ON THE BOUNDARY LAYER FOR DILATANT FLUIDS

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1 Introduction: the boundary layer and the von Mises transformation.

This paper deals with the boundary layer associated to a class of non Newtonian fluids, i. e., fluids for which the stress tensor T , at given temperature and pressure, is not a linear function of the spatial variation of the velocity $\mathrm{L} \doteq \nabla \mathrm{v}$. This class of fluids is relevant in many contexts: chemical engineering (polymer melts, polymer solutions, suspensions, lubricants, paints, etc.), liquid crystals, oriented media, fibrous media, animal blood etc. (see, e. g., Schowalter [28] and Narasimhan [17]). The above notion of non-Newtonian fluids fails to bound the subject. An important subclass is the so called *purely viscous non Newtonian fluids.* To introduce this notion we start from the *Reiner-Rivlin principle of material objectivity*

$$\mathrm{T} = -P\mathrm{I} + \phi_1(\mathrm{I}_1, \mathrm{I}_2)\mathrm{D} + \phi_2(\mathrm{I}_2, \mathrm{I}_3)\mathrm{D}^2,$$

where P is the pressure, $\mathbf{I}$ is the identity tensor and I_i $(i = 1, 2, 3)$ are the principal invariants of $\mathbf{D} \doteq \frac{1}{2}(\nabla \mathrm{v} + \nabla \mathrm{v}^T)$, the symmetric part of $\mathbf{L}$. The special case of ϕ_1 identically constant and $\phi_2 = 0$ corresponds to the case of incompressible Newtonian fluids. The more general case of $\phi_2 = 0$ and non-constant ϕ_1 defines the class of purely viscous non Newtonian fluids (also called generalized Newtonian fluids). It is useful to introduce the *shear stress* function

$$\tau(\kappa) = \frac{1}{2}\phi_1(\kappa)\kappa$$

where κ represents the *shear rate*. The *Power-law* or *Ostwald-de Waele model* is the one associated to the case of

$$\tau(\kappa) = K\,|\kappa|^{p-2}\,\kappa.$$

where $p > 1$ is given as a constitutive property of the fluid. If $p = 2$ we find again the class of Newtonian fluids. The case of $p > 2$ corresponds to the so called *dilatant fluids* and the case $1 < p < 2$ to the *pseudoplastic fluids.*

The Navier-Stokes system associated to a two-dimensional stationary flow of a incompressible dilatant fluid is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial}{\partial x}(|\mathbf{D}|^{p-2}\frac{\partial u}{\partial x}) + \frac{\nu}{2}\frac{\partial}{\partial y}(|\mathbf{D}|^{p-2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}))$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\nu}{2}\frac{\partial}{\partial x}(|\mathbf{D}|^{p-2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})) + \nu\frac{\partial}{\partial y}(|\mathbf{D}|^{p-2}\frac{\partial v}{\partial y})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where $\mathbf{v}$ = (u,v) is the velocity, P the pressure,

$$\mathbf{D} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & \frac{\partial v}{\partial y} \end{pmatrix}$$

and

$$|\mathbf{D}|^2 = u_x^2 + \frac{1}{2}(u_y + v_x)^2 + v_y^2.$$

In 1904, L. Prandtl [22] studied the influence of viscosity in a Newtonian flow at high Reynolds number in the presence of an obstacle. If we assume that the flow is exterior to a body (here represented by the interval (0,X) in the x-axes) and that a representative value of the modulus of the velocity is V, then the Reynolds number is R=$\frac{VX}{\nu}$ (we can assume, for simplicity, that $\rho \equiv 1$). The transition from zero velocity at the wall to the free stream velocity (velocity of the outer flow) (U(x),0) takes place in a very thin layer: *the boundary layer*. To study such a layer, Prandtl used some simplifications. For instance, it is natural to expect that

$$\frac{\delta}{X} << 1,$$

where δ is the boundary layer thickness. It is not difficult to see that this property is equivalent to the condition

$$\left|\frac{\partial u}{\partial y}\right| >> \left|\frac{\partial u}{\partial x}\right|.$$

