On the mathematical treatment of energy balance climate models

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1. Introduction.

Climate models have different characteristics than weather prediction models: the time scale is completely different (centuries versus days or weaks) and their main goal is also complementary (prognostic in the weather prediction and diagnostic in the case of climate models). Climate models were introduced in order to understand past and future climates and their sensitivity on a few of relevant features (which a quantitative analysis reduces to some parameters).

Two of the most important ingredients of the models concern the Solar radiation R_a (short-wave energy from Sun) and the Earth radiation R_e (long-wave radiation escaping into space). The consideration of other features under different degree of accuracy introduces a *hierarchy* in the class of climate models. So, according to the time variable the models are classified into equilibrium and dynamical models. With respect to the space variable the models are called as 0-D zero-dimensional (if only the mean Earth temperature is analized), 1-D latitudinal or vertical models, 2-D horizontal or meridional plane models and so up the most sophisticated 3-D General Circulation model. More complex models have been also considered in the literature by coupling the study of the Earth temperature with different phenomena from the Glaceology, Celestial Mechanics, Geophysics, etc. (see the monographs of Ghill and Childress [1987], North [1993] and Henderson-Sellers and Mc Guffie [1987]).

In this work we shall pay attention to the mathematical treatement of some horizontal energy balance models generalizing the models introduced (independently) by Budyko [1969] and Sellers [1969].

If we represent the Earth by a compact two-dimensional manifold without boundary $\mathcal{M}$ and we denote by $u(t,x)$ the annually (or seasonally) averaged Earth surface temperature, our model is formulated as the reaction-diffusion equation

$$c(t,x)u_t(t,x) - div(k(t,x)grad\ u(t,x)) = R_a(t,x,u(t,x)) - R_e(t,x,u(t,x)) \qquad (1)$$

where the heat capacity $c(t,x)$ is a positive function largely determined by oceans (recall that the 70 per cent of the Earth's surface is covered by oceans). After averaging $c \sim 1.05 \times 10^{23} Jm^{-2}K^{-1}$. The diffusion operator in (1) has a double justification:

$$div(k\ grad\ u) = div(F_c + F_a)$$

with $F_c = k_c\ grad\ u$ the conduction heat flux and F_a the advection heat flux. In Meteorology and Oceanography it is usually assumed $F_a = -\mathrm{v}T$ where v and T are the velocity and temperature of the fluid. In planetary scales O(10^4Km) the velocity is eliminated using the *eddy diffusive approximation*

$$div F_a \simeq div(k_e grad\ u) \qquad (2)$$

where the eddy diffusion coefficient is again a positive number (and more generally a positive function). Obviously the differential operators *div* and *grad* must be suitably understood with respect to the Riemannian metric. An important variant is due to P.H. Stone [1972] who pointed out that in the case of rotating atmospheres the eddy diffusive approximation really leads to a nonlinear diffusion operator of the form

$$div(k_e^*|grad\ u|grad\ u) \qquad (3)$$

for some $k_e^* > 0$ (see Stone [1972] formula 2.24). In terms of equation (2) the nonlinear operator (3) means that the eddy diffusion coeficient k_e must increase as the gradient of the averaged temperature increases.

The solar energy absorbed by the Earth R_a is assumed to be of the form

$$R_a = QS(x)\beta(u) \qquad (4)$$

where Q is the *Solar constant* (i.e. the annual average amount of radiation energy per unit time passing through a unit area perpendicular to the Sun's rays at the Earth orbit). Averaging $Q \sim 1.370\ W/m^2$. $S(x)$ is the distribution of solar radiation over the Earth and $\beta(u)$ is the *planetary coalbedo* representing the fraction absorbed according the average temperature. Usually $\beta(u)$ is assumed to be a non-decreasing function of u taking constant values a_i and a_f (both positive and less than one) for small and respectively large values of u. It is not completely clear how is produced the transition: Budyko [1969] proposes a discontinuity at $u = -10°C$

$$\beta(u) = \begin{cases} a_f & \text{over ice-free} \quad \{x \in \mathcal{M} : u(t,x) > -10\} \\ a_i & \text{over ice-covered} \quad \{x \in \mathcal{M} : u(t,x) < -10\}. \end{cases} \qquad (5)$$

In contrast to that, Sellers [1969] proposes a continuous linear piecewise function with a very large increasing rate near -10. We remark that in seasonally averaged models the terms $QS(x)$ are replaced by a more general function $S(t,x)$ "almost" periodic in time. This is of relevance in the study of *ice ages* since snowcover over the summer is a necessary condition for the growth of continental glaciers as, for instance, the ones of Antarctica and Greenland (see the work by North Mengel and Short [1983] and its references). We also point out that the modeling of clouds is one of the most important open problems in the study of the solar energy absortion.

The mean emmitted energy flux $R_e(t,x,u)$ is determined empirically and depends on the amount of greenhouse gases, clouds and water vapor in the atmosphere. It seems natural to assume that R_e increases with u but the increasing rate is controversial: Sellers [1969] proposes a Stefan-Boltzman radiation law

$$R_e = \sigma u^4 (1 - m \; tanh(\frac{19u^6}{10^6})) \tag{6}$$

where u is represented in Kelvin degrees (here $\sigma > 0$ is the emmisivity and $m > 0$ the atmospheric opacity). Budyko [1969] replaces it by a Newtonnian linear type radiation ansatz

$$R_e = A + Bu \tag{7}$$

which is a linear approximation of (6) near $u = 15^oC$ (the actual mean temperature). Here $A = 210 \; W/m^2$ and $B = 1.9 \; W/^oCm^2$. We point out that the term R_e takes also in account the anthropogenerated changes.

In order to simplify the model we can assume that $\mathcal{M}$ is the unit sphere of $\mathbb{R}^3$ and that the heat capacity coefficient is $c \equiv 1$. We are interested in formulations including the non-linear diffusion proposed by Stone (see (3)) and also the case of a possible discontinuous function β (as, for instance, the one given in (5)). If we denote by φ and λ the colatitude and the longitude then the 1-D model is obtained by introducing $x \in (0,1)$ by $x = \cos\varphi$ and calling $u(x,t)$ to the mean annual temperature average on the latitude circles around the Earth. The model under consideration will be the following

$$(P) \begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x = R_a(x,t,u) - R_e(x,t,u) & x \in I, t > 0, \\ \rho(x)|u_x|^{p-2}u_x = 0 & x \in \partial I, t > 0, \\ u(x,0) = u_0(x) & x \in I, \end{cases}$$

where $I = (-1,1)$. The consideration of the two-dimensional problem on a compact Riemannian manifold without boundary $\mathcal{M}$ is the main objective of the works Diaz - Tello [1993], Diaz - Tello [1996] and Bermejo - Diaz - Tello [1996].

We point out that many of the results of this work will be obtained under the general assumption $1 < p < \infty$ and so they are also of application to the classical models introduced by Budyko [1969] and Sellers [1969] corresponding to the choice $p = 2$ and the one due to Stone [1972] where $p = 3$.

A list of structure assumptions is the following:

$$\rho(x) = k(1-x^2) \text{ with } k>0, \tag{8}$$

$$\left.\begin{array}{l} R_a(x,t,u) = Q(x,t)\beta(u) \text{ where } Q \in C([-1,1]\times I\!R_+) \text{ satisfies} \\ 0 < Q(x,t) \text{ and } \beta \text{ is a nondecreasing function such that} \\ |\beta(u)| \le M \quad \forall u \in I\!R, \text{ for some } M>0, \end{array}\right\} \tag{9}$$

$$\left.\begin{array}{l} R_e(x,t,u) \text{ is a continuous function on } x, \text{ Lipschitz on } t \text{ and } R_e(x,t,\cdot) \\ \text{is nondecreasing as function on } u, \text{ for any fixed } (x,t) \in \overline{I}\times I\!R_+. \end{array}\right\} \tag{10}$$

The rest of the work is organized in the following way: The notion of weak solutions of problem (P) is introduced in Section 2. It is proven that if $u_0 \in L^\infty(I)$ there exists at least one bounded weak solution of (P). This is obtained by two different methods: via a compactness abstract method and via a regularization argument. Due to the presence of the degenerate coefficient $\rho(x)$ the natural energy space is given by $V = \{w \in L^2(I) : w_x \in L^p(I:\rho)\}$, where $L^p(I:\rho)$ is the weighted-Lebesgue space associated to ρ.

The question of the uniqueness of bounded weak solutions is studied in Section 3. The answer is positive for the Sellers model (it is enough to require β be a locally Lipschitz continuous function). As in the case of the homogeneous model (see Díaz[1992]) the Budyko model may have more than one solution. This is explicitly shown in the Subsection 3.1 by means of the construction of a counterexample. Nevertheless, in the Subsection 3.2, it is shown that there is at most one solution of the Budyko model in the class of solutions satisfying a "nondegeneracy property". The free boundary generated in the case of Budyko type models is considered in Section 4. Finally, Section 5 is devoted to the study of the approximate controllability of the problem.

2. On the existence of solutions.

It is well known (see, e.g. Díaz-Herrero [1981] for the special case of $\rho = 1$ and $R_a \equiv 0$) that if $p>2$ the degeneracy of the diffusion operator makes impossible expect the existence of a classical solution of (P) even for a regular initial datum u_0. In order to make precise the notion of solution we shall study, we start by indicating that the eventual discontinuous character of the function R_a will be treated by assuming that

$$\left.\begin{array}{l} R_a(x,t,u) = Q(x,t)\beta(u), \text{ with } Q \text{ as in (4) and } \beta \text{ a} \\ \text{maximal monotone graph of } I\!R^2 \text{ such that } |z| \le M \\ \text{for any } z \in \beta(u), \text{ for any } u \in I\!R \text{ and some } M>0 \end{array}\right\} \tag{11}$$

(i.e. for example, β is given by a nondecreasing real function b as $\beta(r) = \{b(r)\}$ if b is continuous in r or $\beta(r) = [b(r-), b(r+)]$ if b has a jump at the point r: see Brezis[1973]). A usual way to verify the differential equation (at least weakly) is to multiply by a test function followed by an integration by parts. In doing so we obtain

$$\int_I u(x,T)v(x,T)dx - \int_0^T \int_I u(x,t)v_t(x,t)dxdt$$

$$+\int_0^T \int_I \rho(x)|u_x(x,t)|^{p-2}u_x(x,t)v_x(x,t)dxdt$$

$$= \int_0^T \int_I \{Q(x,t)z(x,t) - R_e(x,t,u)\}v(x,t)dxdt + \int_I u_0(x)v(x,0)dx \tag{12}$$

for some function $z(x,t)$ which satisfies that

$$z(x,t) \in \beta(u(x,t)) \text{ a.e. } x \in I \text{ and } t \in (0,T). \tag{13}$$

For several purposes it will be useful to take the solution u as a test function. So, for t fixed, the integrals

$$\int_I \rho|u_x|^p dx \text{ and } \int_I |u|^2 dx$$

must be finite. Then a natural "energy space" associated to (P) is the one defined by

$$V = \{w \in L^2(I) : w_x \in L^p(I:\rho)\},$$

where $L^p(I:\rho)$ is the weighted-Lebesgue space

$$L^p(I:\rho) = \{v : \| v \|_{L^p(I:\rho)} = [\int_I \rho(x)|v(x)|^p dx]^{\frac{1}{p}} < \infty\}.$$

It is easy to see that V is a separable and reflexive Banach space with the norm

$$\| u \|_V = \| u \|_{L^2(I)} + \| u_x \|_{L^p(I:\rho)}.$$

Any weak solution must satisfy $u(\cdot,t) \in V$ for a.e. $t \in (0,T)$. It is not difficult to see that in that case $|u_x(\cdot,t)|^{p-2}u_x(\cdot,t) \in L^{p'}(I:\rho)$, with $p' = p/(p-1)$. We also remark that because of the physical modelling of the problem we shall restrict our study to the class of bounded functions.

