos\gamma - \frac{2A_\rho}{\rho}$. We have thus established the following result.

THEOREM 4.1. *If $V \leq (L\cos\gamma - 2A_\rho/\rho)\kappa^{-1}$, then a free boundary occurs and $D^+ \subset D_\rho$.*

Remarks

(1) Since $\lim_{\rho\to 0} A_\rho/\rho = L$, the assertion makes only sense for $\rho \geq \rho_0 > 0$.

(2) Let $\beta = \max_{0<\rho\leq r_0}(L\cos\gamma - 2A_\rho/\rho)\kappa^{-1}$. Then $V^c \geq \beta$.

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