Using dimensional analysis it can be shown that under this condition

$$\left|\frac{\partial P}{\partial y}\right| << 1.$$

So, following Prandtl, we can assume the Bernouilli equation for the outer flow to be

$$U(x)\frac{dU}{dx}(x) = -\frac{dP}{dx}(x).$$

Neglecting smaller terms, the Navier-Stokes system leads to the following problem:

$$(PS)\begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = UU_x + \nu\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{p-2}\frac{\partial u}{\partial y}\right) & \text{in Q}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 & \text{in Q}, \\ u(0,y) = u_0(y) & y > 0, \\ u(x,0) = 0, v(x,0) = v_0(x) & x \in (0,X), \\ u(x,y) \to U(x) \text{ as } y \to \infty & x \in (0,X). \end{cases}$$

where $Q \doteq \{(x,y) : 0 < x < X, 0 < y\}$. In most physical problems $v_0(x) \equiv 0$; nevertheless, the case $v_0(x) \leq 0$ is also relevant in the so called *suction problems*. To study problem (PS), several reformulations are proposed in the literature. The key point is to work with the *stream function* ψ given by

$$\begin{cases} u = \frac{\partial \psi}{\partial y} \\ v = -\frac{\partial \psi}{\partial x} + v_0, \ \psi(x,0) = 0 \end{cases}$$

Notice that the level lines of ψ coincide with the *current lines* of $\mathrm{v} = (\mathrm{u},\mathrm{v})$. The first mathematical treatment of (PS) is carried out by studying the third order ordinary differential equation satisfied by ψ (see Schlichting [27] for the case of Newtonian flows). The second possibility is to introduce the *von Mises transformation* [34]

$$\begin{array}{ll} \psi = \psi(x,y) & \psi \in (0,\infty), \\ w(x,\psi) \doteq u^2(x,y) & x \in (0,X). \end{array}$$

In this way, we arrive at the scalar problem

$$(P_w)\begin{cases} \frac{\partial w}{\partial x} - \nu\sqrt{w}\frac{\partial}{\partial \psi}\left(\left|\frac{\partial w}{\partial \psi}\right|^{p-2}\frac{\partial w}{\partial \psi}\right) + v_0\frac{\partial w}{\partial \psi} - 2UU_x = 0 & x \in (0,X), \psi \in (0,\infty), \\ w(0,\psi) = w_0(\psi) & \psi \in (0,\infty), \\ w(x,0) = 0 & x \in (0,X), \\ w(x,\psi) \to U^2(x) & as\ \psi \to \infty, \end{cases}$$

where $w_0(\psi)$ is defined through $u_0(y)$. The P.D.E. appearing in (P_w) is a nonlinear degenerate parabolic equation in which the x variable plays the role of *time* and ψ stands for the spatial variable. Some existence and uniqueness results for this problem are due to Oleinik [18], [19] (case of $p = 2$) and Samokhin [26] (case of $p > 2$). The assumptions of those papers are the following:

$$\begin{array}{lr}
U(x) > 0 & \textit{for } x \in (0, X), \\
u_0(0) = 0 \textit{ and } u_0(y) > 0 & \textit{for } y > 0, \\
u_0'(0) = 0\ ,\ (u_0', u_0'') \in L^\infty(0,\infty)^2 & \\
U(0)U_x(0) - v_0(0)\frac{du_0}{dy} + \nu \left|\frac{du}{dy}\right|^{p-2} \frac{d^2u}{dy^2} = 0(y^2) & (\textit{consistency condition}).
\end{array}$$

We also mention the results by Oleinik [19], Serrin [29] and Peletier [21] on the asymptotic behavior when X=$+\infty$.

2 The results

The main goal of this work is to study the *coincidence set*

$$\{(x,\psi) : w(x,\psi) = U^2(x)\}$$

for the case of dilatant fluids. The boundary of this region could be called the *exact boundary layer*.

Remark 1 By the weak maximum principle, it is well known that necessarily $w(x,\psi) \leq U^2(x)$ in $(0, X) \times (0,\infty)$. In fact, if $p = 2$, it can be shown (see Oleinik [19]) that the strong maximum principle also holds and thus $w(x,\psi) < U^2(x)$ in $(0, X) \times (0,\infty)$, i. e. the coincidence set is empty. We recall that there are several attempts to make the boundary layer concept more precise. For instance, in Schlichting [27] it is defined as the zone where u=0. 99U. We must mention also the integral method introduced by von Karman [33] in order to estimate the boundary layer thickness δ.