Definition 1 . *By a bounded weak solution of problem* (P) *we mean a function* $u \in C([0,T] : L^2(I)) \cap L^\infty(I \times (0,T))$ *such that* $u \in L^p(0,T:V), R_e(\cdot,\cdot,u) \in L^1(I \times (0,T))$ *and there exist* $z \in L^\infty(I \times (0,T))$ *satisfying (13) and the identity (12) holds for any* $v \in L^p(0,T:V) \cap L^\infty(I \times (0,T))$ *such that* $v_t \in L^{p'}(0,T:V')$.

The main purpose of this section is to prove the following result

Theorem 1 *For any* $u_0 \in L^\infty(I)$ *there exist at least one bounded weak solution* u *of* (P).

The proof of the above theorem can be carried out by means of different methods. Here we shall present two different type of techniques: (i) a compactness abstract method, and (ii) a regularization method.

2.1. *Existence via a compactness abstract method.*

Problem (P) can be considered as a perturbed problem associated to

$$(P^*)\begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x = 0, & x \in (-1,1), t > 0, \\ \rho(x)|u_x|^{p-2}u_x = 0, & x = \pm 1, t > 0, \\ u(x,0) = u_0(x), & x \in (-1,1). \end{cases}$$

The abstract Cauchy problem associated to (P^*) is given by

$$(CP^*)\begin{cases} \dfrac{du}{dt}(t) + Au(t) = 0, \quad \text{in } L^2(I), \quad \text{for } t > 0, \\ u(0) = u_0 \end{cases}$$

where we are identifying $u(t) \in L^2(I)$ with $u(\cdot,t)$. The operator $A : D(A) \longrightarrow L^2(I)$, with $D(A) \subset L^2(I)$, is described in the following result giving also the existence and uniqueness of the solution of (CP*).

Proposition 1 . *(a) Consider the functional $\varphi : L^2(I) \longrightarrow I\!R \cup \{+\infty\}$ given by*

$$\varphi(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_I \rho(x)|u_x|^p dx & \text{if } u \in V \\ = +\infty & \text{otherwise.} \end{cases} \tag{14}$$

Then $\varphi \not\equiv +\infty$, φ is convex and lower semicontinuous.
(b) Let $A(u) = \partial\varphi(u)$. Then $D(A) \subset V$, $D(A)$ is dense in $L^2(I)$ and

$$Au = -(\rho(x)|u_x|^{p-2}u_x)_x \text{ for any } u \in D(A). \tag{15}$$

(c) For any $u_0 \in L^2(I)$ there exists a unique function $u \in C([0,T] : L^2(I))$, for $T > 0$ arbitrary, such that $u(t) \in D(A)$ for a.e. $t > 0$, $t^{\frac{1}{2}}\frac{du}{dt} \in L^2(0,T : L^2(I))$ and satisfies (CP^*). Moreover if $u_0 \in L^q(I)$ with $1 \leq q \leq +\infty$ then $u(t) \in L^q(I)$. Finally, the application $S(t)u_0 = u(t)$ is a semigroup of contractions on $L^2(I)$.

Proof. (a) To prove that $\varphi \not\equiv +\infty$ and that φ is convex is obvious. The lower semicontinuity of φ can be shown, for instance, using the reflexivity of the space $L^p(I : \rho)$, and that the norm is l.s.c. for the weak convergence.
(b) It is clear that $V = D(\varphi)(\equiv \{w \in L^2(I) : \varphi(w) < \infty\})$ is a dense subspace of $L^2(I)$ (notice that $C_0^\infty(I) \subset V$). Then as $D(\partial\varphi) \subset D(\varphi)$ and $\overline{D(\partial\varphi)} = \overline{D(\varphi)}$ (see Brezis [1973]) we have that $\overline{D(\partial\varphi)} = L^2(I)$. On the other hand it is a routin matter to see that φ is Gateux differentiable in V and that

$$< \varphi'(u), h >_{V'V} = \lim_{\lambda \searrow 0} \frac{\varphi(u + \lambda h) - \varphi(u)}{\lambda} = \int_I \rho(x)|u_x|^{p-2}u_x h_x dx.$$

As $\partial\varphi(u)$ is a maximal monotone operator we obtain (15).
(c) The existence of u with the indicated regularity is now a consequence of the abstract Hille-Yosida theorem given in Brezis [1973]. If $u_0 \in L^q(I)$ we multiply the equation by

the test function $|u|^{q-1}\text{sign}u$ (more precisely, by a smooth approximation of this function) and a simple integration by parts shows that

$$\frac{d}{dt}\int_I |u|^q dx \leq 0,$$

which gives the result. ∎

Theorem 1 can be obtained from an abstract perturbation result (see Vrabie [1987] and Díaz-Vrabie [1987]) assuming that the operator $A = \partial\varphi$ generates a compact semigroup. By a result due to H.Brezis (see the reference in the book of Vrabie [1987]) this condition is equivalent to know that

$$\left.\begin{array}{l}\text{“for any } K > 0 \text{ the set } \{w \in L^2(I) : \| w \|^2_{L^2(I)} + \varphi(w) \leq K\} \\ \text{is relatively compact in } L^2(I)\text{”.}\end{array}\right\} \tag{16}$$

This is proved in the following auxiliary result:

Lemma 1 . *(i) Let ρ given by (8) and assume $p > 2$. Then for any $q \in [1, p/2)$ we have that*

$$V \subset \{w \in L^2(I) : w_x \in L^q(I)\} \tag{17}$$

with continuous imbedding. Moreover, for any $r \in [1, \infty]$ we have

$$V \subset L^r(I), \tag{18}$$

where the imbedding is continuous and compact for any $r \in [1, \infty]$.
(ii) If $1 < p \leq 2$, then we have the continuous imbedding $V \subset L^q(I)$ for any $q \in [1, \infty)$ if $p = 2$ and any $q \in [1, p^*)$ with $p^* = 2p/(2-p)$.
(iii) If $1 < p \leq 2$, the imbedding $V \subset L^2(I)$ is always compact.

Proof. (i) Let $w \in L^p(I : \rho)$ and $q \in [1, p/2)$. By the Hölder inequality with $p_1 = p/q$ and $p_1' = p/(p-q)$

$$\int_I |w(x)|^q dx = \int_I |w(x)|^q \rho(x)^{q/p} \rho(x)^{-q/p} dx \leq$$

$$\leq \left(\int_I |w(x)|^p \rho(x) dx\right)^{q/p} \left(\int_I \frac{dx}{\rho(x)^{q/(p-q)}}\right)^{(p-q)/p}.$$

But

$$\int_I \frac{dx}{\rho(x)^{q/(p-q)}} \leq \frac{1}{K_0^{q/p-q}} \int_{-1}^{1} \frac{dx}{(1-x^2)^{q/p-q}} < \infty$$

since $(1-x^2) \geq Cd(x, \partial I)$ and $q/(p-q) < 1$. This proves the first part of the statement. This also shows the continuous imbedding $V \subset W^{1,1}(I)$ and so (17) holds by a well-known result (see, e.g., Brezis [1983], Theorem VIII.7). Then $V \subset W^{1,q}(I)$ for any $q \in [1, p/2)$ and by the mentioned result the imbedding (17) is also compact for $r = +\infty$. The proof of (ii) can be found in Adams [1980] or Rakotoson-Simon [1993]. Part (iii) is shown in Meyer [1967] for $p = 2$. His proof can be extended to any $p \in (1, 2)$ using part (ii). ∎

Corollary 1 . *Assume (8), (10), (11), (14) and $p \geq 2$. Then for any $u_0 \in L^2(I)$ there exists a function $u \in C([0,T] : L^2(I))$ such that $u(t) \in D(A)$ a.e. $t \in (0,T]$, $t^{\frac{1}{2}}\frac{du}{dt} \in L^2(0,T : L^2(I))$, $\varphi(u) \in L^1(0,T : I\!R)$ and it satisfies (P) a.e. $t \in (0,T)$ on $L^2(I)$ as well as in the sense of (12). Moreover, if $u_0 \in V$ then $\frac{du}{dt} \in L^2(0,T : L^2(I))$ and $u \in C([0,T] : V)$. Finally, if $u_0 \in L^\infty(I)$ then $u \in L^\infty(I \times (0,T))$.*

Proof. The existence of u satisfying (P) a.e. $t \in (0,T)$ on $L^2(I)$ is a consequence of the application of a suitable fixed theorem for a compact operator (see, e.g., Vrabie [1987], Corollary 2.3.2). The application of such results is guaranteed by Proposition 1, Lemma 1 and the assumptions (10) and (11). This function obviously satisfies trivially (12) (take integrals on $(\tau, T) \times I$ and make $\tau \searrow 0$). The boundedness of u, assumed $u_0 \in L^\infty(I)$, is proved as in Proposition 1 if the right hand side of the equation is a bounded term. ∎

Remark 1. The above method can be applied to two-dimensional problems (on a compact Riemannian manifold without boundary): see Hetzer [1990] (for the Sellers type model) and Díaz-Tello [1993], [1996] when $c \equiv 1$ and Bermejo - Diaz - Tello [1996] when $c \in L^\infty(\mathcal{M})$ (for the Budyko model).

2.2. *Existence via a regularization method.*

The existence of a bounded weak solution of (P) can be also obtained by approximating the multivalued (discontinuous) term $\beta(\cdot)$ by a regular function $\beta_\epsilon \in C^\infty(I\!R)$ with the properties

$$\beta_\epsilon'(s) \geq 0 \text{ and } |\beta_\epsilon(s)| \leq M \quad \forall s \in I\!R. \tag{19}$$

It is also usefull to remove the degeneracy at ∂I by replacing $\rho(x)$ by

$$\rho_\epsilon(x) = \rho(x) + \epsilon. \tag{20}$$

In order to approximate u by classical solutions of a related problem we also replace the data u_0, Q and R_e by C^∞ functions $u_{0,m}$, Q_n, $R_{e,k}$ such that

$$u_{0,m}(\pm 1) = 0, \quad \| u_{0,m} \|_{L^\infty(I)} \leq \| u_0 \|_{L^\infty(I)},$$

and

$$u_{0,m} \longrightarrow u_0 \text{ in } L^2(I), \text{ as } m \to \infty,$$

$$Q_n \longrightarrow Q \text{ in } C(\overline{I} \times [0,T]),$$

$$\left.\begin{array}{l} R_{e,k} \text{ satisfies (4)}, \ R_{e,k}(\cdot,\cdot,u) \longrightarrow R_e(\cdot,\cdot,u) \text{ in } C(\overline{I} \times [0,T]) \\ \text{for any fixed } u \in I\!R \text{ and } R_{e,k}(x,t,\cdot) \longrightarrow R_e(x,t,\cdot) \text{ in } C(J) \text{ for} \\ \text{any compact } J \subset I\!R \text{ and any fixed } (x,t) \in \overline{I} \times [0,T]. \end{array}\right\}$$

Given ϵ, m, n and k positive numbers we consider the problem (P_ϵ)

$$\begin{cases} u_t - [\rho_\epsilon(x)|u_x|^{p-2}u_x]_x - \epsilon u_{xx} = Q_n(x,t)\beta_\epsilon(u) - R_e(x,t,u), & x \in I \times (0,T), \\ \rho_\epsilon(x)(|u_x|^{p-2}u_x + \epsilon u_x) = 0 & \text{on } \partial I \times (0,T), \\ u(x,0) = u_{0,m}(x) & \text{on } I. \end{cases}$$

The partial differential equation is now uniformly parabolic and so by well-known results (see e.g. Ladyzenskaja-Solonnikov-Ural'ceva [1968], Chapt.V) there exists a unique classical solution $U = u_{\epsilon,m,n,k}$. In order to study the convergence, when $\epsilon \searrow 0$ and $m, n, k \to +\infty$ we need some a priori estimates.

Lemma 2 . *The solution U of (P_ϵ) satisfies (for n and k large enough)*

$$\| U \|_{L^\infty(I\times(0,T))} \leq C, \tag{21}$$

$$\| \rho_\epsilon U_x \|_{L^p(0,T:L^p(I))} \leq C, \tag{22}$$

where C denotes a positive constant independent of ϵ, m, n and k.

