On a degenerate evolution equation modelling large ice sheets dynamics

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Abstract

We present the mathematical analysis of a cold ice sheet flow model. The model combines the assumptions of slow, gravity driven non-newtonian viscous flow as appropriate to the solid state creep of ice. In order to prove the well-posedness of the model we introduce a weak formulation of multivalued type. The existence and location of the *free boundary* generated by the support of the solution are also considered and a waiting time property for the response of the ice sheet is proved.

1 Introduction

Modelling ice-sheet flow dynamics has been a challenging problem since the beginning of the century. Nevertheless the application of the shallow ice approximation is quite recent and respond to the empirical observation that typical ice sheets (Antarctica and Greenland, for example) have thicknesses much less than their horizontal extent and respond to this type of problems. The Antarctic and Greenland ice sheets are the two mayor present day examples of ice sheets. During the last ice age (terminating about 10.000 years ago) ice sheets existed in North America (the Laurentide) and northern Europe (the Fennoscandian), the ice extending into Southern England and

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Northern Europe. These ice sheets interact with climate, and may be responsable for sudden shifts in climate in the recent geological past. The ice sheet model is the simplest of its type, and assume a nonlinear, temperature independent flow law. The governing equations of conservation of mass and momentum, together with the kinematic wave equation for the top (free) surface have been scaled, using appropriate depth, length, stress and velocity scales. The resulting model can be formulated as a free boundary problem associated to a nonlinear diffusion equation for the thickness h of the form

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}(h^{n+2}|\frac{\partial h}{\partial x}|^{n-1}\frac{\partial h}{\partial x}) = a \quad \text{over } I(t) \tag{1.1}$$

where $I(t)$ denotes the (unknown) region where $h(t,.) > 0$ and $a(x,t)$ is the (dimensionless) accumulation rate. In particular $a \leq 0$ represents an ablation. Equation (1.1) represents a two-dimensional ice sheet. The three dimensional model is obtained by replacing $\frac{\partial}{\partial x}$ by ∇ or $\nabla\cdot$ as appropriate. The usual situation for (1.1) is that a is positive over an interval, and negative outside this. An exaustive asymptotic analysis in his natural language of matched asymptotic expansions was developed in [F]. Here we are interested in the mathematical treatment of such a problem. In order to prove the well-posedness of the model we start by introducing some weaker formulations which, in fact, are equivalent to the formulation (1.1) under suitable regularity on h. This is presented in Section 2 where we collect some results in the literature. In particular we obtain the well-posedness in a suitable framework for the weak formulation by applying the recent results by Diaz and Padial (1993),(1995). The rest of the paper is devoted to the study of the free boundary. We obtain different estimates on its location and we analyze its behaviour for initial times (t near zero). More details on these type of results can be found in [S] where the behaviour of the free boundary for large times is also considered together with additional information about its location.

2 Weak formulations

The original *strong* formulation can be stated in the following terms: let $T > 0$ and D be an open bounded interval of $\mathbb{R}$. Given an accumulation rate function $a(t,x)$ defined on $(0,T) \times D$ and an initial thickness $h_0(x) \geq 0$

on D, find two curves $S_+, S_- \in C^0([0,T])$, with $S_-(t) \leq S_+(t)$, $\Omega(t) := (S_-(t), S_+(t)) \subset D$ for any $t \in [0,T]$, and a sufficently smooth function $h(t,x)$ defined on the set $Q_T := \bigcup_{t\in(0,T)} \Omega(t)$ such that

$$\begin{cases} \frac{\partial h}{\partial t} &= \frac{\partial}{\partial x}\left(h^{n+2}\left|\frac{\partial h}{\partial x}\right|^{n-1}\frac{\partial h}{\partial x}\right) + a(t,x) \quad \text{in } Q_T \\ h &= 0 \quad \text{on } \{S_-(t)\} \cup \{S_+(t)\} \qquad t \in (0,T) \\ h(0,x) &= H_0(x) \qquad \text{on D} \end{cases} \tag{2.1}$$

and $h(t,x) > 0$ on Q_T.