Our main results are the following

Theorem 1 (Existence of the coincidence set).

Assume p> 2, $v_0(x) \leq 0$, and there exists $\psi_0 \in (0,\infty)$ such that $w_0(\psi) = U^2(0)$ for any $\psi \geq \psi_0$; Then there exists $C > 0$ and $\mu \in (0,1)$ such that

$$w(x,\psi) = U^2(x) \textit{ for any } (x,\psi) \textit{ such that } \psi \geq \psi_0 + Cx^\mu.$$

Theorem 2 (Waiting distance along a streamline)

Assume $p > 2$, $v_0(x) \leq 0$ and that there exists $\psi_0 \geq 0$, $C > 0$ and $\sigma \in (0,1)$ such that,

$$\int_\psi^\infty (U^2(0) - w_0(\tau))^2 d\tau \leq C(\psi_0 - \psi)_+^{\sigma/(1-\sigma)}$$
$$\textit{for any } \psi \in (\psi_0 - \varepsilon, +\infty).$$

Then, there exists $x_0 \in (0, X)$ such that

$$w(x,\psi_0) = U^2(x) \textit{ for any } x \in [0, x_0].$$

Sketch of the proof of Theorem 1. It is based on a general *Energy Method* first introduced by one of the authors [1] and later improved and developed in [2], [3], [4], [5], [6], [7], [8], [9], [11], [12], [13], [14], [15], [16], [20] and [30] (see also [10], [23], [24], [31] and [32]).

First step. We introduce the homogenized unknown

$$z(x,\psi) \doteq U^2(x) - w(x,\psi).$$

We remark that by the comparison principle $z(x,\psi) \geq 0$ on $(0,X)\times(0,\infty)$. We also point out that arguing as in [18], or [26], it is possible to obtain the *a priori estimate*

$$0 < C_1 \leq U^2(x) - z(x,\psi) \leq C_2 \text{ for any } x \in (0,X), \psi \in (0,\infty),$$

for some constants $C_1 \leq C_2$. On the other hand, it is easy to see that z satisfies

$$(P_z)\begin{cases} \frac{\partial z}{\partial x} - \nu\sqrt{U^2 - z}\frac{\partial}{\partial \psi}\left(\left|\frac{\partial z}{\partial \psi}\right|^{p-2}\frac{\partial z}{\partial \psi}\right) + v_0\frac{\partial z}{\partial \psi} = 0 & x \in (0,X), \psi \in (0,\infty), \\ z(0,\psi) = U^2(0) - w_0(\psi) & \psi \in (0,\infty), \\ z(x,0) = U^2(x) & x \in (0,X), \\ z(x,\psi) \to 0 & as\ \psi \to \infty. \end{cases}$$

We remark that $z(x,\psi) = 0$ on the coincidence set.

Second step: *Integration by parts formula.* We introduce the one-parameter energy domain

$$Q_\rho = (0,x)\times(\rho,\infty)$$

where $\rho \geq \psi_0$ is arbitrary. Multiplying by z and by integrating (formally) by parts we obtain that

$$\begin{gathered} \frac{1}{2}\int_\rho^\infty z^2(x,\psi)d\psi + \int_0^x\int_\rho^\infty \theta(s,\psi)\left|\frac{\partial z}{\partial \psi}(s,\psi)\right|^p ds d\psi + \\ \frac{1}{2}\int_0^x(-v_0(s)z^2(s,\rho)ds = \frac{1}{2}\int_\rho^\infty z^2(0,\psi)d\psi - \int_0^x \sqrt{w}z\left|\frac{\partial z}{\partial \psi}\right|^{p-2}\frac{\partial z}{\partial \psi}\Big|_{\psi=\rho} ds \end{gathered}$$

where

$$\theta(x,\psi) \doteq \frac{2U^2(x) - 3z(x,\psi)}{2\sqrt{U^2(x) - z(x,\psi)}}.$$

It is easy to see that

$$0 < C_3 \leq \theta(x,\psi) \leq C_4 \text{ for any } x \in (0,X), \psi \in (0,\infty).$$

