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On the Approximate Controllability for Higher Order Parabolic Nonlinear Equations of Cahn-Hilliard Type

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ABSTRACT. We prove the approximate controllability property for some higher order parabolic nonlinear equations of Cahn-Hilliard type when the nonlinearity is of sublinear type at infinity. We also give a counterexample showing that this property may fail when the nonlinearity is of superlinear type.

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ of class C^{2m}, $T > 0$, ω a nonempty open subset of Ω, f a continuous real function and $k \in \mathbb{N}$ such that $0 \leq 2k \leq m$. The main goal of this work is the study of the approximate controllability of the following semilinear equation with Dirichlet boundary conditions:

$$(1.1) \qquad \begin{cases} y_t + (-\Delta)^m y + (-\Delta)^k f(y) = h + v\chi_\omega & \text{in } Q := \Omega \times (0,T), \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad , \quad j = 0, 1, \ldots, m-1 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

where v is a suitable output control, χ_ω is the characteristic function of ω, ν is the unit outward normal vector, $h \in L^2(0,T : H^{-m}(\Omega))$ and $y_0 \in L^2(\Omega)$. Due to the term χ_ω the controls are assumed supported on the set $\mathcal{O} := \omega \times (0,T)$. Problems as (1.1), sometimes known as Cahn-Hilliard problems, appear, with $m = 2$, in the study of phase separation in cooling binary solutions and in other contexts generating spatial pattern formation (see [6], [8] and the references cited therein).

We recall that problem (1.1) satisfies the approximate controllability property, at time T with states space X and controls space Y, if the set

$$\{ \, y(T, \cdot : v) : \; v \in Y, \; y \text{ solution of (1.1)} \}$$

is dense in X.

The main goal of this paper is to extend the approximate controllability results on second order problems, $m = 1$ and $k = 0$ (see e.g. [9], [10] and [7]) to the case of higher order equations for which the maximum principle does not hold, in general.

Our first result gives a positive answer when f is assumed to be sublinear at the infinity:

Theorem 1.1. *Assume that f satisfies the following conditions: there exist some positive constants c_1 and c_2 such that*

$$|f(s)| \leq c_1 + c_2|s| \quad \text{for all } s \in \mathbb{R} \tag{1.2}$$

and

$$\text{there exists } f'(s_0) \text{ for some } s_0 \in \mathbb{R}. \tag{1.3}$$

Then problem (1.1) satisfies the approximate controllability property at time T with states space $X = L^2(\Omega)$ and controls space $Y = L^2(\mathcal{O})$.

In contrast to the above result, we shall prove that when f is superlinear the approximate controllability property does not hold in general, as explained in Section 4. Therefore if, for instance, $f(s) = |s|^{p-1}s$ Theorem 1.1 gives a positive approximate controllability result for $0 < p \leq 1$. The results of section 6 provide a negative approximate controllability answer when $1 < p < \infty$. The similar alternative was obtained in Díaz-Ramos [7] for second order parabolic semilinear problems.

We remark that the existence of solutions in the class

$$y \in L^2(0,T; H_0^m(\Omega)) \cap C([0,T]; L^2(\Omega)), \; f(y) \in L^2(Q), \; \Delta^k f(y) \in L^2(0,T; H^{-m}(\Omega)),$$

is also obtained as a by-product of Theorem 1.1 for a suitable subclass of controls. The uniqueness of solutions can be easily proved if, for instance, f is nondecreasing or Lipschitz continuous. Those uniqueness results are not needed in our arguments.

2. Approximate controllability for an associated linear problem

In order to prove Theorem 1.1 we follows the same scheme of proof than in [9], [10] and [7]. We define the function

$$g(s) = \frac{f(s) - f(s_0)}{s - s_0}.$$

From assumptions (1.2) and (1.3) we have that $g \in L^\infty(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$. The conclusion will be derived from a fixed point argument. As $f(s) = f(s_0) + g(s)s - g(s)s_0$, we shall start by considering the approximate controllability for a linear problem obtained by replacing the term $f(y)$ by

$$g(z)y + f(s_0) - g(z)s_0,$$

where z is an arbitrary function in $L^2(Q)$. Notice that when $z = y$ this expression coincides with $f(y)$ and that if we denote $g(z(t,x)) := a(t,x)$ and

$$h(a) := -(-\Delta)^k f(s_0) + (-\Delta)^k (a(t,x)s_0), \tag{2.1}$$

then $a \in L^\infty(Q)$ and $h(a) \in L^\infty(0,T; H^{-2k}(\Omega))$. More in general, given $a \in L^\infty(Q)$ and $h(a)$ defined by (2.1), we consider the approximate controllability property corresponding to the linear problem

$$\begin{cases} y_t + (-\Delta)^m y + (-\Delta)^k (a(t,x)y) = h + h(a) + u\chi_\omega & \text{in } Q := \Omega \times (0,T), \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad , \quad j = 0,1,\ldots,m-1 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

