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On the multiplicity of stationary solutions of a global two-dimensional climate model.

We present some recent results on the mathematical study of a global two-dimensional stationary climate model. We prove the multiplicity of solutions for some values of a solar parameter. Then, we obtain a S-shaped bifurcation branch.

1. Introduction and Preliminares

The so called *climate energy balance models* were introduced, independently, by M. Budyko [2] and W. Sellers [10]. The energy balance is stated in the following terms:

$$\text{Heat variation} = R_a - R_e + D,$$

where R_a represents the solar energy absorbed by the Earth, R_e is the energy emitted by the Earth to the outer space and D is the temperature diffusion. If we denote by u the mean surface temperature of the Earth, then it is usual to take $R_a = QS(x)\beta(u)$ with $Q > 0$ the Solar Constant, $S(x) > 0$ the insolation function and $\beta(u)$ the coalbedo function (a nondecreasing function of u such that $\beta(u) = 0,7$ if $u > -10 + \epsilon$, $\beta(u) = 0,4$ if $u < -10 - \epsilon$, for some $\epsilon \geq 0$). The term R_e is also assumed to be a nondecreasing function on u. Assuming (for simplicity) the heat capacity and the diffusion coefficient equal to one, we obtain an energy balance model of the type

$$(P)\begin{cases} u_t - \Delta_p u + R_e(u) \in QS(x)\beta(u) & \text{in } (0,\infty)\times\mathcal{M} \\ u(0,x) = u_0(x) & \text{on } \mathcal{M} \end{cases}$$

where

- $(\mathcal{M}, \mathrm{g})$ is a compact bidimensional Riemannian manifold without boundary (as, for instance, $\mathcal{M} = \mathbf{S}^2$ the unit sphere of $\mathbb{R}^3$), simulating the Earth.
- $\Delta_p u = \mathrm{div}_{\mathcal{M}}(|\mathrm{grad}_{\mathcal{M}} u|^{p-2}\mathrm{grad}_{\mathcal{M}} u)$ for some $p \geq 2$, where grad $_{\mathcal{M}}$ is understood in the sense of the Riemannian metric. Budyko [2] and Sellers [10] considered $p = 2$. Later, Stone [11] proposed the case $p = 3$ arguing that the diffusion coefficient must increase as the gradient of the temperature increases.
- In the Budyko model R_e is represented by a Newton cooling law as $Bu + C$ with B and C positive parameters. Stefan - Bolzman law modelizes R_e as $C|u|^3u$ (Sellers [10]).
- $S : \mathcal{M} \to \mathbb{R}$, $0 < \underline{S} \leq S(x) \leq \overline{S}$, $S \in L^\infty(\mathcal{M})$.
- β is a bounded maximal monotone graph of $\mathbb{R}^2$, and $m \leq b \leq M$ for any $b \in \beta(s)$ for any $s \in \mathbb{R}$ (sometimes β is assumed to be either *multivalued* at $u = -10$, Budyko [2] or a *locally Lipschitz* function, Sellers [10]).

The general theory (existence and uniqueness of weak solutions) for this class of problems was carried out in Díaz [4] for the one-dimensional model and then generalized in Díaz-Tello [6] to the two-dimensional case. The existence of solutions was obtained in the space $C([0,\infty); L^2(\mathcal{M})) \cap L^p_{loc}(0,\infty; V)$, where $V = \{u \in L^2(\mathcal{M}) : \mathrm{grad}_{\mathcal{M}} u \in L^p(T\mathcal{M})\}$.

2. Stationary solutions

We consider the quasilinear elliptic problem (P_Q) $\quad -\Delta_p u + Bu + C \in QS(x)\beta(u)$ in $\mathcal{M}$.

Theorem 1. *(Díaz - Hernández - Tello [5])*

There exist four explicit values of Q, $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$ such that

i) If $0 < Q < Q_1$ then (P_Q) has a unique solution.

ii) If $Q_2 < Q < Q_3$ then (P_Q) has at least three solutions.

iii) If $Q_4 < Q$ then (P_Q) has a unique solution.

R e m a r k . The result seems to be new even for $p = 2$ (Laplacian operator). Previous results are due to Hetzer [8] for the Sellers model and $p = 2$.

R e m a r k . The result also holds for $\Omega \subset \mathbb{R}^2$ regular bounded domain and adding Neumann boundary conditions.

P r o o f. The proof of (i) and (ii) is based in sub - supersolutions theory and the comparison principle for the Δ_p operator. In order to prove (ii) we consider a sequence of lipschizian functions β_ϵ which converges to the maximal monotone graph β. Firstly, we prove the result for the approximate model

$$(P_Q^\epsilon) \qquad -\Delta_p u + Bu + C \in QS(x)\beta_\epsilon(u) \text{ in } \mathcal{M}$$

by using topological degree techniques. Secondly, we pass to the limit in $L^\infty(\mathcal{M})$ in the three sequences u_1^ϵ, u_2^ϵ and u_3^ϵ of solutions of (P_Q^ϵ), if $Q_2 < Q < Q_3$, and we use suitable a priori estimates to separate the limits.

3. Bifurcation branch.

We define $\Sigma := \{(Q,u) \in \mathbb{R}^+ \times V : -\Delta_p u + Bu + C \in QS(x)\beta(u) \text{ in } \mathcal{M}\}$.

T h e o r e m 2. *(Arcoya - Díaz - Tello [1])*

Σ has a S-shaped unbounded component starting in $(0, \frac{-C}{B})$.

P r o o f. First, we prove the result for the aproximated model (P_Q^ϵ) previously defined. This proof uses Rabinowitz bifurcation theorem and the strong maximum principle in order to compare the bifurcation curve of zero-dimensional models (P_0^1) and (P_0^2) with the bifurcation branch of (P_Q^ϵ), where

$$\begin{array}{lrcl} (P_0^1) & Bu + C & = & Q\overline{S}\beta_\epsilon(u) \\ (P_0^2) & Bu + C & = & Q\underline{S}\beta_\epsilon(u). \end{array}$$

If we call Σ_ϵ the branch of (P_Q^ϵ), we can obtain the convergence of Σ_ϵ with $\epsilon \to 0$ (in a suitable sense) to Σ by a topological argument preserving the S-shape.

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