Here the exponent n represents the Glen exponent and it is usually assumed $n = 3$. It is well-known ([D]) that this class of degenerate equations are typical of slow phenomena and satisfy the finite speed of propagation property: assuming, for instance, $a \equiv 0$ if $h(0,x)$ has compact support then $h(t,x)$ has also a compact support in $I\!R$, for any $t \in [0,T]$. So, if $a \equiv 0$, the domain Q_T can be found through the support of the solution $h(t,x)$ of the nonlinear parabolic equation over the whole space $(0,T) \times I\!R$ and satisfying the initial condition $h(0,x) = h_0(x)$, $x \in I\!R$. Unfortunately, the physically relevant case, $a \not\equiv 0$, is much more complicated. Indeed, the finite speed of propagation still holds if $a(t,.)$ has compact support in $I\!R$ (for fixed $t \in (0,T)$). Moreover, in that case, it can be shown that support $h(t,.) \subset$ support $a(t,.)$ and so $a(t,.)$ vanishes on the free boundary. Nevertheless, in glaceological models it is well known that usually $a(t,.) < 0$ near the free boundary (i.e. near the boundary of the ice-sheet) and so there must exist another reason (other than the degenerate character of the equation) justifying the occurrence of the free boundaries $S_-(t), S_+(t)$.

The new model we present here is based upon the fact that we can extend the function $h(t,x)$ outside of Q_T by zero on $(0,T) \times D \setminus Q_T$ and that this extension still satisfies a nonlinear equation (this time of multivalued type) having the great advantage of being defined on an *a priori* known domain $(0,T) \times D$. This type of problem is known in the literature as an *obstacle problem* (in our case the obstacle function is $\psi \equiv 0$) and it arises in many contexts related to friction, elasticity, thermodynamics and so on. The associated **Complementary Formulation** is the following: given D, a and h_0

as before, find a sufficently smooth function h such that:

$$\begin{cases} h_t & - & \frac{\partial}{\partial x}(h^{n+2}|\frac{\partial h}{\partial x}|^{n-1}\frac{\partial h}{\partial x}) - a(t,x) \geq 0 & \text{in } (0,T)\times D \\ (h_t & - & \frac{\partial}{\partial x}(h^{n+2}|\frac{\partial h}{\partial x}|^{n-1}\frac{\partial h}{\partial x}) - a(t,x))h = 0 & \text{in } (0,T)\times D \\ h & \geq & 0 & \text{in } (0,T)\times D \\ h & = & 0 & \text{on } \ (0,T)\times \partial D \\ h(0,x) & = & h_0(x) & \text{on } \ D \end{cases} \tag{2.2}$$

It is obvious that if a regular function H verifies the strong formulation then its extension by zero over $[0,T]\times D \setminus Q_T$ (which we will denote again by H) satisfies trivially the complementary formulation, assuming that $a(t,x)$ satisfies the condition

$$a(t,x) \leq 0 \quad \text{on } (0,T)\times D\setminus Q_T. \tag{2.3}$$

2.1 A Comparison Principle

Defining $\phi(r) = |r|^{n-1}r$, $r \in I\!R$, $n > 0$ and $\psi(s) = \frac{1}{m}s^{\frac{1}{m}}$ with $s > 0$ and $m = 2(n+1)/n$ the above obstacle problem can be also formulated as: given $D \subset I\!R$, $a \in L^\infty$ and $h_0 \in L^\infty$ with compact support, find a sufficently smooth function h solution of

$$\begin{cases} h_t & - & \phi(\psi(h)_x)_x - a(t,x) \geq 0 & \text{in } (0,T)\times D \\ h & \geq & 0 & \text{in } (0,T)\times D \\ (h_t & - & \phi(\psi(h)_x)_x - a(t,x))h = 0 & \text{in } (0,T)\times D \\ h & = & 0 & \text{on } (0,T)\times \partial D \\ h(0,x) & = & h_0(x) & \text{on } D \end{cases} \tag{2.4}$$