Third step. We introduce the *energy functions*

$$b(x,\rho) \doteq ess \sup_{0\leq s\leq x}\frac{1}{2}\int_\rho^\infty z^2(x,\psi)d\psi$$

$$E(x,\rho) \doteq \int_0^x \int_\rho^\infty \theta(s,\psi) \left| \frac{\partial z}{\partial \psi}(s,\psi) \right|^p ds d\psi.$$

Applying the Holder inequality, we get that

$$\left| \int_0^x \sqrt{w} z \left| \frac{\partial z}{\partial \psi} \right|^{p-2} \frac{\partial z}{\partial \psi} |_{\psi=\rho} ds \right| \leq C_5 (-\frac{\partial E}{\partial \rho}(x,\rho))^{\frac{p-1}{p}} (\int_0^x |z(s,\psi)|^p ds)^{\frac{1}{p}}.$$

Now we need a technical result

Lemma 3 (Trace-interpolation inequality, [16]). *Let $\sigma \in (0,1)$ be given by $\sigma = (p+2)/3p$. Then*

$$(\int_0^x |z(s,\psi)|^p ds)^{\frac{1}{p}} \leq C_6 x^{(1-\sigma)/p} (E^{1/p} + x^{1/p} b^{1/2})^\sigma b^{(1-\sigma)/2}.$$

End of the proof of Theorem 1. By using the above inequalities we can find an exponent $\mu \in (0,1)$ and a positive constant C_7 such that

$$E^\mu \leq (E+b)^\mu \leq C_7 x^{(1-\mu)/(p-1)} (-\frac{\partial E}{\partial \rho}(x,\rho)).$$

This inequality implies the conclusion due to the following easy result

Lemma 4 ([1]) *Let $y \in C([0,t_1] \times [0,\rho_0+\delta]), y \geq 0$ such that*

$$\Phi(y(t,\rho)) \leq C t^\omega \frac{\partial y}{\partial \rho}(t,\rho)$$

for a. e. $\rho \in [0,\rho_0+\delta]$ and for any $t \in [0,t_1]$, where $\omega \geq 0$ and Φ is a continuous nondecreasing function such that $\Phi(0) = 0$ and

$$\int_{0+} \frac{ds}{\Phi(s)} < \infty.$$

Then there exists $t_0 \in (0,t_1]$ and a function $\rho(t)$ with $0 \leq \rho(t) \leq \rho_0 + \delta$ such that $y(t,\rho) = 0$ for any $t \in [0,t_0]$ and any $\rho \in [0,\rho(t)]$.

Remark 2 A different proof of Theorem 1, based upon the comparison principle, and under additional conditions, is due to [26].

Idea of the proof of Theorem 2. Using the same type of arguments and the assumption at $x = 0$ we obtain the differential inequality

$$E^\mu \leq C_7 x^{(1-\mu)/(p-1)} (-\frac{\partial E}{\partial \rho}(x,\rho)) + C_8 (\psi_0 - \psi)_+^{\mu/(1-\mu)}$$

for any $\psi \in (\psi_0 - \varepsilon, +\infty)$. The conclusion comes now from an analysis of this differential inequality

Lemma 5 ([2]) *Let $y \in C([0,t_1]\times[0,\rho_0+\delta]), y \geq 0$ such that*

$$\Phi(y(t,s)) \leq Ct^{\omega}\frac{\partial y}{\partial \rho}(t,s) + G((\rho-\rho_0)_+)$$

for a. e. $\rho \in [0,\rho_0+\delta]$ and for any $t \in [0,t_1]$, where $\omega \geq 0$ and Φ is as in Lemma 4 and

$$\exists \mu > 0 \text{ and } \varepsilon > 0 \text{ such that } G(s) \leq \varepsilon\Phi(\eta_\mu(s)), \ \text{a.e. } s \in (0,\rho)$$

with

$$\eta_\mu(s) = \Theta_\mu^{-1}(s), \qquad \Theta_\mu(\tau) = \int_{0+}^{\tau}\frac{ds}{\mu\Phi(s)}.$$

Then there exists $t^* \in (0,t_1]$ such that $y(t,\rho) = 0$ for any $t \in [0,t^*]$ and any $\rho \in [0,\rho_0]$.

