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On the uniqueness of solutions of a nonlinear elliptic problem arising in the confinement of a plasma in a Stellarator device.

We obtain the uniqueness of solutions of a nonlocal elliptic problem when the nonlinear terms at the right hand side are assumed to be prescribed. The problem arises in the study of the magnetic confinement of a plasma in a Stellarator device.

1. Introduction.

The main goal of this communicztion is to prove the uniqueness of the solution of a two dimensional free boundary problem modeling the magnetic confinement of a plasma in a Stellarator device. The model consists of a second order partial differential equation of elliptic type, obtained from the 3-D ideal MHD system by Hender and Carreras [4] by using toroidal averaging arguments and a suitable system of coordinates. This problem has recently been studied by Díaz [1] who introduced the following formulation in the form of a *free boundary problem.* Let Ω be an open, bounded, regular and connected set contained in $\mathbb{R}^2$, and let

$$\lambda > 0, F_v > 0, a, b \in L^\infty(\Omega), a \geq 0, b > 0 \text{ a.e. in } \Omega.$$

Given $\gamma \in \mathbb{R}_- := \{t \in \mathbb{R} : t < 0\}$, the problem is to find

$$u : \Omega \to \mathbb{R} \text{ and } F : \mathbb{R} \to \mathbb{R}_+$$

such that $F(s) = F_v$ for any $s \leq 0$ and the following conditions hold:

$$(\mathcal{P}) \left\{ \begin{array}{rcll} -\Delta u &=& aF'(u) + F(u)F'(u) + \lambda b u_+ & \text{in } \Omega \\ u &=& \gamma & \text{on } \partial\Omega \\ 0 &=& \int_{\{u>t\}} \{F(u)F'(u) + \lambda u_+ b\} & \forall t \in [\text{essinf} u,\ \text{esssup} u] \end{array} \right.$$

In the sequel we will refer to the family of integral identities stated in $(\mathcal{P})$ as the *Stellarator Condition.*

In order to characterize the unknown function F, the above problem was reformulated in Díaz [1] using the notion of *relative rearrangement.* There, he proved that if (u, F) is a solution of $(\mathcal{P})$ such that $u \in \mathcal{U}$ where

$$\mathcal{U} = \{u \in W^{2,p}(\Omega), \text{ for any } 1 \leq p < \infty : meas\,\{x \in \Omega : \nabla u(x) = 0\} = 0\}$$

then u satisfies the following uncoupled non local equation

$$-\Delta u = a\left[F_v^2 - 2\lambda \int_0^{u_+(x)} \sigma b_{*u}(|u > \sigma|)d\sigma\right]_+^{1/2} + \lambda u_+[b - b_{*u}(|u > u(x)|)] \quad \text{in } \Omega \tag{1}$$

where we denote $meas\,\{x \in \Omega : u(x) > t\}$ by $|u > t|$, u_* represents the decreasing rearrangement of u and b_{*u} is the relative rearrangement of b with respect to u (the definition of these functions and some of their properties can be found, for instance, in [6] and in its references). Moreover, if u satisfies (1) then the function $F = F^u$ is given by

$$F^u(t) = \left[F_v^2 - 2\lambda \int_0^{t_+} \sigma b_{*u}(|u > \sigma|)d\sigma\right]_+^{1/2} \quad \text{for any } t \in [\text{essinf} u,\ \text{esssup} u].$$

The existence of u, solution of $(\mathcal{P})$ in the class of functions $\mathcal{U}$ was proved by Díaz and Rakotoson [2, 3]under some additional assumptions.

Here we give a partial result to the uniqueness question. Notice that the equation of $(\mathcal{P})$ involves nonlinear terms which do not need to be convex neither concave functions. Our proof uses some a priori estimates, some properties of the relative rearrangement and the study of a suitable weighted eigenvalue problem. The idea of using an auxiliary linear eigenvalue problem is inspired by the technique used in Puel [5] to establish the uniqueness of solution of a different free boundary problem arising in the study of the plasma confinement in Tokamak devices.

1 The main result.

To state the uniqueness result we shall need to refer to the weighted eigenvalue linear problem

$$(\mathcal{P}_g^\mu)\begin{cases} -\Delta w &= \mu g(x) w \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega \end{cases}$$

as well as to a suitable positive constant $\lambda_0 + \lambda_0(a, b, F_v, |\Omega|, \mu_2)$ which depends on the data of the problem a, b, F_v, on the constants of Poincaré and of a Sobolev's Imbedding and on μ_2, the second eigenvalue of $(\mathcal{P}_g^\mu)$.

Theorem Let (u, F) with $u \in \mathcal{U}$ be a solution of $(\mathcal{P})$. Suppose that FF' is Lipschitz on $I\!R$, i.e.,

$$|F(t)F'(t) - F(\hat{t})F'(\hat{t})| \leq \lambda K|t - \hat{t}|$$

for every $t, \hat{t} \in I\!R$ and for some positive constant K. Suppose also that the parameter $\lambda > 0$ is such that

$$\lambda < \lambda_0 \ ,$$

where λ_0 is the above mentioned constant and g is defined by

$$g(x) := C_2 ||b||_{L^\infty(\Omega)} a(x) + b(x) + K$$

for some known constant $C_2 > 0$. Then, if (v, F) is an other solution of $(\mathcal{P})$, then, necessarily, $v \equiv u$. .

Proof. Suppose that there exist two solutions u, v of $(\mathcal{P})$. The proof consists in two main steps:

- To verify that for small values of the parameter λ ($\lambda < \lambda_0$) the solutions are necessarily ordered, for instance, $u \leq v$. To do this we adapt a technique used by Puel [5] as well as some technical results on the regularity and positiveness of the function F.
- To derive a contradiction: if $u \geq v$ in Ω then, as $u = v$ on $\partial\Omega$ we obtain $\nabla u \cdot n \leq \nabla v \cdot n$. But integrating the equation of $\mathcal{P}$ in Ω and using the the Stellarator Condition , the Divergence theorem and the strictly decreasing character of F we arrive to

$$\int_{\partial\Omega} \nabla u \cdot n > \int_{\partial\Omega} \nabla v \cdot n$$

leading then to a contradiction.

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