Putting $u := h^m = \psi(h)$, and $u^{1/m} = h = \psi^{-1}(u) := b(u)$ we have $\phi(\nabla(\psi(h))) = \phi(\nabla u) = |\nabla u|^{p-2}\nabla u$. The mentioned multivalued formulation is the following: determine a function $u(t,x)$ solution of

$$\begin{aligned} b(u)_t - \operatorname{div}\phi(\nabla u) + \beta(u) \ &\ni a(t,x) && \text{in} \quad Q = D\times(0,\infty) \\ u(t,x) \ &= 0 && \text{on} \quad \Sigma = \partial D \times (0,+\infty) \\ u(0,x) \ &= u_0(x) && \text{on} \quad D \end{aligned} \tag{2.5}$$

where β is the maximal monotone graph

$$\beta(r) = O \quad \text{if} \quad r < 0, \quad \beta(0) = (-\infty, 0], \quad \beta(r) = 0 \quad \text{if} \quad r > 0.$$

A more general framework is obtained by assuming that $D \subset I\!R$ is a regular domain, ϕ a real continuous strictly increasing convex function such that $\phi(0) = 0$, and β as before. Given $h_0 \in L^\infty(Q)$ and $a \in L^\infty(Q)$ the results of Díaz-Padial ([DP]) lead to the existence of a (unique) solution using the class of bounded variation functions

$$\mathrm{BV}_t(Q) = \left\{ u \in L^1(Q) : \frac{\partial u}{\partial t} \in \mathcal{M}_b(Q) \right\}$$

where $\mathcal{M}_b(Q)$ is the space of bounded Radon measure over Q. We notice that if $n \simeq 3$ then $p \simeq 4$ and $m \simeq \frac{8}{3} > 1$ and the principal difficulty in our case is due to the regularity of the function (or distribution) $b(u)_t$. Existence of solutions for $b(u)_t \in L^{p'}\left(0, T; W^{-1,p'}(\Omega)\right)$ only need $u_0 \in L^\infty(\Omega)$. Comparison (and uniqueness) for weak bounded solutions of (2.5) has been obtained first by [DT] assuming $b(u)_t \in L^1(0, T; L^1(\Omega))$ and improved by [DP] under the more general hipotesis $b(u)_t$ bounded Radon measure: i.e.; $b(u) \in M_b(Q)$. For our purposes the following Comparison Principle (details may be found in [P]) is enough:

Theorem 2.1 *Let $b, (a_1, u_{0_1}), (a_2, u_{0_2})$ verifying the above structural hipotesis and let β a maximal monotone graph in $I\!R^2$. If u_1 y u_2 are two solutions of (2.5) associated to the data (a_1, u_{0_1}) and (a_2, u_{0_2}), with α_1 and α_2 belonging to $L^1(Q)$ ($\alpha_1(t,x) \in \beta(u_1)(t,x)$ and $\alpha_2(t,x) \in \beta(u_2)(t,x)$ a.e.$(x,t) \in Q$ respectively) then, $\forall t \in]0, T[$ we have*

$$\int_\Omega \Big[b(u_1(t,x)) - b(u_2(t,x))\Big]_+ dx \leq \left\{ \int_\Omega \Big[b(u_{0_1}(x)) - b(u_{0_2}(x))\Big]_+ dx + \int_0^t \int_\Omega \Big[a_1(s,x) - a_2(s,x)\Big]_+ dx ds \right\}.$$

3 On the free boundary

3.1 Existence and location

In this section we will prove the existence of a null set

$$N(h(t,.)) := \{x \in D/\, h(t,x) = 0\}$$

for the unique solution $h(t,x)$ of problem

$$\begin{aligned} h_t - \Delta_p \psi(h) + \beta(h) \ni a(t,x) \quad &\text{in} \quad Q = D \times (0,\infty) \\ h(t,x) = 0 \quad &\text{on} \quad \Sigma = \partial D \times (0,+\infty) \\ h(0,x) = h_0(x) \quad &\text{on} \quad D \end{aligned} \tag{3.6}$$