Remark 3 In the case of pseudo-plastic fluids ($1 < p < 2$), it is possible to apply another kind of energy method (now using a suitable global energy) which leads to a different estimate on the location of the exact boundary layer: if X is large enough and U(x)$\equiv 0$ $\forall x \geq x_U$ for some $x_U > 0$, then there exists $x_0 \geq x_U$ such that $w(x,\psi) = U^2(x) = 0$, $\forall x \geq x_0$, $\forall \psi \in (0,\infty)$ (see also [25]).

Acknowledgments. *The research of the authors is partially sponsored by the project INTAS-94-2187. The second author is also supported by the DGICYT (Spain) project PB93-0443.*

References

1. Antontsev, S.N.: On the localization of solutions of nonlinear degenerate elliptic and parabolic equations. *Dokl. Akad. Nauk. SSSR*, 1981, 260, No. 6, pp. 1289-1293. (in Russian; English translation in *Soviet Math. Dokl.* , 1981, 24, No. 2, pp. 420-424).

2. Antontsev, S.N. and Diaz, J.I.: Application of an Energy Method in the Localization of the Solutions of the Equations of Continuous medium mechanics. (Russian). *Dokl. Akad. Nauk. SSSR.* 1988. Vol 303. No 2. pp. 320 - 325. (English transl. in *Soviet Phys. Dokl.* 1988. Vol 33. No 11. pp. 813 - 816.).

3. Antontsev, S.N. and Diaz, J.I.: New results on localization of solutions of nonlinear elliptic and parabolic equations obtained by the energy method. *Dokl. Akad. Nauk. SSSR.* 1988, Vol. 303, No. 3, pp. 320-325. (English translation in *Soviet Math. Dokl.* 1988, Vol. 33, No. 3,pp. 535-539).

4. Antontsev, S.N. and Diaz, J.I.: The Energy Method and the Localization of Solutions of Equations in Continuum Mechanics. (Russian). *Zh. Prikl. Mekh. i. Tekhn Fiz.* 1989. No 2. pp. 18-25. (English transl. in *J. Appl. Mech. Tech. Phys.* 1989. Vol 30. No 2. pp. 182-189).

5. Antontsev, S.N. and Diaz, J.I.: On Space or Time Localization of Solutions of Nonlinear Elliptic or Parabolic Equations Via Energy Methods. (English). In *Recent Advances In Nonlinear Elliptic and Parabolic Problems*, Ph. Benilan, L.C. Evans and M. Pierre eds. Pitman Research Notes In Mathematics. Series 208, 1989, pp. 3-14.

6. Antontsev, S.N. and Diaz, J.I.: Space and Time Localization in the Flow of Two Immiscible Fluids Through a Porous medium: Energy Method Applied to Systems. *Nonlinear Analysis, Theory, Methods and Applications*, 1991, Vol 16. No 4. pp. 299-313.

7. Antontsev, S.N., Diaz, J.I. and Shmarev, S.I.: New applications of energy methods to parabolic free boundary Problems. *Uspekhi Mat. Nauk.*, 1991, 46, No. 6 (282), pp. 181-182. (in Russian; English translation: in *Russian Mathematical Surveys*, 1991, 46, No. 6, pp. 193-194).

8. Antontsev, S.N., Diaz, J.I. and Shmarev, S.I.: The support shrinking properties for solutions of quasilinear parabolic equations with strong absorption terms. *Annales de la Fac. des Sciences de Toulouse,* 1995, 4, No. 1, pp. 5-30.

9. Antontsev, S.N., Shmarev, S.I.: Local energy method and vanishing of weak solutions of nonlinear parabolic equations. *Dokl. Akad. Nauk. SSSR*, 1991, 318, No. 4. pp. 777-781. (in Russian; English translation in *Soviet Math. Doklady*, 1991, No. 43).

10. Benilan, Ph. and Crandall, M.G.: The continuous dependence on φ of the solutions of $u_t - \Delta\varphi(u) = 0$, *Indiana Univ. Math. J.*, No. 30, 1981, pp. 161-177.

11. Bernis, F.: Compactness of the support for some nonlinear elliptic problems of arbitrary order in dimension n, *Comm. Partial Differential Equations* 1984, 9, pp. 271-312.

12. Bernis, F.: Finite speed of propagation and asymptotic rates for some nonlinear higher order parabolic equations with absorption, *Proc. Royal Soc. Edinburgh 104A*, 1986, pp. 1-19.