Proof. Estimate (21) is derived from the maximum principle (see e.g. Ladyzenskaja-Solonnikov-Ural'ceva [1968]). To obtain (22) we multiply the equation by U. Integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\int_I U^2(x,t)dx + \int_I \rho_\epsilon |U_x|^p dx + \epsilon \int_I |U_x|^2 dx \leq C$$

(where we have used (19) and (21)). ∎

Using the a priori estimates and the assumption (11) the proof of the convergence $U \longrightarrow u$, $\beta_\epsilon(U) \longrightarrow z$ with $z \in \beta(u)$ and that u is a bounded weak solution of (P) is standard (notice that this is not the case if we want obtain more regularity on u_t as, for instance, that given in Corollary 1).

Remark 2. The regularization of the multivalued term $\beta(u)$ was already carryed out in Xu [1991] for $p = 2$ (see also Feireisl-Norbury [1991] for some related problems). We also point out that the existence of a weak solution can be obtained by the method of upper and lower solutions combined with monotone iteration arguments (see e.g. Carl [1989] and Díaz-Stakgold [1989] for other related problems).

3. On the uniqueness of solutions: positive and negative answers.

The type of answer to the question of the uniqueness of solutions to problem (P) is rather different in the cases of the *Sellers model* (where $R_a(x,t,u)$ is a smooth function) and the *Budyko model* (where $R_a(x,t,u)$ is a discontinuous function of u).

3.1. *The Sellers model.*

The following result shows the uniqueness and others properties of solutions for the Sellers model.

Theorem 2 *Let $p > 1$, and assume that*

$$R_a \text{ satisfies (11) with } \beta \text{ a locally Lipschitz function of } u. \tag{23}$$

Then given $u_0 \in L^\infty(I)$ there exists at most one bounded weak solution of (P).

Idea of the proof. First of all we point out that $u_t \in L^{p'}(0,T:V')$. This can be obtained from the definition of bounded weak solutions and the characterization of the dual space V' (see e.g. Ivanov [1981], Lemma V.2.1). Moreover, if we define $w = e^{-Ct}u$, w satisfies (in a weak sense) the equation

$$w_t - e^{-C(p-2)t}(\rho(x)|w_x|^{p-2}w_x)_x = e^{-Ct}Q(x,t)\beta(we^{Ct}) - e^{-Ct}R_e(x,t,we^{Ct}) - Cw.$$

Since β is assumed locally Lipschitz we can choose C large enough such that the function

$$F(x,t,w) = e^{-Ct}Q(x,t)\beta(we^{Ct}) - Cw$$

is a strictly decreasing function of v for fixed (x,t). Now assume that we have another solution u^* of (P) corresponding to the same datum u_0. We take $w - w^*$ ($w^* = e^{-Ct}u^*$) as test function in the difference of the identities satisfied by w and w^* (see the definition of bounded weak solution). We have that

$$< w_t(t) - w_t^*(t), w(t) - w^*(t) >_{V'V} = \frac{d}{dt}\int_I |w(t) - w^*(t)|^2 dx$$

(see e.g. Temam [1988]). Moreover, there exists $K > 0$ such that if $p \geq 2$

$$\int_I \rho(x)(|w_x|^{p-2}w_x - |w_x^*|^{p-2}w_x^*)(w_x - w_x^*)dx \geq K\int_I \rho(x)|w_x - w_x^*|^p dx \tag{24}$$

For $1 < p < 2$ the right-hand side term must be replaced by

$$K\int_I \rho(x)|w_x - w_x^*|^2(|w_x|^{2-p} + |w_x^*|^{2-p})dx$$

(see, e.g., Díaz[1985] Lemma 4.10). Using the monotonicity of $R_e(\cdot,\cdot,u)$ and $F(\cdot,\cdot,w)$ we obtain that

$$\frac{d}{dt}\int_I |w(t) - w^*(t)|^2 dx \leq 0$$

and so necessarily $u = u^*$.

■

Corollary 2 . *Assume (23). Let $u_0, \hat{u}_0 \in L^\infty(I)$ and let $u, \hat{u}$ be weak solutions of (P) corresponding to the energy emmision functions $R_a(x,t,u) = \gamma(u) + f(x,t)$ and $\hat{R}_a(x,t,u) = \gamma(u) + \hat{f}(x,t)$ satisfying the condition (11). Then there exists a constant $K = K(T) \geq 0$ such that*

$$\| [u(t) - \hat{u}(t)]_+ \|_{L^2(I)} \leq$$
$$\leq e^{Kt}\left(\| [u_0 - \hat{u}_0]_+ \|_{L^2(I)} + \int_0^t e^{-Kt} \| [f(s) - \hat{f}(s)]_+ \|_{L^2(I)} ds\right). \tag{25}$$

In particular $u_0 \leq \hat{u}_0$, $f \leq \hat{f}$ imply $u \leq \hat{u}$.

Proof. It suffices to use now $(w - w^*)_+$ $(= max(w - w^*, 0))$ as a test function. Indeed, by a variant of a result due to Stampacchia we know that $(w - w^*)_+ \in L^p(0, T : V)$. Moreover

$$< w_t(t) - w_t^*(t), (w(t) - w^*(t))_+ >_{V'V} = \frac{d}{dt}\int_I |[w(t) - w^*(t)]_+|^2 dx$$

and inequality (25) follows.

■

3.2. *A non uniqueness result for the Budyko model.*

The discontinuity of the coàlbedo function $\beta(u)$ and its role as a source term in the equation may lead to the existence of multiple (even infinite) solutions of the problem. This has already been shown in Díaz [1992] for the case of the homogeneous (zero-dimensional) balance model

$$\frac{du}{dt} = R_a(u) - R_e(u).$$

The main purpose of this subsection is to show that this situation may also occurs for problem (P). Our presentation is inspired in the work of Feireisl-Norbury [1991] (see also Feireisl [1991]). We fix our attention in the special case of Budyko model i.e., R_a and R_e are given by (4), (5) and (7) respectively. We shall also assume that

$$Q(x,t) \equiv Q \text{ and } Qa_i < A - 10B. \tag{26}$$

Consider a function u_0 such that

$$\left.\begin{array}{l} u_0 \in C^\infty(I),\ u_0(x) = u_0(-x) \text{ for all } x \in [0,1], \\ u_0^{(k)}(0) = 0 \text{ for } k = 1, 2,\ u_0(0) = -10 \\ u_0'(x) < 0 \text{ if } x \in (0,1),\ u_0'(1) = 0 \end{array}\right\} \tag{27}$$

(in this hypothetical case the maximum of the distributed temperature is $-10°C$ and it is only attained at the equator). We first show the existence of a "completely ice covered" solution u^*.

Proposition 2 . *Let R_a, R_e given by (4),(5) and (7) respectively. Assume that (26) and (27) holds. Then there exist at least one solution u^* of (P) such that $u^*(x,t) < -10$ for any $x \in (-1,1)$ and $t \in (0,T]$.*

Proof. Let u^* be the unique solution of the problem

$$(P)\begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x + Bu = -A + Qa_i, & x \in I, t > 0, \\ \rho(x)|u_x|^{p-2}u_x = 0 & x \in \partial I, t > 0, \\ u(x,0) = u_0(x) & x \in I. \end{cases}$$

The existence and uniqueness can be shown again by different methods (for instance, it is a trivial consequence of Proposition 1). The function $z = -10 - u^*$ satisfies that

$$z_t - (\rho(x)|z_x|^{p-2}z_x)_x + f(z) = 0$$

with

$$f(z) = Bz + 10B - A + Qa_i.$$

Moreover $z(x,0) > 0$ and $z(0,0) = 0$. Then from (26) and the strong maximum principle (see Vazquez [1984]) we deduce that $z(x,t) > 0$ [i.e. $u^*(x,t) < -10$] for all $(x,t) \in (-1,1) \times (0,T]$.

■

The nonuniqueness of the solutions will be a consequence of the existence of solutions which exhibit the presence of "free-ice zones".

Theorem 3 . *Under the assumptions of Proposition 2 there exists at least one weak solution u of (P) such that* $\{(x,t) : u(x,t) > -10\}$ *is not empty for any* $t > 0$ *small enough.*

To carry out the proof of Theorem 3 we shall construct a family of auxiliary functions v^λ depending on a parameter $\lambda > 0$ in the following way. We first introduce the partition $(-1,1) \times [0,\lambda] = Q_1^\lambda \cup Q_2^\lambda \cup Q_3^\lambda$ by

$$\begin{aligned} Q_1^\lambda &= \{(x,t) \in (0,1) \times [0,\lambda], x > t/\lambda\} \\ Q_2^\lambda &= \{(x,t) \in (-1,1) \times [0,\lambda], -t/\lambda \leq x \leq t/\lambda\} \\ Q_3^\lambda &= \{(x,t) \in (-1,0) \times [0,\lambda], x < -t/\lambda\}. \end{aligned}$$

Now we define v^λ on Q_1^λ as the unique solution of the problem

$$P(Q_1^\lambda) \begin{cases} v_t - (\rho(x)|v_x|^{p-2}v_x)_x + Bv = -A + Qa_i, & (x,t) \in Q_1^\lambda, \\ v_x(1,t) = 0, \quad v(\frac{t}{\lambda},t) = -10, & t \in [0,\lambda], \\ v(x,0) = u_0(x) & x \in [0,1]. \end{cases}$$

The existence and uniqueness of a solution of $P(Q_1^\lambda)$ is an easy modification of the results of Friedman [1964] (see also Idrissi [1983]). Finally

$$v^\lambda(x,t) = -10 + C^\lambda(t)(x - t/\lambda)(x + t/\lambda) \text{ for all } (x,t) \in Q_2^\lambda, \tag{28}$$

$$v^\lambda(x,t) = v^\lambda(-x,t) \text{ if } (x,t) \in Q_3^\lambda.$$

We have

Proposition 3 . *It is possible to choose* $C^\lambda(t)$ *in (28) such that*
(i) $v^\lambda \in C([-1,1] \times [0,\lambda])$, $v_x^\lambda \in C((-1,1) \times [0,\lambda])$.
(ii) v^λ *is a bounded weak solution of the associated problem*

$$\begin{cases} v_t - (\rho(x)|v_x|^{p-2}v_x)_x + Bv = -A + h^\lambda(x,t) & \text{in } I \times (0,\lambda), \\ \rho(x)|v_x|^{p-2}v_x = 0 & \text{on } \partial I \times (0,\lambda), \\ v(x,0) = u_0(x) & \text{on } I, \end{cases}$$

where $h^\lambda \in L^\infty(I \times (0,\lambda))$ *satisfies that* $h^\lambda \equiv Qa_i$ *in* $Q_i^\lambda \cup Q_3^\lambda$ *and*

$$h(x,t) \leq Q(a_f - a_i)/2 \text{ for } x \in I \text{ and } t \in (0,T_\lambda) \text{ with } T_\lambda \text{ small enough.} \tag{29}$$

(iii) $v^\lambda(x,t) > -10$ *on* Q_2^λ *and* $v^\lambda < -10$ *on* $Q_1^\lambda \cup Q_3^\lambda$.

Proof.(i) The continuity of v^λ follows from the continuity of the solution of $P(Q_1^\lambda)$ (any $w \in L^\infty(J)$ such that $\rho(x)w' \in L^p(J)$ satisfies $w \in C^0(J)$, for any open interval $J \subset (0,1)$). Moreover, by (27), the solution v^λ of $P(Q_1^\lambda)$ is regular on the segment $\{(t/\lambda, t) : t \in (0,\lambda)\}$ and the function

$$g^\lambda(t) = v_x^\lambda(t/\lambda, t)$$

satisfies that $g^\lambda \in C^1((0,\lambda))$, $g^\lambda(0) = (g^\lambda)'(0) = 0$ and from (26) and the strong maximum principle (see e.g. Vazquez [1984]) $g^\lambda(t) < 0$ if $t \in (0,\lambda]$. Then choosing

$$C^\lambda(t) = \frac{g^\lambda(t)\lambda}{2t}$$

we obtain that $v_x^\lambda \in C((-1,1) \times [0,\lambda])$. From the strong maximum principle and (27) we deduce (iii). To complete the proof we only need to show that the (multivalued) equation also holds on Q_2^λ. So it suffices to show that if u^λ is given by (27) then the function

$$h^\lambda(x,t) = v_t^\lambda - (\rho(x)|v_x^\lambda|^{p-2}v_x^\lambda)_x + Bv^\lambda$$

satisfies (29). A strightforward computation yields

$$h^\lambda(x,t) = \frac{\lambda(x - t/\lambda)(x + t/\lambda)}{2t^2}[g(t)(Bt+2) - g'(t)t]$$

$$-\left(\frac{g(t)\lambda}{2t}\right)^{p-1} 2^{p-1}kx^{p-2}\left[(p-1) - (p+1)x^2\right] - \frac{g(t)}{\lambda}$$

(where g denotes g^λ). The bound

$$\left|\frac{(x - t/\lambda)(x + t/\lambda)}{2t^2}\right| \leq C(\lambda) \text{ on } Q_2^\lambda$$

with $C(\lambda)$ independent of, allows to choose T_λ so small such that the function h^λ satisfies (29).