We are able now to analyse a great number of geophysical fenomenous related to location and evolution of the free boundary and associated with the behaviour of the function $a(t,x)$. We will use a technique based on the comparison result of section (2) consisting in the construction of appropriated local super-sub solutions having compact support. We define, $\forall \epsilon > 0$ the set

$$N_\epsilon(a(t,.)) := \{(t,x) \in D \times \{t\}/\, a(t,x) \leq -\epsilon\} \tag{3.7}$$

and also $S_\epsilon(a(t,.) := Q_t \setminus N_\epsilon(a(t,.)$. We have

Theorem 3.1 *Let $h \in C(\bar{Q})$, $h \geq 0$ a solution of (3.6). Let $\epsilon > 0$ such that the set $N_\epsilon(a(t,.))$ is not empty. Then there exists $T_0 \geq 0$ such that $\forall t \geq T_0$ we have*

$$N(u(t,.)) \supset \{(t_0,x_0) \in N_\epsilon(a(t_0,.))/\, d(x_0, Q_t \setminus N_\epsilon(a(t_0,.))) \geq R\}$$

Proof: We consider the set $N_\epsilon(a(t,.)$ and define the function

$$\tilde{h}(t,x) = \psi^{-1}\left(\eta(|x-x_0|) + \psi(U(t))\right)$$

where

$$\eta(r) = c\, r^{\frac{p}{p-1}}, \qquad c = \frac{p-1}{p}\left(\frac{\epsilon}{2n}\right)^{\frac{1}{p-1}} \tag{3.8}$$

and $U(t)$ the solution of the initial value problem

$$U_t + \frac{1}{2}\beta(U) \ni -\frac{\epsilon}{2} \tag{3.9}$$

$$U(0) = ||u_0||_{L^\infty}. \qquad (3.10)$$

It is easy to see that $U(t) = [-\frac{\epsilon}{2}t + ||h_0||_{L^\infty}]^+$ whence

$$U(t) \equiv 0 \qquad \forall t \geq T_0 = \frac{2}{\epsilon}||h_0||_{L^\infty(\Omega)}$$

On the other hand $\Delta_p(\eta) = +\frac{\epsilon}{2}$ whence (in $N_\epsilon(a(t,.))$)

$$\begin{aligned}
&\tilde{h}_t - \Delta_p\psi(\tilde{h}) + \beta(\tilde{h}) = \\
&= \frac{d}{dt}[\psi^{-1}(\eta(x) + \psi(U(t)))] - \Delta_p\eta + \beta(\psi^{-1}(\eta(x) + \psi(U(t)))) \geq \\
&\geq \frac{\psi'(U)}{\psi'(\psi^{-1}(\eta(x) + \psi(U(t)))} \frac{dU}{dt} - \Delta_p\eta + \frac{1}{2}\beta(\psi^{-1}(\eta)) + \frac{1}{2}\beta(U) \geq \\
&\geq U_t + \frac{1}{2}\beta(U) - \Delta_p\eta + \frac{1}{2}\beta(\psi^{-1}(\eta)) \geq -\epsilon \geq a(t,x)
\end{aligned}$$

Using (Proposition 2.4 [B]) the following estimate holds

$$||h||_{L^\infty(Q)} \leq ||h_0||_{L^\infty(D)} + \int_0^t ||a||_{L^\infty(D)} = M(t). \qquad (3.11)$$

Then

$$\tilde{h} \geq M(t) \geq ||h||_{L^\infty(Q)} \qquad (3.12)$$

iff $\psi^{-1}(\eta(x) + \psi(U(t))) \geq M(t)$ i.e.; $\eta + \psi(U) \geq \psi(M(t))$. In particular this is true if $c\,|x - x_0|^{\frac{p}{p-1}} \geq \psi(M(t))$; by (3.8) the above reads

$$|x - x_0| \geq \frac{\psi(M(T))^{\frac{p-1}{p}}}{(\frac{p-1}{p})^{\frac{p-1}{p}} (\frac{\epsilon}{2n})^{\frac{1}{p}}} = R \qquad (3.13)$$

and (3.13) implies $\tilde{h} \geq h$ on $\partial N_\epsilon(a(t,.))$ At $t = 0$ we use the monotonicity of ψ^{-1}:

$$\tilde{h}(0,x) = \psi^{-1}(\eta(x) + \psi(U(0))) = \psi^{-1}(\eta(x) + \psi(||h_0||_{L^\infty})) \geq$$

$$\geq \psi^{-1}(\psi(||h_0||_{L^\infty})) = ||h_0||_{L^\infty} \geq h_0(x) \geq 0$$

