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On a nonlinear stationary free boundary problem arising in the magnetic confinement of a plasma in a Stellarator.

We prove the existence of a solution (u, F) of an inverse stationary two-dimensional problem arising in the mathematical modeling of the magnetic confinement of a plasma in a Stellarator.

1. Modeling.

By using the Boozer vacuum coordinates and averaging the poloidal magnetic flux over the toroidal angle, Hender and Carreras [8] obtained a nonlinear elliptic equation for the magnetic poloidal flux function $\psi(\rho,\ \theta)$ associated to the magnetic field for the confinement of a plasma in a Stellarator. The free boundary formulation is as follows (Díaz [2]): Given $R > 0$, let $\Omega = \{(\rho,\ \theta) : R > \rho \geq 0,\ \theta \in (0, 2\pi]\}$ and let $\lambda > 0$, $F_v > 0$, a, $b \in L^\infty(\Omega)$ with $b > 0$ a.e. in Ω. Given $\gamma \in \mathbb{R}$ we ask for the existence of $\psi : \Omega \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}_+$, with $F(s) = F_v$ for any $s \leq 0$, satisfying

$$
(\mathcal{P}_1)\begin{cases}
-\mathcal{L}\psi = a(\rho,\theta)F(\psi) + F(\psi)F'(\psi) + \lambda b(\rho,\theta)\psi_+ & \text{in } \Omega \quad (1)\\
\psi = \gamma \text{ on } \Gamma_R, \quad \psi(\rho,0) = \psi(\rho,2\pi) \text{ for } \rho \in (0,R), \ \frac{\partial u}{\partial \theta} = 0 \text{ on } \Gamma_0 & (2)\\
0 = \int_{\{\psi \geq t\}} [F(\psi)F'(\psi) + \lambda b\psi_+]\rho d\rho d\theta & \text{for all } t \in (\inf\psi, \sup\psi), \quad (3)
\end{cases}
$$

where $\Gamma_R = \{(R,\ \theta) : \theta \in (0, 2\pi]\}$ and $\Gamma_0 = \{(0,\ \theta) : \theta \in (0, 2\pi]\}$. Here $\mathcal{L}$ is a linear second order elliptic operator with coefficients determined through the metric of the vacuum magnetic surfaces, a and b are given functions, F represents the averaged covariant coordinate of the magnetic field and we are assuming the constitutive law $p(\psi) = \frac{\lambda}{2}\psi_+^2$ (where $\psi_+ = \max(\psi,\ 0)$) and that $\partial\Omega$ is a perfect conducting wall containing the plasma region $\Omega_p = \{(\rho,\ \theta) \in \Omega : \psi(\rho,\ \theta) > 0\}$ in the interior of Ω. The condition (3) is typical of Stellarators and express the zero net current within each flux surface.

The main purpose of this communication is to present the results of Díaz - Rakotoson [4], [5] showing that problem $(\mathcal{P}_1)$ is (mathematically) well possed at least for small values of λ (i.e. the usual parameter β is not too large). This give a positive answer to the question raised in the literature since some time ago (e.g. Shoet [12], Freidberg [6], Garabedian [7]...).

2. Formulation in terms of relative rearrangement.

If we assume that the measure of the set $\{(\rho, \theta) : \nabla\psi(\rho, \theta) = 0\}$ is zero it was shown in Díaz [3] that if we denote ψ by u then u satisfies the problem

$$
(\mathcal{P}_2)\begin{cases}
-\mathcal{L}u = a(\rho,\ \theta)\Big(F_v^2 - \lambda \int_{m(0)}^{m(u_+(\rho,\ \theta))} \frac{d}{d\sigma}(u_{*+})^2(\sigma)b_{*u}(\sigma)d\sigma\Big)_+^{\frac{1}{2}} + \\
\quad +\lambda u_+(\rho,\ \theta)[b(\rho,\ \theta) - b_{*u}(m(u(\rho,\ \theta))] & \text{in } \Omega \quad (4)\\
u = \gamma & \text{on } \partial\Omega,
\end{cases}
$$

where $m(t) = \int_{\{u>t\}} \rho d\rho d\theta$ is the distribution function of u, u_* is the decreasing rearrangement and b_{*u} is the relative rearrangement of b with respect to u (in all of the cases for the measure $\rho d\rho d\theta$) (see Mossino-Temam [9] and Rakotoson-Temam [11]). It is possible to show that $(\mathcal{P}_2)$ and $(\mathcal{P}_1)$ are in fact equivalent.