∎

Proof of Theorem 3. We consider a regular approximation β_ϵ of β (e.g. $\beta_\epsilon \in C^\infty(I\!R)$) satisfying (19) and also

$$a_i + \frac{a_f - a_i}{2} \leq \beta_\epsilon(s) \leq a_f \text{ if } s \geq -10 \text{ and } a_i \leq \beta_\epsilon(s) \leq a_i + \frac{a_f - a_i}{2} \text{ if } s < -10. \tag{30}$$

By theorems 1 and 2 we know the existence and uniqueness of a solution u_ϵ of the problem

$$\begin{cases} u_t - (\rho(x)|u_x|^{p-2}u_x)_x + Bu = -A + Q\beta_\epsilon(u) & \text{in } I \times (0,T), \\ \rho(x)|u_x|^{p-2}u_x = 0 & \text{on } \partial I \times (0,T), \\ u(x,0) = u_0(x) & \text{on } I. \end{cases}$$

On the other hand, from Proposition 3, (29) and (30) we know that v^λ satisfies

$$\begin{cases} v_t - (\rho(x)|v_x|^{p-2}v_x)_x + Bv \leq -A + Q\beta_\epsilon(v) & \text{in } I \times (0, T_\lambda), \\ \rho(x)|v_x|^{p-2}v_x = 0 & \text{on } \partial I \times (0, T_\lambda), \\ v(x,0) = u_0(x) & \text{on } I. \end{cases}$$

and then by Theorem 2 we conclude that $u^\epsilon \geq v^\lambda$ on $\overline{I} \times [0, T_\lambda]$. Using the same kind of a priori estimates as in Lemma 2 we have that $u^\epsilon \longrightarrow u$ (weakly in $L^p(0, T : V)$ and weakly in $L^\infty(0, T : V)$) as $\epsilon \downarrow 0$, with u a bounded weak solution of (P) such that

$$u \geq v^\lambda \text{ on } \overline{I} \times [0, T_\lambda], \text{ for any } \lambda > 0, \tag{31}$$

and the conclusion follows from (28). ∎

Remark 3. It is not difficult to show (see Feireisl-Norbury [1991]) that (27) implies that the solution u of Theorem 3 satisfies $u_x(x,t) > 0$ for any $x \in (-1, 0) \cup (0, 1)$ and $t > 0$. Then by the *Implicit Function Theorem* there exists a continuous function $\zeta : [0, T] \longrightarrow [0, 1]$, defining completely *the free boundary* associated to u i.e. such that for any fixed $t \in [0, T]$

$$\{x \in \overline{I} : u(x,t) = 1\} = \{-\zeta(t)\} \cup \{\zeta(t)\}. \tag{32}$$

Clearly $\zeta \in C^1((0, T])$. Moreover (31) implies that

$$\zeta(t) \geq t/\lambda \text{ for any } \lambda > 0.$$

As $\zeta(0) = 0$ we deduce that necessarily $\zeta'(t) \uparrow +\infty$ as $t \downarrow 0$.

3.3. *On the uniqueness of solutions of the Budyko model.*

We have proved that the mere presence of a "bad point" x_0 where $u(t_0, x_o) = -10$ and $u_x(t_0, x_0) = 0$ can be the reason of multiple solutions for $t \geq t_0$. The following result shows that if the initial datum u_0 leads to a solution u never flat at the level $u = -10$ then in fact u is the unique solution. We introduce the following notation:

Definition 2 . *Let $w \in L^\infty(I)$. We say that w satisfies the strong (resp. weak) p-nondegeneracy property if there exists $C > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$|\{x \in I : |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$$

(resp. $|\{x \in I : 0 < |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$).

Theorem 4 *Assume $p \geq 2$. Let R_e satisfying (10) and R_a given by (4) and (5). Let $u_0 \in L^\infty(I)$.*

(i) Assume that there exists a solution $u(\cdot, t)$ satisfying the strong p-nondegeneracy property for any $t \in [0, T]$. Then u is the unique bounded weak solution of (P).

(ii) At most there is a unique solution among the class of bounded weak solutions satisfying the weak p-nondegeneracy property.

We start by proving that under the nondegeneracy property the multivalued term generates a continuous operator from $L^\infty(I)$ into $L^q(I)$, for any $q \in [1, \infty)$.

Lemma 3 *(i) Let $w, \hat{w} \in L^\infty(I)$ and assume that w satisfies the strong p-nondegeneracy property. Then for any $q \in [1,\infty)$ there exists $\tilde{C} > 0$ such that for any $z, \hat{z} \in L^\infty(I)$, $z(x) \in \beta(w(x))$, $\hat{z}(x) \in \beta(\hat{w}(x))$ a.e. $x \in I$ we have*

$$\| z - \hat{z} \|_{L^q(I)} \leq (a_f - a_i) \min\{\tilde{C} \| w - \hat{w} \|_{L^\infty(I)}^{(p-1)/q} \ , \ 2^{1/q}\}. \tag{33}$$

(ii) If $w, \hat{w} \in L^\infty(I)$ and satisfy the weak p-nondegeneracy property then

$$\int_I (z(x) - \hat{z}(x))(w(x) - \hat{w}(x))dx \leq (a_f - a_i)C \| w - \hat{w} \|_{L^\infty(I)}^p . \tag{34}$$

Proof of Lemma 3. If $\| w - \hat{w} \|_{L^\infty(I)} > \epsilon_0$ then

$$\| z - \hat{z} \|_{L^q(I)} \leq (a_f - a_i)2^{1/q} \leq (a_f - a_i)\frac{2^{1/q}}{(\epsilon_0)^{(p-1)/q}} \| w - \hat{w} \|_{L^\infty(I)}^{(p-1)/q} .$$

Assume now that $\| w - \hat{w} \|_{L^\infty(I)} \leq \epsilon_0$. Define the coincidence sets

$$A = \{x \in I : w(x) = -10\} \quad \hat{A} = \{x \in I : \hat{w}(x) = -10\},$$

as well as the descomposition

$$\Omega = A \cup \Omega_+ \cup \Omega_- \quad \Omega = \hat{A} \cup \hat{\Omega}_+ \cup \hat{\Omega}_-,$$

where

$$\Omega_+ = \{x \in I : w(x) > -10\} \quad \Omega_- = \{x \in I : w(x) < -10\}$$

and $\hat{\Omega}_+, \hat{\Omega}_-$ are defined similarly replacing w by $\hat{w}$. Let $z, \hat{z}$ defined as in the statement. Then

$$\begin{array}{ll} |z(x) - \hat{z}(x)| \leq (a_f - a_i) & \text{on} \quad A \cup \hat{A} \cup (\Omega_+ \cap \hat{\Omega}_-) \cup (\Omega_- \cap \hat{\Omega}_+) \\ z(x) = \hat{z}(x) & \text{on} \quad (\Omega_+ \cap \hat{\Omega}_+) \cup (\Omega_- \cap \hat{\Omega}_-) \end{array}$$

Thus as $|I| = 2$

$$\| z - \hat{z} \|_{L^q(I)} \leq (a_f - a_i) \min\{|A \cup \hat{A} \cup (\Omega_+ \cap \hat{\Omega}_-) \cup (\Omega_- \cap \hat{\Omega}_+)|^{1/q}, 2^{1/q}\}. \tag{35}$$

But we have

$$\left(A \cup \hat{A} \cup (\Omega_+ \cap \hat{\Omega}_-) \cup (\Omega_- \cap \hat{\Omega}_+)\right) \subset B_\epsilon \equiv \{x \in \Omega : -10 - \epsilon \leq w(x) \leq -10 + \epsilon\}.$$

Indeed; it is clear that $A \subset B_\epsilon$. Moreover,

$$\hat{w}(x) - \| w - \hat{w} \|_{L^\infty(I)} \leq w(x) \leq \| w - \hat{w} \|_{L^\infty(I)} + \hat{w}(x) \text{ a.e. } x \in I.$$

Then the inclusion $\hat{A} \subset B_\epsilon$ is obvious. If $x \in \Omega_+ \cap \hat{\Omega}_-$, $-10 < w(x) \leq \epsilon + \hat{w}(x) < -10 + \epsilon$ and so $x \in B_\epsilon$. Finally if $x \in \Omega_- \cap \hat{\Omega}_+$, $-10 - \epsilon \leq -10 - |w(x) - \hat{w}(x)| \leq \hat{w}(x) + w(x) -$

$\hat{w}(x) \leq w(x) < -10$ and $x \in B_\epsilon$. Consequentely, inequality (33) follows from the strong p-nondegeneracy assumption on w.
Let $w, \hat{w}$ satisfying the weak p-nondegeneracy property. As before we can assume that $\| w - \hat{w} \|_{L^\infty(I)} \leq \epsilon_0$. Then remarking that

$$(z(x) - \hat{z}(x))(w(x) - \hat{w}(x)) = 0 \text{ if } x \in A \cap \hat{A}$$

and that if $w(x) \neq -10$ (resp. $\hat{w}(x) \neq -10$) and $x \in \hat{A}$ (resp. $x \in A$) we have that

$$x \in \{x \in I : 0 < |w(x) + 10| \leq \epsilon\} \text{ (resp. } \{x \in I : 0 < |\hat{w}(x) + 10| \leq \epsilon\})$$

we obtain (34). ∎

Proof of Theorem 4. Let $\hat{u}$ be any other bounded weak solution of (P). Then, as in the proof of Theorem 2, using the monotonicity of R_e

$$\frac{d}{dt}\int_I |u(t) - \hat{u}(t)|^2 dx + \int_I \rho(x)(|u_x(t)|^{p-2}u_x(t) - |\hat{u}_x(t)|^{p-2}\hat{u}_x(t))(u_x(t) - \hat{u}_x(t))dx$$

$$\leq Q \int_I (z(x,t) - \hat{z}(x,t))(u(x,t) - \hat{u}(x,t))dxdt$$

for some $z, \hat{z} \in L^\infty(I \times (0,T))$ with $z(x,t) \in \beta(u(x,t))$, $\hat{z}(x,t) \in \beta(\hat{u}(x,t))$ for a.e. $(x,t) \in I \times (0,T)$. Now assume $p > 2$. Then by (24) we obtain that

$$\frac{d}{dt}\int_I |u(t) - \hat{u}(t)|^2 dx + \| (u(t) - \hat{u}(t))_x \|^p_{L^p(I:\rho)} \leq$$

$$\leq Q \| z(t) - \hat{z}(t) \|_{L^1(I)} \| u(t) - \hat{u}(t) \|_{L^\infty(I)} .$$