Summarizing we have, that if $x \in N_\epsilon(a(t,.))$ such that $|x - x_0| \geq R$ then

$$h_t - \Delta_p\psi(h) + \beta(h) \leq a \leq \tilde{h}_t - \Delta_p\psi(\tilde{h}) + \beta(\tilde{h}) \text{ in } N_\epsilon(a(t,.)) \qquad (3.14)$$

$$h(t,x) \leq \tilde{h}(t,x) \text{ on } \partial N_\epsilon(a(t,.)) \quad (3.15)$$

$$h_0(x) \leq \tilde{h}(0,x) \text{ on} N_\epsilon(a(0,.)) \quad (3.16)$$

It follows from comparison results for problem (3.6) that

$$0 \leq h(t,x) \leq \tilde{h}(t,x) \quad (3.17)$$

and we end up observing that $h(t,x_0) = 0 \ \forall t \geq T_0 = \frac{2}{\epsilon}||h_0||_{L^\infty}$ and $x_0 \in \{N_\epsilon(a)/|x-x_0| \geq R\}$.

3.2 The waiting time property

The following property applies if the initial data is sufficently 'flat' in the 'ablation region'.

Theorem 3.2 *let $h \in C(\bar{Q})$, $u \geq 0$ a solution of problem (3.6) Define $\delta = \eta^{-1}(\psi(M))$ and $B_\delta^+(x_0) = \{x \in D/\ x \in [x_0, x_0+\delta)\}$ being $M = ||h||_{L^\infty}$, $x_0 = S_+(0)$, $\tilde{c} = (\frac{p-1}{p})(\frac{\epsilon}{n})^{\frac{1}{p-1}}$ and $\eta(|x-x_0|) = \tilde{c}\,|x-x_0|^{\frac{p}{p-1}}$. Assume that there exists $T^* > 0$ such that $a(t,x) \leq -\epsilon$ a.e.$x \in B_\delta^+(x_0)$ and $t \in (0,T^*)$. If $x_0 \in D$ satisfy $0 \leq h_0(x_0) \leq \psi^{-1}(\eta(|x-x_0|))$ then*

$$\exists t^*,\ 0 < t^* \leq T^* \text{ such that } S_0 = S_t\ \forall t \in (0,T^*)$$

Proof. We define the function on the set $B_\delta^+(x_0) \times [0,T^*]$

$$\tilde{h} = \psi^{-1}(\eta(|x-x_0|)).$$

Then

$$\tilde{h}_t - \Delta_p\psi(\tilde{h}) + \beta(\tilde{h}) \geq -\epsilon \geq a \geq h_t - \Delta_p\psi(h) + \beta(h)$$

On $\partial B_\delta^+(x_0) \times [0,t^*]$ we have to verify that

$$h \leq M \leq \tilde{h} = \psi^{-1}(\eta) \quad (3.18)$$

and this is true if and only if

$$\psi(M) \leq \eta = \tilde{c}|x-x_0|^{\frac{p}{p-1}}$$

On ∂B_δ^+ this reads

$$\psi(M) \leq \tilde{c}\delta^{\frac{p}{p-1}}.$$

Using that $\delta = \eta^{-1}(\psi(M))$ then

$$\psi(M) \leq \tilde{c}[\eta^{-1}(\psi(M))]^{\frac{p}{p-1}}$$

$$\iff [\frac{\psi(M)}{\tilde{c}}]^{\frac{p-1}{p}} \leq \eta^{-1}(\psi(M)) \iff \eta\left([\frac{\psi(M)}{\tilde{c}}]^{\frac{p-1}{p}}\right) \leq \psi(M).$$

In conclusion we have

$$\left\{\begin{array}{rcll} h_t - \Delta_p\psi(h) + \beta(h) & \leq & a \leq \tilde{h}_t - \Delta_p\psi(\tilde{h}) + \beta(\tilde{h}) & \text{in } B_\delta^+(x_0) \times (0, t^*) \\ h(x_0, 0) = h_0(x_0) & \leq & \tilde{h}(x) = \psi^{-1}(\eta(|x - x_0|)) & \text{on } B_\delta^+(x_0) \\ h(t, x) & \leq & M \leq \tilde{h}(x) & \text{on } \partial B_\delta^+(x_0) \times (0, t^*) \end{array}\right. \tag{3.19}$$

Then a comparison argument gives that $0 \leq h(t, x) \leq \tilde{h}(x)$ and so $h(t, x_0) \equiv 0 \quad \forall t \in (0, t^*)$.