3. Existence of solutions giving rise to a free boundary.

We introduce the functional spaces

$$L^2(\Omega,\ \rho) = \{u : \Omega \to \mathbb{R} \text{ measurable, s.t. } \int_\Omega |u(\rho,\ \theta)|^2 \rho d\rho d\theta < \infty\},$$

$$H^1(\Omega;\rho) = \{u \in L^2(\Omega;\rho);\ \frac{\partial u}{\partial \rho} \in L^2(\Omega;\rho);\ \frac{1}{\rho}\frac{\partial u}{\partial \theta} \in L^2(\Omega;\rho)\}$$

which is a Hilbert space with

$$(f,\ g) = \int_\Omega \frac{\partial f}{\partial \rho}\frac{\partial g}{\partial \rho}\rho d\rho d\theta + \int_\Omega \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \theta}\frac{d\rho d\theta}{\rho} + \int_\Omega fg\rho d\rho d\theta,$$

and finally $H_0^1(\Omega;\rho) = \overline{C_c^\infty(\Omega)}^{H^1(\Omega;\rho)}$. Thanks to the Sobolev-Poincaré inequality over $H_0^1(\Omega;\ \rho)$ (see Rakotoson-Simon [10]) $\mathcal{L}$ is coercive. We denote by $\lambda_1 = \inf\{<-\mathcal{L}\varphi, \varphi>,\ \varphi \in H_0^1(\Omega;\ \rho) \text{ s.t. } \int_\Omega \varphi^2 \rho d\rho d\theta = 1\}$. We have

Theorem 1. *Assume that* $\lambda|b|_\infty < \lambda_1\varepsilon$ *for a suitable* $\varepsilon \in (0,\ 1)$. *Then there exists* $u \in W^{2,\ p}_{loc}(\Omega) \cap W^{1,\ \infty}(\Omega)$ *(for any* $1 \le p < \infty$*) solution of* $(\mathcal{P}_2)$ $(u - \gamma \in H_0^1(\Omega;\ \rho))$. *Moreover* $\text{meas}\{(\rho,\ \theta) \in \Omega :\ \nabla u(\rho,\ \theta) = 0\} = 0$.

The physical case corresponds to $\gamma < 0$. The following result gives some conditions for the existence of a free boundary (the boundary of the plasma region $\Omega_p = \{(\rho,\ \theta) \in \Omega :\ u(\rho,\ \theta) > 0\}$).

Theorem 2. *Let* ψ_1 *be the unique positive function satisfying*

$$\begin{cases} -\mathcal{L}\psi_1 = \lambda_1\psi_1 \\ \psi_1 \in\ H_0^1(\Omega;\ \rho) \\ \lambda_1 \int_\Omega \psi_1\ \rho d\rho d\theta =\ 1. \end{cases}$$

If $-\gamma < F_v \int_\Omega a(\rho,\ \theta)\psi_1\ (\rho,\ \theta)\rho d\rho d\theta$ *then* $0 < \text{meas}\{(\rho,\ \theta) \in \Omega :\ u(\rho,\ \theta) > 0\} < meas(\Omega)$.

The relation between the constant γ and the measure of the plasma region is given in our last result

Theorem 3. *Let* $\gamma_0 = -F_v \int_\Omega a(\rho,\ \theta)\phi_1(\rho,\ \theta)\rho d\rho d\theta$. *Assume* $\gamma > \gamma_0$ *and* $\lambda|b|_\infty < \lambda_1\varepsilon$ *for a suitable* $\varepsilon \in (0,\ 1)$. *Then there exists an increasing function* $M : (\gamma_0,\ 0) \to (0,\ \infty)$, *with* $M(\gamma) \to 0$ *if* $\gamma \to \gamma_0$, *such that* $m(0) \equiv \int_{\{u>0\}} \rho d\rho d\theta \ge M(\gamma)$.

Remark *The proof of Theorem 1 is carried out by means of an iterative algorithm. The most delicate point is the passing to the limit which is justified by generalizing a result due to Almgren and Lieb (1989) on the continuity of the decreasing rearrangement.*

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