From Theorem 4 of Rakotoson-Simon [1993] we have the estimate

$$\| v \|_{L^\infty(I)} \leq C_1 \| v_x \|_{L^p(I:\rho)} + |I|_\rho^{-1} \| v \|_{L^1(I:\rho)}, \quad \forall v \in V \tag{36}$$

where

$$C_1 = |I|_p^{(p-2)/2p} C_0$$

with $C_0 > 0$ independent of I and

$$|I_\rho| = \int_{-1}^1 \rho(x)dx = k\int_{-1}^1 (1 - x^2)dx = 4k/3.$$

Then by Lemma 3 and using $(a+b)^p \leq 2^p(a^p + b^p)$ we get

$$Q \| z(t) - \hat{z}(t) \|_{L^1(I)} \| u(t) - \hat{u}(t) \|_{L^\infty(I)} - \| (u(t) - \hat{u}(t))_x \|^p_{L^p(I:\rho)} \leq$$

$$\leq \| u(t) - \hat{u}(t) \|^p_{L^\infty(I)} \left(QC(a_f - a_i) - \frac{1}{2^p C_1^p} \right) + (C_1 |I|_\rho)^{-p} \| u(t) - \hat{u}(t) \|^p_{L^1(I:\rho)} \leq$$

$$\| u(t) - \hat{u}(t) \|^p_{L^\infty(I)} \left(QC(a_f - a_i) - \frac{1}{2^p C_1^p} \right) + C_2 \| u(t) - \hat{u}(t) \|^2_{L^2(I)}$$

where

$$C_2 = \frac{(\int_I \rho(x)^2 dx)^{p/2}}{C_1^p (\int_I \rho(x) dx)^p} \parallel u - \hat{u} \parallel^{p-2}_{L^\infty((0,T):L^2(I))} \leq C_3$$

for some C_3 independent of u and $\hat{u}$ (that can be obtained from the estimates as (25) in terms of the data, $\parallel u_0 \parallel_{L^2(I)}$, $Q(a_f - a_i)$ and $\parallel R_e(x,t,\cdot) \parallel_{L^\infty(0,T:L^2(I))}$). Assume now that

$$QC(a_f - a_i) - \frac{1}{2^p C_1} \leq 0.$$

Then we conclude that

$$\frac{d}{dt} \parallel u(t) - \hat{u}(t) \parallel^2_{L^2(I)} \leq C_3 \parallel u(t) - \hat{u}(t) \parallel^2_{L^2(I)} . \tag{37}$$

Setting $U(t) = \parallel u(t) - \hat{u}(t) \parallel^2_{L^2(I)}$ we obtain that $U(t) \leq U(0)e^{C_3 t}$ but as $U(0) = 0$ we deduce that $u(t) = \hat{u}(t)$ for any $t \in [0,T]$. If (37) does not hold we introduce the rescaling $y = \alpha x$ with $\alpha > 0$. Given a function $h(x,t)$ we define $h(y,t)$ by $h(y,t) = h(\alpha x, t)$. Then the functions $u(y,t)$ and $\hat{u}(y,t)$ satisfy

$$\frac{\partial u}{\partial t} - \alpha^p (\rho_\alpha(y)|u_y|^{p-2}u_y)_y = Qz(y,t) - R_e(\frac{y}{\alpha}, t, u)$$

$$\frac{\partial \hat{u}}{\partial t} - \alpha^p (\rho_\alpha(y)|\hat{u}_y|^{p-2}\hat{u}_y)_y = Q\hat{z}(y,t) - R_e(\frac{y}{\alpha}, t, \hat{u})$$

in $(-\alpha, \alpha) \times (0,T)$, where

$$\rho_\alpha(y) = K\left(1 - \frac{y^2}{\alpha^2}\right).$$

Arguing as in the case $\alpha = 1$ we have

$$\frac{d}{dt} \parallel u(t) - \hat{u}(t) \parallel^2_{L^2(-\alpha,\alpha)} + \alpha^p \parallel (u(t) - \hat{u}(t))_y \parallel^p_{L^p((-\alpha,\alpha):\rho_\alpha)} \leq$$
$$\leq Q \parallel z(t) - \hat{z}(t) \parallel_{L^1(-\alpha,\alpha)} \parallel u(t) - \hat{u}(t) \parallel_{L^\infty(-\alpha,\alpha)} .$$

Estimate (36) remains true when one replaces I by $I_\alpha (= (-\alpha, \alpha))$ and ρ by ρ_α. So a simple computation leads to $|I_\alpha|_{\rho_\alpha} = \alpha|I|_\rho$ and thus

$$\parallel v \parallel_{L^\infty(-\alpha,\alpha)} \leq \alpha^{(p-2)/2p} C_1 \parallel v_y \parallel_{L^p((-\alpha,\alpha):\rho_\alpha)} + (\alpha|I|_\rho)^{-1} \parallel v \parallel_{L^1((-\alpha,\alpha):\rho_\alpha)} .$$

Then by Lemma 3

$$Q \parallel z(t) - \hat{z}(t) \parallel_{L^1(-\alpha,\alpha)} \parallel u(t) - \hat{u}(t) \parallel_{L^\infty(-\alpha,\alpha)} - \alpha^p \parallel (u(t) - \hat{u}(t))_y \parallel^p_{L^p((-\alpha,\alpha):\rho_\alpha)} \leq$$
$$\leq \parallel u(t) - \hat{u}(t) \parallel^p_{L^\infty((-\alpha,\alpha))} (QC(a_f - a_i)\alpha - \frac{\alpha^{p-(p-2)/2}}{2^p C_1^p}) +$$
$$+ C_4(\alpha) \parallel u(t) - \hat{u}(t) \parallel^2_{L^2(-\alpha,\alpha)} .$$

Taking α large enough we obtain that $U_\alpha(t) = \parallel u(t) - \hat{u}(t) \parallel$ satisfies $U_\alpha \leq U_\alpha(0)e^{C_4(\alpha)t}$ and so again $u(t) = \hat{u}(t)$ for any $t \in [0,T]$.

If $p = 2$ the estimate (36) must be replaced by

$$\| v \|_{L^r(I,\rho)} \leq C_1 \| v_x \|_{L^p(I:\rho)} + |I|_\rho^{(1/r)-1} \| v \|_{L^1(I:\rho)} \tag{38}$$

for any $r \in [1, \infty)$ where

$$C_1 = |I|_\rho^{1/r} C_0$$

with $C_0 > 0$ independent of I (see Rakotoson-Simon [1993]). But as $u(t) - \hat{u}(t) \in L^\infty(I)$ we know that for any $\delta > 0$ there exists $n(\delta) > 0$ such that for any $r \in [n(\delta), +\infty)$

$$\left| \| u(t) - \hat{u}(t) \|_{L^\infty(I)} - \| u(t) - \hat{u}(t) \|_{L^r(I:\rho)} \right| \leq \delta \tag{39}$$

and so

$$\| u(t) - \hat{u}(t) \|^p_{L^\infty(I)} \leq 2^p \| u(t) - \hat{u}(t) \|^p_{L^r(I:\rho)} + 2^p \delta^p \leq$$
$$\leq 2^p C_1^p \| (u(t) - \hat{u}(t))_x \|^p_{L^p(I:\rho)} + 2^p |I|^{[(1/r)-1]p} \| u(t) - \hat{u}(t) \|_{L^1(I:\rho)} + 2^p \delta^p.$$

Arguing as in the case $p > 2$ we obtain

$$\frac{d}{dt} \| u(t) - \hat{u}(t) \|^2_{L^2(I)} \leq \| u(t) - \hat{u}(t) \|^p_{L^\infty(I)} (QC(a_f - a_i) - \frac{1}{2^p C_1^p})$$
$$+ C_3 |I|^{p/r} + 2^p \delta^p.$$

Making $\delta \downarrow 0$ as C_3 is independent of r we obtain (37) and the proof of *(i)* ends. Part *(ii)* is obtained in a similar way by using now *(ii)* of Lemma 3. ∎

To complete the study of the uniqueness of solutions of (P) we concentrate our attention on the nondegeneracy properties. The local character of those conditions is pointed in the next result.

Proposition 4 .*(i) Let $w \in C^0(I)$. Assume that the set $A = \{x \in I : w(x) = -10\}$ has a finite number of connected components and that there exists $\epsilon > 0$ and a positive constant K such that for any $\epsilon \in (0, \epsilon_0)$ and $x \in \hat{B}_\epsilon \equiv \{x \in I : 0 < |w(x) + 10| \leq \epsilon\}$*

$$|w(x) + 10| \geq K |x - x_i|^{1/(p-1)}, \quad \forall x_i \in \partial A. \tag{40}$$

Then w satisfies the weak p-nondegeneracy property. Furthermore, if $|A| = 0$ then w satisfies the strong p-nondegeneracy property.

(ii) Let $W^{1,\infty}_{loc}(I)$ and assume that A has a finite number of connected components and that there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ $\exists \delta = \delta(\epsilon)$ such that

$$|w_x(x)| \geq \delta \text{ a.e. } x \in \{x \in I : |w(x) + 10| \leq \epsilon\} \tag{41}$$

then w satisfies the strong 2-nondegeneracy property.

Proof. From (40) we deduce that if $x \in \hat{B}_\epsilon$ then $|x-x_i| \leq \epsilon^{p-1}/K$. Thus $|B_\epsilon| \leq (N/K)\epsilon^{p-1}$ where N is the number of points of ∂A.

(ii) It is clear that (41) implies that meas $|A| = 0$. Let $[a,b] \subset \overline{I}$ a connected component of $B_\epsilon = \{x \in I : |w(x)+10| \leq \epsilon\}$. Assume that $w_x(x) \geq \delta$ on (a,b) [the other case $w_x(x) \leq -\delta$ on (a,b) is treated in a similar way]. Then $w(a) = -10-\epsilon$, $w(b) = -10+\epsilon$ and there exists $x_0 \in (a,b)$ such that $w(x_0) = -10$. Then for any $x \in [x_0, b]$ we have

$$\epsilon \geq w(x) + 10 = \int_{x_0}^{x} w_x(s)ds \geq \delta(x-x_0).$$

Analogously, for any $x \in [a, x_0)$,

$$\epsilon \geq -10 - w(x) = \int_{x}^{x_0} w_x(s)ds \geq \delta(x_0 - x)$$

and thus (40), with $p = 2$, holds. ∎

Remark 4. The nondegeneracy properties of the solutions of (P) can be obtained under some additional assumptions on the initial datum. Let $u_0 \in C^1(\overline{I})$ such that

$$A_0 = \{x \in I : U_0(x) = -10\} \text{ has a finite number of connected components,} \tag{42}$$

and

$$\left.\begin{array}{l} \exists \epsilon_0 > 0 \text{ and } K > 0 \text{ such that } \forall \epsilon \in (0,\epsilon_0)\text{and any} \\ x \in \hat{B}_{\epsilon,0} = \{x \in I : 0 < |u_0(x)+10| \leq \epsilon\} \\ \text{ we have } |u_0(x)+10| > K|x-x_i|^{1/(p-1)} \quad \forall x \in \partial A \end{array}\right\} \tag{43}$$

Then there exists a $T_u \in (0,T]$ such that $u(t)$ satisfies the weak non-degeneracy property for any $t \in [0,T_u)$ where u is any continuous weak solution of (P). In particular if u and $\hat{u}$ are continuous weak solutions of (P) there exists a $T^* \in (0,T]$ such that $u = \hat{u}$ on $[0,T^*) \times I$. Indeed; let $u, \hat{u}$ be continuous bounded weak solutions of (P), by the continuity near $t = 0$ we deduce that there exist T_u, $T_{\hat{u}} \in (0,T]$ such that $u(t)$, $\hat{u}(t)$ satisfy (40) and that the set where they take the value -10 has the same (finite) nomber of connected components for any $t \in [0,T_u)$, $[0,T_{\hat{u}})$ respectively. Taking $T^* = \min\{T_u, T_{\hat{u}}\}$ the conclusion follows from part (ii) of Theorem 4.

Remark 5. Let $u_0 \in C^1(\overline{I})$ such that u_0 is an even function, $u_{0x}(x) > 0$ for any $x \in (-1,0)$, $u_0(0) > -10$, $u(-1) < -10$. Then (42) and (43) holds for $p = 2$. Moreover, if u is the solution built in the section *3.2* for $p = 2$ then $u(t)$ satisfies the strong 2-nondegeneracy property for any $t \in [0,T]$. Finally, if $p = 2$ problem (P) has a unique bounded solution on $[0,T] \times I$. Indeed; it is an easy modification of Lemma 6.2 and Corollary 6.3 of Feireisl-Norbury [1991]. For some other criterium on u_0 see Diaz - Tello [1996].

Remark 6. It should be interesting to know if the techniques on non-degeneracy properties for the parabolic obstacle problem (see,e.g., Pietra-Verdi [1985]) can be applied to obtain the p-nondegeneracy properties for the solutions of (P).