3.3 On the inital growing of the free boundaries

We are interested in the evolution of function $S(t)$ for $0 < t \ll 1$.

Theorem 3.3 *Let $h \in C(\bar{Q})$, $h \geq 0$ a solution of problem (3.6). Assume that $\exists T > 0$ such that $S(t) \setminus S(h_0) \subset N_\epsilon(a(t, .)) \quad \forall t \in (0, T)$. If, in addition, $h_t \in L^\infty(Q)$ then we have the estimate*

$$S(h(t, .)) \subset S(h_0) + B(\psi(\eta^{-1}(Ct_0))).$$

for any $t \in (0, t_0)$ y some $C > 0$ depending on $||h_t||_{L^\infty}$

Proof: Let $t_0 \in (0, T)$ and $x_0 \in S(h(t_0, .)) \setminus S(h_0)$. We consider the (open) region

$$R(t_0) = \{(t, x)/\, 0 < t < t_0, h(t, x) > 0, x \notin s(h_0)$$

and the function

$$\bar{h}\,(x) = \psi^{-1}(\eta(|x - x_0|)) = \tilde{c}r^{\frac{p}{(p-1)m}}$$

where $\tilde{c}$ is given by (3.8). We have (on $R(t_0)$):

$$h_t - \Delta_p\psi(h(x)) = a \tag{3.20}$$

$$\bar{h}_t - \Delta_p \psi(\bar{h}(x)) = -\epsilon \geq a \tag{3.21}$$

One can see that the difference $w(x,t) = h(x,t) - \bar{h}(x)$ satisfies a linear parabolic equation of the form

$$w_t = A(x,t)w_{xx} + B(x,t)w_x + C(x,t)w \equiv \mathcal{L}\sqsupseteq$$

where the coefficients A, B, C can be derived from (3.20), (3.21) by a linearization procedure with respect to w. $\mathcal{L}$ is an elliptic operator ([GT]) and by the strong maximun principle one derives that w takes its maximum on the parabolic boundary of $R(t_0)$: nevertheless on $\partial_p R(t_0) \setminus \partial S \times (0, t_0)$ we have:

$$0 = h(t,x) \leq \bar{h}(x)$$

i.e.; $w \leq 0$ while on (t_0, x_0) $(\in S(t_0))$

$$h(t_0, x_0) - \bar{h}(x_0) = h(t_0, x_0) > 0$$

i.e.; $w(x_0, t_0) > 0$ hence there exists a point $(\bar{x}, \bar{t})$ in $\partial S \times (0, t_0)$ where $w > 0$. This means

$$\bar{h}(\bar{x}) < h(\bar{t}, \bar{x})$$

$$\tilde{c}|\bar{x} - x_0|^{\frac{p}{(p-1)m}} < h(\bar{x}, \bar{t})$$

$$|\bar{x} - x_0| < \frac{1}{\tilde{c}} h(\bar{t}, \bar{x})^{\frac{m(p-1)}{p}} = \psi(\eta^{-1}(h(\bar{t}, \bar{x})))$$

and

$$d(x_0, S) \leq d(x_0, \partial S) = |\bar{x} - x_0| \leq \psi(\eta^{-1}(h(\bar{t}, \bar{x}))) \leq$$

$$\leq \psi(\eta^{-1}(h(\bar{t}, \bar{x}) - h(\bar{x}, 0))) \leq$$

$$\leq \psi(\eta^{-1}(C\,\bar{t})) \leq \psi(\eta^{-1}(Ct_0)) = [Ct_0]^{\frac{(p-1)m}{p}}$$

As x_0 is an arbitrary point of $S(t_0)$ we have the result:

$$S(t) \subset S + B(Ct_0^{\frac{(p-1)m}{p}}). \tag{3.22}$$

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