13. Bernis, F.: Qualitative properties for some nonlinear higher order degenerate parabolic equations, *Houston J. Math.*, 14, 1988, pp. 319-352.

14. Diaz, J.I.: *Nonlinear partial differential equations and free boundaries.* Research Notes in Math. No. 106. Pitman, 1985.

15. Diaz, J.I. and Liñan, A.: Movimiento de descarga de gases en conductos largos: Modelizacion y estudio de una ecuacion parabolica doblemente no lineal. *Actas de la Reunion Matematica en Honor de A. Dou*, J.I. Diaz and J.M. Vegas eds., Editorial Universidad Complutense, Madrid, 1989, pp. 95-119.

16. Diaz, J.I. and Veron, L.: Local vanishing properties of solutions of elliptic and parabolic quasilinear equations. *Trans. Amer. Math. Soc.*, 1985, 290. No. 2, pp. 787-814.

17. Narasimhan, M.N.L.: *Principles of Continuum Mechanics*, John Wiley and Sons, Inc. New York, 1993.

18. Oleinik, O. A.: On a system of equations in boundary layer theory, *Zhurn. Vyvhslit. Mat. Mat. Fiz.* 1963, 3, pp. 489-507 [Russian]. Engl. transl. USSR *Comput. Math. Math. Phys.* 1963, 3, pp. 650-703.

19. Oleinik O. A.: *Mathematical problems of boundary layer theory.* Minneapolis. Univ. of Minnesota. 1969.

20. Palymsky I. B.: Some qualitative properties of the solutions of equations of nonlinear heat conductivity with absorption. *Chislennye metody mekhaniki sploshnoi sredy,* 1985, Vol 16. No. 1, pp. 136-145.

21. Peletier, L.A.: On the asymptotic behavior of velocity profiles in laminar boundary layers, *Arch. Rat. Mech. Anal.* 45, 1972, pp. 110-119.

22. Prandtl, L.: Über Flussigkeits-be wegungen bei sehr kleiner Reibung. In *Verh. II. Int. Math-Kongr. Heidelberg,* 1904, pp. 484-494. Leipzig: Teubner.

23. Rykov, Yu.G.: On the behavior of generalized solutions of degenerating parabolic equations with anisotropic non-power nonlinearities with the growth of time. *Differential Equations and Their Applications.* Moscow. Gos. Univ. Moscow. 1984, pp. 160-164.

24. Rykov, Yu.G.: On the finite velocity of propagation of perturbations and inertia for generalized solutions of degenerating quasilinear equations. *Differential Equations and Their Applications.* Moscow. Gos. Univ. Moscow. 1984, pp. 160-164.

25. Samokhin, V.N.: Generalized solutions and extension of the boundary layers of a pseudo-plastic liquid. *Trudy. Sem. im. I. G. Petrovskogo,* No. 3, 1978, pp. 161-175.

26. Samokhin, V.N.: A system of steady–state boundary layer for a dilatantous medium. *Trudy Seminara imeni I. G. Petrovskogo.* No. 14, 1989, pp. 89-109. (English transl. 1990, pp. 2358-2373).

27. Schlichting, H.: *Boundary-Layer Theory,* McGraw-Hill Company, New York, 7th edition, 1979

28. Schowalter, R.: *Mechanics of non-Newtonian Fluids,* Pergamon Press, Oxford, 1978.

29. Serrin, J.: On the mathematical basis for Prandtl's boundary layer theory: An example. *Arch. Rat. Mech. Anal.* 28, 1967, pp. 217-225.

30. Shmarev, S.I.: Local properties of solutions of nonlinear equation of heat conduction with absoption. *Dinamika Sploshnoi Sredy,* Novosibirsk, 1990, 95, pp. 28-42. (in Russian).

31. Straughan, B.: *Instability, nonexistence and weighted energy methods in fluid dynamics and related theories.* Pitman Advanced Publishing Program. Research Notes in Math., 74, 1982.

32. Straughan, B.: *The energy method, stability and nonlinear convection.* Applied Mathematical. Sciences. 1991. Springer Verlag.

33. von Karman, Th.: Über laminare und turbulente Reibung, *Z. Angew. Math. Mech.* 1 1927, pp. 233-252.

34. von Mises, R.: Bemerkungen zur Hydrodynamic, *Z. Angew. Math. Mech.* 7 1927, pp. 425-431.