4. On the free boundary for Budyko type models.

This section is devoted to present some qualitative properties of the solutions associated to the Budyko type model. The discontinuity of the albedo function assumed in the Budyko model generates a natural *free boundary* or interface $\zeta(t)$ between the ice-covered area $(\{x \in I : u(x,t) < -10\})$ and the ice-free area $(\{x \in I : u(x,t) > -10\})$. The free boundary is then given as $\zeta(t) = \{x \in I : u(x,t) = -10\}$. In Xu [1991] the Budyko model for $p = 2$ is considered. He shows that if the initial datum u_0 satisfies

$$\begin{aligned}
&u_0(x) = u_0(-x),\ u_0 \in C^3([-1,1]),\ u_0'(x) < 0 \text{ for any } x \in (0,1) \\
&\text{and there exists } \zeta(0) \in (0,1) \text{ such that } (u_0(x)+10)(x-\zeta(0)) < 0 \\
&\text{for any } x \in [0,\zeta(0)) \cup (\zeta(0),1],
\end{aligned}$$

then there exists a bounded weak solution u of (P) for which the set $\zeta(t) = \{\zeta_+(t)\} \cup \{\zeta_-(t)\}$ with $x = \zeta_+(t)$ a smooth curve, $\zeta_-(t) = \zeta_+(t)$ and $\zeta_+(\cdot) \in C^\infty([0,T^*))$ where T^* is characterized as the first time t for which $\zeta_+(t) = 1$. He also gives an expression for the derivative $\zeta_+'(t)$ (some related results for a model corresonding to $\rho(x) = 1$ can be found in Feireisl-Norbury [1991]). We point out that the uniqueness result (Theorem 4) can be applied for such an initial datum (see Remark 4).

The size of the separating zone $\zeta(t)$ for other models is in fact a controversial question. So, some satellite pictures (Image of the Weddell sea taken by the satellite Spot on December 10, 1987: Lions [1991]) show that the separating region between the ice-free and the ice-covered zones is not a simple line on the Earth (i.e. a point in $(-1,0)$ or $(0,1)$) but a narrow zone where ice and water are mixed. Mathematically it corresponds to say that the set

$$M(t) = \{x \in I : u(x,t) = -10\}$$

is a positively measured set. In the following we shall denote this set as the *mushy region* (since it plays the same role than in changing phase problems, see e.g. Díaz-Fasano-Meirmanov [1992]).

Using the strong maximum principle (see e.g. Vazquez [1984]) it is possible to show that if $p = 2$ (or more in general if $1 < p \leq 2$) the interior set of the mushy region $M(t)$ is empty even if the interior of $M(0)$ is a nonempty open set. The main goal of the next result is to show that this is not the case when $p > 2$ (as it happens for the Stone model : $p = 3$). A necessary condition for $\overset{\circ}{M}(t) \neq \emptyset$ is that $R_a(x,t,-10) - R_e(x,t,-10) \ni 0$ for any $x \in \overset{\circ}{M}(t)$ and $t \in [0,T]$. In the case of the Budyko model R_a is defined by (4) and (5), R_e by (7) and the necessary condition can be written in the following terms

$$A - 10B \in [a_i Q(x,t), a_f Q(x,t)] \text{ for a.e. } x \in I, \text{ a.e. } t \in [0,T] \tag{44}$$

We shall show that if $p > 2$ this condition is also sufficient.

Theorem 5 . *Let $p > 2$, R_a given by (4) and (5) and R_e given by (7). Assume (44) and $u_0 \in L^\infty(I)$ such that there exist $x_0 \in I$ and $R_0 > 0$ satisfying*

$$M(0) = \{x \in I : u_0(x) = -10\} \supset B(x_0, R_0)(= \{x \in I : |x - x_0| < R_0\}).$$

If u is the bounded weak solution of (P) satisfying the weak p-nondegeneracy property then there exists $T^* \in (0, T]$ and a nonincreasing function $R(t)$ with $R(0) = R_0$ such that

$$M(t) = \{x \in I : u(x,t) = -10\} \supset B(x_0, R(t))$$

for any $t \in [0, T^*)$.

Proof. We shall use an energy method as developped in Díaz-Veron [1985]. Given u bounded weak solution of (P) we define $v = u+10$. As in Lemma 3.1 of the above reference multiplying the partial differential equation by v we obtain that for a.e. $R \in (0, R_0)$ and $t \in (0, T)$ we have

$$\frac{1}{2}\int_{B(x_0,R)} |v(x,t)|^2 dx + \int_0^t \int_{B(x_0,R)} \rho(x)|v_x|^p dx d\tau + B\int_0^t \int_{B(x_0,R)} |v(x,\tau)|^2 dx d\tau \leq$$

$$\leq \int_0^t \int_{S(x_0,R)} \rho(x)|v_x|^{p-2} v_x \cdot \vec{n} v ds d\tau + \int_0^t \int_{B(x_0,R)} \{Q(x,\tau)z(x,\tau) - A + 10B\} v dx d\tau =$$

$$= I_1 + I_2. \tag{45}$$

where $S(x_0, R) = \partial B(x_0, R) = \{x_0 - R\} \cup \{x_0 + R\}$ and $z(x,t) \in \beta(u(x,t))$ for a.e. $x \in B(x_0, R)$ and $t \in (0, T]$. We introduce the energy functions

$$E(R,t) = \int_0^t \int_{B(x_0,R)} \rho(x)|v_x|^p dx d\tau$$

$$b(R,t) = \sup_{0 \leq \tau \leq t} \text{ess} \int_{B(x_0,R)} |v(x,\tau)|^2 dx.$$

Using Holder's inequality and the interpolation-trace Lemma of Díaz-Veron [1985] (since $p > 2$) we get

$$I_1 \leq \left(\frac{\partial E}{\partial R}(R,t)\right)^{(p-1)/p} \left(\int_0^t \int_{S(x_0,R)} |v|^p dx d\tau\right)^{1/p} \leq$$

$$\leq Ct^{(1-\theta)/p} \left(\frac{\partial E}{\partial R}(R,t)\right)^{(p-1)/p} \left(E(R,t)^{1/p} + R^\delta t^{1/p} b(R,t)^{1/2}\right)^\theta b(R,t)^{(1-\theta)/2},$$

where

$$\theta = p/(3p-2) \text{ and } \delta = -(3p-2)/2p.$$

Using the assumption (44) we have that

$$\hat{z}(\cdot) = [(A - 10B)/Q(\cdot, t)] \in \beta(-10). \tag{46}$$

Then applying Lemma 3 to $w(\cdot) = u(\cdot, t)$, $z(\cdot) = B(\cdot, t)$, $\hat{w}(\cdot) = -10$ and $\hat{z}(\cdot)$ given by (46) we get that

$$I_2 \leq (a_f - a_i) \parallel Q \parallel_{L^\infty(I\times(0,T))} C \int_0^t \parallel v(\tau) \parallel^p_{L^\infty(B(x_0,R))} d\tau.$$

Using the inequality (36) on $B(x_0, R)$ we obtain

$$I_2 \leq (a_f - a_i) \parallel Q \parallel_{L^\infty(I\times(0,T))} C(C_1 E(R,t) + tC_2(R) b(R,t)),$$

where now

$$C_2(R) = \frac{\left(\int_{B(x_0,R)} \rho(x)^2 dx\right)^{p-2}}{C_1^p \left(\int_{B(x_0,R)} \rho(x) dx\right)^p} \parallel u + 10 \parallel^p_{L^\infty((0,T):L^2(I))}.$$

As in the proof of Theorem 4, without loss of generality we can assume C_1 small enough. Then, there exists $T^* \in (0, T]$ and $\lambda \in (0,1]$ such that

$$\lambda(E(R,t) + b(R,t)) \leq I_1$$

which implies that

$$\lambda E^\mu \leq t^{(1-\theta)/p} \frac{\partial E}{\partial R}$$

for some $\mu \in (0,1)$ and for any $t \in [0, T^*)$ and the proof ends as in Díaz-Veron [1985] (proof of Theorem 3.1). ∎

Remark 7. The existence of the mushy region (for any value of $p \in (1, \infty)$) can be proved for a different class of models by taking into account a discontinuous diffusivity (see Held-Linder-Suarez [1981]). In that case the problem is a variant of the Stefan problem (see, e.g., Díaz-Fasano-Meirmanov [1992]). We also point out that if we define the mushy region associated to a temperature u_c, with $u_c \neq -10$, by

$$M(t : u_c) = \{x \in I : u(x,t) = u_c\},$$

then the results of Díaz-Veron [1985] and Antonsev-Díaz [1989] allows to obtain the same type of conclusions than in Theorem 5 (but without the non-degeneracy assumption on the solution) for suitable functions $Q(x,t)$.

5. Obstruction and Controllability in Energy Balance Models.

In 1955, John von Neumann wrote: *Probably intervention in atmospheric and climate matters will come in a few decades, and will unfold on a scale difficult to imagine at present* ([1955]). Today one phase of this programme is almost a dream come true: the "rain making" research initiated by I. Langmuir and coworkers have originated already sucessful experiences (see Dennis [1980]). While is not easy to evaluate the significance of the efforts made thus far, the evidence seems to indicate that the aim is an attainable one.

The main goal of this section is to carry out a theoretical study on the remainig part of the von Neumann programme: the control of the climate. Our modest goal is to study such a *general philosophy* by considering the simple climate models introduced by M.I. Budyko and W. D. Sellers.

Continuing our previous research (see Díaz [1994]) in which it was shown how the *obstruction phenomenon* leads to the general uncontrollability of the Sellers model, we show here that a chance still remains: *the restricted (approximate) controllability.* We will show that a very large class of desired climate states are attainable (in a weak sense) by introducing suitable spatially localized controls on the climate system.

Our main goal is to study if possible antropogenerated actions on the climate system allows to carry the average temperature from a given distribution $y(0,x)$ to a desired distribution $y_d(x)$ after a given period of time T. Such type of questions was already considered by J. Fourier [1824] and some of the most relevant applied mathematicians of this century (J. von Neumann [1955] and J.L. Lions [1990] [1992] among them). The connection between this question and the study of the irreversibility of the antropogenetic changes already introduced in the atmosphere since the beginings of the Industrial Society is obvious. It is also clear that many of the actual world decissions on greenhouse gases emmision norms follow also this philosofy.

A mathematical statement of the question under consideration can be the following: given ω an open submanifold of $\mathcal{M}$, $T > 0$, an initial distribution of temperatures $u_0 : \mathcal{M} \to I\!R$ and a desired temperature $y_d : \mathcal{M} \to I\!R$, we want to find a control $v : (0,T)\times\omega \to I\!R$ such that $y(T : v) = y_d$ where $y(\cdot : v)$ denotes the solution of problem $(\mathcal{P})$ replacing $f(t,x)$ by $f(t,x) + v(t,x)\chi_\omega$ with χ_ω the characteristic function of ω. When the answer is positive we say that $(\mathcal{P})$ *is controllable.* Nevertheless, the parabolic character of the equation of $(\mathcal{P})$ implies some regularizing effects making impossible our goal except for a very limited class of desired states y_d. A relaxed statement comes in a natural way: *the approximate controllability.* Given $\varepsilon > 0$ we seek now a control v_ε (defined again on $(0,T) \times \omega$) such that $d(y(T,v_\varepsilon), y_d) \leq \varepsilon$. In the above expression $d(\cdot,\cdot)$ represents the distance measured in some space of functions defined on $\mathcal{M}$ (usually $L^2(\mathcal{M})$, or, more generally, $L^p(\mathcal{M})$ with $1 \leq p \leq \infty$).

The nature of our spatial domain $\mathcal{M}$ leads to some additional (and technical) difficulties in our study. A simpler formulation which still gives a representative idea of the treatement in more complex situations corresponds to the case in which we replace $\mathcal{M}$ by an open regular bounded set Ω of $I\!R^2$ (here $I\!R^2$ can be also substitued by $I\!R^N$ with $N \in I\!N$). As boundary condition on $(0,T) \times \partial\Omega$ we can chose the one of Neumann type since it leads to a set of test functions for the weak formulation very similar to the one corresponding to the case of a Riemannian manifold without boundary. Another unrelevant simplification is to assume $f \equiv 0$. Thus the new formulation is the following: given ω an open bounded subset of Ω, y_0, $y_d : \Omega \to I\!R$ and $\varepsilon > 0$ find $v_\varepsilon : (0,T) \times \omega \to I\!R$ such that $d(y(T : v_\varepsilon), y_d) \leq \varepsilon$ where, in general, $y(T : v)$ represents the solution of problem

$$(\mathcal{P}_\omega) \begin{cases} y_t - \Delta y + g(y) \in QS(x)\beta(y) + v\chi_\omega & \text{in } (0,T) \times \Omega \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0,\cdot) = y_0(\cdot) & \text{on } \Omega, \end{cases}$$

where n is the outer unit vector to $\partial\Omega$.

In a previous work (Díaz [1994]) it was shown that the answer to the approximate controllability property depends on the asymptotic behaviour of the nonlinearities of the equation (and not on its regularity). So, a positive answer is collected in the following result

Theorem 6 (Díaz [1994])

Assume y_0, $y_d \in L^2(\Omega)$, β satisfying (9) and g a nondecreasing function such that

$$|g(s)| \leq C_1 + C_2|s| \quad \forall\, s \in I\!R,\ |s| > \overline{M} \tag{47}$$

for some nonnegative constants C_1, C_2 and $\overline{M}$. Then problem $(\mathcal{P}_\omega)$ is approximate controllable in $L^2(\Omega)$, i.e. there exist $v_\varepsilon \in L^2((0,T) \times \omega)$ such that

$$\| y(T : v_\varepsilon) - y_d \|_{L^2(\Omega)} \leq \varepsilon.$$

The above theorem can be easily extended to the case in which we replace $L^2(\Omega)$ by $L^p(\Omega)$ with $1 \leq p < \infty$ or $\mathcal{C}(\overline{\Omega})$. The main idea of the proof is the application of the Kakutani fixed point theorem similarly to the work Fabré, Puel and Zuazua [1992] (see also Henry [1978], Lions [1968] [1991], Díaz [1993] and Díaz and Ramos [1993] [1994] for other related works).

We point out that Theorem 6 applies to the special case of the Budyko model since there $g(y) = By$ and (47) fails for the Sellers model (assume $m = 0$ in (6) and also $u > 0$ in order to reduce the study to a nondecreasing function g). In fact, it was shown in Díaz [1994] (see also [1991]) that if we assume

$$g(y) = \lambda |y|^{p-1} y \text{ for } y \in I\!R \text{ and some } \lambda > 0 \text{ and } p > 1 \tag{48}$$

then an *obstruction phenomenon* appears

Theorem 7

Assume (48) and that $\partial\omega$ satisfies the interior and exterior sphere condition. Let $y_0 \in L^\infty(\Omega)$. Then, there exists a function $Y_\infty \in C([0,T] \times (\Omega - \overline{\omega}))$ such that for any $v \in L^2((0,T) \times \omega)$ and any solution $y(t, x : v)$ of (P_ω) we have

$$|y(t, x : v)| \leq Y_\infty(t, x) \text{ for } (t, x) \in (0, T] \times (\Omega - \overline{\omega}). \tag{49}$$

The obstruction function Y_∞ in (49) was constructed in Díaz [1994] such that

$$\begin{array}{ll} Y_\infty(t,x) = +\infty & \text{on } (0,T)\times\partial\omega \\ \frac{\partial Y_\infty}{\partial n}(t,x) = 0 & \text{on } (0,T)\times\partial\Omega. \end{array}$$

In consequence, condition (48) implies that problem (P_ω) is not (in general) approximate controllable since if $|y_d(x)| > Y_\infty(T,x)$ a.e. x in a positively measured subset D of $\Omega - \overline{\omega}$ then for any $v \in L^2((0,T)\times\omega)$

$$\|y(T;v) - y_d\|_{L^2(\Omega)} \geq \|Y_\infty(T,\cdot) - y_d\|_{L^2(\Omega)}$$

and so, if $\epsilon > 0$ is small enough, it is imposible to choose v satisfying the required properties. We remark that a previous uniform estimate (independetly of the control) for superlinear equations but when the control acts on the boundary was due to A. Bamberger (see Henry [1978]). Due to the relevance of the Sellers model, a natural question arises: is problem P_ω approximate controllable in a smaller class of desired states y_d?

The main contribution of this work is to give a positive answer to the above question. For the sake of the exposition we shall simplify, even more, the model under consideration to

$$(P_p)\begin{cases} y_t - \Delta y + \lambda|y|^{p-2}y = v\chi_\omega & \text{in } (0,T)\times\Omega \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,\cdot) = y_0(\cdot) & \text{on } \Omega. \end{cases}$$

The extension of the following results to the case of problem (P_ω), assumed (48), is merely a technical matter and can be carried out as in Díaz [1994].

The starting point of our approach consists in improving the estimate (49) by obtaining some *sharp obstruction functions.* This is the objective of the next result

Proposition 5

Given $y_0 \in L^1(\Omega)$ there exist $\underline{Y}_\infty$, $\overline{Y}_\infty \in C((0,T]\times\Omega-\overline{\omega})$ such that $\underline{Y}_\infty$ is a weak solution to the problem

$$\begin{cases} \underline{Y}_t - \Delta\underline{Y} + \lambda|\underline{Y}|^{p-2}\underline{Y} = 0 & \textit{in } (0,T)\times(\Omega-\overline{\omega}) \\ \underline{Y}_\infty = -\infty & \textit{on } (0,T)\times\partial\omega \\ \frac{\underline{Y}_\infty}{\partial n} = 0 & \textit{on } (0,T)\times\partial\Omega \\ \underline{Y}_\infty(0,\cdot) = y_0(\cdot) & \textit{on } \Omega \end{cases}$$

and $\overline{Y}_\infty$ satisfies the same conditions except that $\overline{Y}_\infty = +\infty$ on $(0,T)\times\partial\omega$.

Idea of the proof. As in Bandle, G. Díaz and J.I. Díaz [1994], given $N \in I\!N$ we define $\underline{Y}_N$ as the (unique) solution of the problem

$$\begin{array}{lll} Y_t - \Delta Y + \lambda|Y|^{p-2}Y = 0 & \text{in} & (0,T)\times(\Omega-\overline{\omega}) \\ Y = -N & \text{on} & (0,T)\times\partial\omega \\ \frac{\partial Y}{\partial n} = 0 & \text{on} & (0,T)\times\partial\Omega \\ Y(0,\cdot) = \mathrm{Sup}\{(y_0)_-(\cdot), -N\} & \text{on} & \Omega. \end{array}$$

where $(y_0)_-(x) = \inf\{y_0(x), 0\}$. Using the maximum principle and the assumption $p > 1$ it is easy to see that there exists $\underline{Z}_\infty \in C((0,T) \times (\Omega - \overline{\omega}))$ such that $\underline{Z}_\infty \leq .... \leq \underline{Y}_2 \leq \underline{Y}_1 \leq 0$. Then we can define $\underline{Y}_\infty(t,x) = \lim_{N\to\infty} \underline{Y}_N(t,x)$ and by duality arguments it is proven that $\underline{Y}_\infty$ satisfies the requiered conditions. The arguments for $\underline{Y}_\infty$ are completely similar.

■

We point out that if we assume, formally, $Q = 0$ in P_ω then the obstruction functions of Proposition 1 is sharper than the ones given in Theorem 7, i.e.

$$\underline{Y}_{\ \infty}(t,x) \leq -Y_{\ \infty}(t,x) \leq \underline{Y}_{\ \infty}(t,x) \leq \overline{Y}_\infty(t,x).$$

Now we are in a condition to state our *restricted approximate controllability criterion:*

Theorem 8

Let $y_0 \in C(\overline{\Omega})$ and consider $y_d \in C(\overline{\Omega})$ such that

$$\underline{Y}_{\ \infty}(T,x) < y_d(x) < \overline{Y}_\infty(T,x) \quad \forall x \in \Omega - \overline{\omega}. \tag{50}$$

Then for any $\epsilon > 0$ there exists $v_\epsilon \in C([0,T] \times \overline{\omega})$ such that if $y(t:v)$ is the corresponding solution of (P_p) we have

$$\|y(T:v_\epsilon) - y_d\|_{C(\overline{\Omega})} \leq \epsilon. \tag{51}$$

The above statement is an obvious consequence of the following more general result:

Theorem 9

Let $y_0 \in C(\overline{\Omega})$ and let $\epsilon > 0$ fixed. Consider $y_d \in C(\overline{\Omega})$ such that

$$\underline{Y}_{\ \infty}(T,x) - \epsilon < y_d(x) < \overline{Y}_\infty(T,x) + \epsilon \quad \forall x \in \Omega - \overline{\omega} \tag{52}$$

Then there exists $v_\epsilon \in C([0,T] \times \overline{\omega})$ satisfying (51).

Remark. The assumption (52) is optimal. Indeed, assume v_ϵ such that (51) holds. Then by the comparison principle

$$\underline{Y}_{\ \infty}(t,x) < y(t,x:v_\epsilon) < \overline{Y}_\infty(t,x) \quad \forall (t,x) \in [0,T] \times (\Omega - \overline{\omega})$$

and so

$$\underline{Y}_{\ \infty}(T,x) - \epsilon < y(T,x:v_\epsilon) - \epsilon \leq y_d(x) \leq y(T,x:v_\epsilon) + \epsilon < \overline{Y}_\infty(T,x) + \epsilon$$

which proves (50).

The proof of Theorem 9 consists of several steps. We start by proving the *restricted approximate controllability* for an auxiliary control problem with controls actuing on the boundary

Theorem 10

Let $y_0 \in C(\overline{\Omega} - \omega)$, $\epsilon > 0$ fixed and let $y_d \in C(\overline{\Omega} - \omega)$ satisfying (52). Then there exists $u_\epsilon \in C([0,T] \times \partial\omega)$ such that if $\tilde{y}(t, x : u_\epsilon)$ denotes the solution of the problem

$$(\mathcal{P}_{\Omega-\omega}) \begin{cases} \tilde{y}_t - \Delta\tilde{y} + \lambda|\tilde{y}|^{p-2}\tilde{y} = 0 & \text{in } (0,T) \times (\Omega - \omega) \\ \tilde{y} = u_\epsilon & \text{on } (0,T) \times \partial\omega \\ \frac{\partial\tilde{y}}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega \\ \tilde{y}(0,x) = y_0(x) & \text{on } \Omega - \overline{\omega}, \end{cases}$$

we have

$$\|\tilde{y}(T, \cdot : u_\epsilon) - y^d(\cdot)\|_{C(\Omega-\omega)} \leq \epsilon.$$

The proof of Theorem 10 uses another auxiliary result:

Lemma 4

Let G be an open regular bounded set of $I\!R^N$. For $a \in L^\infty((0,T) \times G)$ and $y_0 \in C(\bar{G})$ given we denote by $y(t, x : u)$ the solution of the linear control problem

$$(PL) \begin{cases} y_t - \Delta y + ay = 0 & \text{in } (0,T) \times G \\ y = u & \text{on } (0,T) \times \partial_D G \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T) \times \partial_N G \\ y(0,\cdot) = y_0(\cdot) & \text{on } G. \end{cases}$$

where $\partial G = \partial_D G \cup \partial_N G$. Let $\epsilon > 0$ and $y_d \in C(\bar{G})$. Then (i) There exists $u_\epsilon \in C^0([0,T] \times \partial_D G)$ such that

$$\|y(T, \cdot : u_\epsilon) - y_d(\cdot)\|_{C(\bar{G})} \leq \epsilon. \tag{53}$$

Moreover, there exists $q = q(N) > 0$ and two functions $\hat{\varphi}$ and h such that

$$u_\epsilon(t,x) = h(t,x)\|(T-t)^q \frac{\partial\hat{\varphi}}{\partial n}(t,\cdot)\|_{L^1((0,T)\times\partial_D G)} \tag{54}$$

with

$$h(t,x) \in sign(\frac{\partial\hat{\varphi}}{\partial n}(t,x)) \qquad \forall(t,x) \in (0,T) \times \partial_D G. \tag{55}$$

(ii) If $a \geq 0$ a.e. on $(0,T) \times G$, the function u_ϵ given in (54) satisfies that

$$\|u_\epsilon\|_{C([0,T]\times\partial_D G)} \leq C \tag{56}$$

for some $C > 0$ independent of a.

Sketch of the proof. Part (i) is an adaptation of the duality method introduced in Lions [1991] and Lions [1992]. We start by defining the space $V = C(\bar{G})$ and let V' its dual (i.e. the set of Baire measures of bounded variation: Yosida [1974] p. 119). Given $\varphi_0 \in V'$ we consider the retrograde problem

$$(PLR)\begin{cases} -\varphi_t - \Delta\varphi + a\varphi = 0 & in\ (0;T)\times G \\ \varphi = 0 & on\ (0,T)\times\partial_D G \\ \frac{\partial\varphi}{\partial n} = 0 & on\ (0,T)\times\partial_N G \\ \varphi(T,x) = \varphi_0(x) & on\ G. \end{cases}$$

As in Proposition 5.5 of Fabré - Puel - Zuazua [1992] it can be shown that there exists a positive number q (depending on the dimension N) such that the solution φ of (PLR) satisfies that

$$L(\varphi_0, a) := (T-t)^q \frac{\partial\varphi}{\partial n} \in L^1((0,T)\times\partial_D G). \tag{57}$$

We introduce the functional

$$J(\varphi_0 : a, y_d) := \frac{1}{2}\left(\int_0^T \int_{\partial_D G} |(T-t)^q \frac{\partial\varphi}{\partial n}(\sigma,t)| d\sigma dt\right)^2 + \epsilon\|\varphi_0\|_{V'} - < y_d, \varphi_0 >_{V\times V'}$$

It is clear that J is a strictly convex and continuous function on V'. Moreover using the unique continuation theorem (see Mizohata [1958] and Saut - Scheurer [1987]) J is a coercive functional

$$\liminf_{|\varphi_0|_{V'}\to\infty} \frac{J(\varphi_0 : a, y_d)}{|\varphi_0|_{V'}} \geq \epsilon \tag{58}$$

(see Proposition 2.1 of Fabré - Puel - Zuazua [1992]) and so J achieves its minimum at a unique point $\hat{\varphi}_0$ in V'. The associated Euler-Lagrange equation implies the existence of h satisfying

$$h \in sign(L(\hat{\varphi}_0 : a))\chi_{[0,T]\times\partial_D G}$$

and

$$0 = \int_0^T \int_{\partial_D G} (T-t)^q \frac{\partial\theta}{\partial n} h d\sigma dt + \epsilon[|\hat{\varphi}_0 + \theta_0|_{V'} - |\hat{\varphi}_0|_{V'}] - < y_d, \theta_0 >_{VV'} \tag{59}$$

for any $\theta_0 \in V'$ and where θ denotes the solution of (PLR) replacing φ_0 by θ_0. On the other hand, multiplying by θ the equation of (PL) (with u given by (54))

$$\int_G y(T,x)\theta_0(x)dx = -\int_0^T \int_{\partial_D G} u_\epsilon(\sigma,t)\frac{\partial\theta}{\partial n}(\sigma,t)d\sigma dt. \tag{60}$$

From (59) and (60) we get

$$< y_d - y(T,\cdot), \theta_0 >_{V'V} \leq \epsilon(|\hat{\varphi}_0 + \theta_0|_{V'} - |\hat{\varphi}_0|_{V'}) \leq \epsilon|\theta_0|_{V'}$$

and in consequence

$$\|y(T,\cdot) - y_d\|_{C(\overline{G})} \leq \sup_{\theta_0\in V'} \frac{< y_d - y(T,\cdot), \theta_0 >_{V'V}}{\|\theta_0\|_{V'}} \leq \epsilon.$$

In order to prove part (ii) we denote by $\hat{\varphi}_{0,+}$ and $\hat{\varphi}_{0,+}$ the positive and negative parts of $\hat{\varphi}_0$. Let $\hat{\varphi}$ the solution of (PLR) corresponding to the initial datum $\hat{\varphi}_0$ and let ψ_+ and let

ψ_- the solutions of (PLR) *assuming* $a = 0$ *in the equation* and corresponding to initial data $\hat{\varphi}_{0,+}$ $\hat{\varphi}_{0,-}$ respectively. Then, by the comparison principle we have $\psi_- \leq \hat{\varphi} \leq \psi_+$, $\psi_- \leq 0$ and $\psi_+ \geq 0$ in $(0,T) \times G$. Besides

$$\frac{\partial \psi_-}{\partial n} \geq \frac{\partial \hat{\varphi}}{\partial n} \geq \frac{\partial \psi_+}{\partial n} \quad \text{on } (0,T) \times \partial_D G. \tag{61}$$

Then, for any $\varphi_0 \in V'$ we have

$$J(\varphi_0 : a, y_d) \leq I(\varphi_0 : y_d) \tag{62}$$

where I is the functional (independent of a) given by

$$I(\varphi_0 : y_d) := \frac{1}{2}\left(\int_0^T \int_{\partial_D G} (T-t)^q \max\{|\frac{\partial \psi_-}{\partial n}(\sigma,t)|,\ |\frac{\partial \psi_+}{\partial n}(\sigma,t)|\} d\sigma dt\right)^2$$

$$+\varepsilon \parallel \varphi_0 \parallel_{V'} - < y_d, \varphi_0 >_{V \times V'} .$$

From (58) we deduce that

$$\liminf_{|\varphi_0|_{V'} \to \infty} \frac{I(\varphi_0 : y_d)}{|\varphi_0|_{V'}} \geq \varepsilon.$$

So, there exists $M > 0$ (independent of a) such that

$$I(\varphi_0 : y_d) \geq \frac{\varepsilon}{2} \parallel \varphi_0 \parallel_{V'} \text{ assumed } \parallel \varphi_0 \parallel_{V'} \geq M.$$

This implies that if $\hat{\varphi}_0$ is the minimum of J in V' then there exists $\hat{M} > 0$ independently of a such that

$$\parallel \hat{\varphi}_0 \parallel_{V'} \leq \hat{M}. \tag{63}$$

Using (61), (63) and (54) we get (56).

■

Proof of Theorem 10. From assumption (52) and the construction of $\underline{Y}_\infty$ and $\overline{Y}_\infty$ we deduce that there exists $N_0 \in I\!N$ such that

$$\underline{Y}_N(T,x) - 2\varepsilon \leq y_d(x) \leq \overline{Y}_N(T,x) + 2\varepsilon \quad \forall x \in \Omega - \overline{\omega}.$$

Let $N \geq N_0$ large enough and define

$$f_N(s) = \begin{cases} -\lambda N^p & \text{if} \quad s \leq -N \\ \lambda |s|^{p-1} s & \text{if} \quad -N \leq s \leq N \\ \lambda N^p & \text{if} \quad s \geq N. \end{cases}$$

Since f_N is a (globally) Lipschitz function and bounded, as in Theorem 1.2 of Fabré - Puel - Zuazua [1992], there exists $u_\varepsilon^N \in C([0,T] \times \partial\omega)$ such that if $y^*(t, x : u_\varepsilon^N)$ denotes the solution of

$$\begin{array}{ll} y_t^* - \Delta y^* + f_N(y^*) = 0 & \text{in } (0,T) \times (\Omega - \overline{\omega}) \\ y^* = u_\varepsilon^N & \text{on } (0,T) \times \partial\omega \\ \frac{\partial y^*}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega \\ y^*(0,x) = y_0(x) & \text{on } \Omega - \overline{\omega} \end{array}$$

the we have

$$\| y^*(T, \cdot : u_\varepsilon^N) - y^d(\cdot) \|_{\mathcal{C}(\overline{\Omega}\setminus\omega)} \leq \varepsilon.$$

Moreover such a control u_ε^N can be found as a fixed point of the application $\Lambda : \mathcal{C}([0,T] \times (\overline{\Omega} - \omega)) \to \mathcal{P}(\mathcal{C}([0,T] \times (\overline{\Omega} - \omega))$ defined by

$$\Lambda(z) = \{y(\cdot, \cdot : u) \text{ solution of (PL) with } a = \frac{f_N(z)}{z} \text{ and } u \text{ satisfying (53), (54)}\}.$$

From estimate (56) of Lemma 1 we deduce that if u_ε^N is a fixed point of Λ it must satisfy

$$\| u_\varepsilon^N \|_{\mathcal{C}([0,T]\times\partial_D G)} \leq C$$

with C (independent of N) given in (56). Then, by the maximum priciple we conclude that if $N \geq N_0$ is large enough then the function $y^*(t, x : u_\varepsilon^N)$ satisfies

$$|y^*(t, x : u_\varepsilon^N)| \leq N \quad \forall\, (t,x) \in [0,T] \times (\overline{\Omega} - \omega)$$

and so, in fact, $y^*(0, \cdot : u_\varepsilon^N)$ satisfies the requirements of the statement of Theorem 10.

■

In order to complete the proof of Theorem 7 we need to use some other auxiliary results.

Lemma 5 *(Díaz and Fursikov [1994])*

Let $u_\varepsilon \in \mathcal{C}([0,T] \times \partial\omega)$ fixed. There exists $\hat{v}_\varepsilon \in \mathcal{C}([0,T] \times \overline{\omega})$ such that the solution $\hat{y}(t : \hat{v}_\varepsilon)$ of

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + \lambda|\hat{y}|^{p-1}\hat{y} = \hat{v}_\varepsilon & \text{in } (0,T) \times \omega \\ \hat{y} = u_\varepsilon & \text{on } (0,T) \times \partial\omega \\ \hat{y}(0,\cdot) = y_0(\cdot) & \text{on } \omega \end{cases}$$

satisfies

$$\| \hat{y}(T : \hat{v}_\varepsilon) - y_d \|_{\mathcal{C}(\overline{\omega})} \leq \varepsilon.$$

We would need to regularize the matching between the functions $\tilde{y}$ and $\hat{y}$ given in Theorem 10 and Lemma 4 respectively.

Lemma 6

Let ω_ε be an open regular subset of ω such that $d(\omega_\varepsilon, \partial\omega) \leq \varepsilon$. Then there exists $y^* \in \mathcal{C}([0,T] \times \overline{\Omega}) \cap C^2((0,T) \times \Omega)$ such that $y^* = \hat{y}$ on $[0,T] \times \overline{\omega}_\varepsilon$ and $y^* = \tilde{y}$ on $[0,T] \times (\overline{\Omega} - \omega)$.

The proof of this result uses standard regularization techniques and the details are left to the reader. The last technical result is consequence of the continuous dependence of the solutions of problem (P_p) with respect to different initial data.

Lemma 7

Let y^* be the function given in Lemma 3. Define

$$v_\varepsilon := y_t^* - \Delta y^* + \lambda |y^*|^{p-1} y^* \quad in\ (0,T) \times \Omega.$$

Then $v_\varepsilon = 0$ on $(0,T) \times (\Omega - \overline{\omega})$ and if $y(t, \cdot : v_\varepsilon)$ is the corresponding solution of (P_p) we have

$$\| y^*(t,\cdot) - y(t,\cdot : v_\varepsilon) \|_{C(\overline{\Omega})} \leq \varepsilon \qquad \forall\, t \in [0,T].$$

Proof of Theorem 7. Let v_ε be the function defined in Lemma 6. Then using Theorem 10 and lemmas 4, 5 and 6 we have that

$$\begin{aligned} \| y(T : v_\varepsilon) - y_d \|_{C(\overline{\Omega})} &\leq \| y^*(T,\cdot) - y(T,\cdot : v_\varepsilon) \|_{C(\overline{\Omega})} + \| y^*(T,\cdot) - y_d \|_{C(\overline{\Omega})} \\ &\leq \| y^*(T,\cdot) - y(T,\cdot : v_\varepsilon) \|_{C(\overline{\Omega})} + \| \tilde{y}(T,\cdot) - y_d \|_{C(\overline{\Omega} - \omega)} \\ &\quad + \| y^*(T,\cdot) - y_d \|_{C(\overline{\omega})} + \varepsilon \leq 4\varepsilon \end{aligned}$$

and the conclusion holds. ∎

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