Before stating an approximate controllability result for this problem, following Lions [14] and Fabre-Puel-Zuazua [9], [10], we consider $\varepsilon > 0$ and $y_d \in L^2(\Omega)$ and we introduce the functional $J = J(\cdot; a, y_d) : L^2(\Omega) \to \mathbb{R}$ defined by

$$J(\varphi_0; a, y_d) = J(\varphi^0) = \frac{1}{2}\left(\int_{\mathcal{O}} |\varphi(t,x)| dx dt\right)^2 + \varepsilon \parallel \varphi^0 \parallel_{L^2(\Omega)} - \int_\Omega y_d \varphi^0 ; dx \tag{2.3}$$

where $\varphi(t,x)$ is the solution of the backward problem

$$\begin{cases} -\varphi_t + (-\Delta)^m \varphi + a(t,x)\Delta^k \varphi = 0 & \text{in } Q := \Omega \times (0,T), \\ \dfrac{\partial^j \varphi}{\partial \nu^j} = 0 \quad , \quad j = 0,1,\ldots,m-1 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ \varphi(T) = \varphi^0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

To study the above backward problem we introduce the space

$$W := \{y \in L^2(0,T; H_0^m(\Omega)), \; y_t \in L^2(0,T; H^{-m}(\Omega))\}.$$

The following result will be used later

Proposition 2.1. *Given $h \in L^2(0,T; H^{-m}(\Omega))$ and $y_0 \in L^2(\Omega)$, there exists a unique function $y \in W$ satisfying*

$$\begin{cases} y_t + (-\Delta)^m y + a(t,x)\Delta^k y = h & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad , \quad j = 0,1,\ldots,m-1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{2.5}$$

Furthermore, we have the estimate

$$\|y\|_{L^2(0,T;H_0^m(\Omega))} + \|y_t\|_{L^2(0,T;H^{-m}(\Omega))} \leq C\left(\|h\|_{L^2(0,T;H^{-m}(\Omega))} + \|y_0\|_{L^2(\Omega)}\right), \tag{2.6}$$

where the constant C depends only on $M := \parallel a \parallel_{L^\infty(Q)}$ (provided that Ω, T and m are kept fixed). Moreover, if $h \in L^2(Q)$, the solution y also satisfies that

$$y \in L^2(\delta, T; H^{2m}(\Omega)) \quad \text{and} \quad y_t \in L^2((\delta,T) \times \Omega) \quad \text{for all } \delta \in (0,T). \tag{2.7}$$

Proof. For all $n \in \mathbb{N}$ we define y^{n+1} as the solution of the following iterative problem

$$\begin{cases} y_t^{n+1} + (-\Delta)^m y^{n+1} = h - a(t,x)\Delta^k y^n & \text{in } Q, \\ \dfrac{\partial^j y^{n+1}}{\partial \nu^j} = 0 \quad , \quad j = 0,1,\ldots,m-1 & \text{on } \Sigma, \\ y^{n+1}(0) = y_0 & \text{in } \Omega, \end{cases}$$

where $y^0(t) := 0$ for all $t \in [0,T]$. The existence of a solution $y^n \in W$ can be found, for instance, in Theorem 3.4.1 of Lions-Magenes [15]. Thus, for all $n \in \mathbb{N}\backslash\{0,1\}$, $y^{n+1} - y^n$ satisfies

$$(2.8)\quad \begin{cases} (y^{n+1}-y^n)_t + (-\Delta)^m(y^{n+1}-y^n) = -a(t,x)\Delta^k(y^n - y^{n-1}) & \text{in } Q,\\ \dfrac{\partial^j(y^{n+1}-y^n)}{\partial\nu^j} = 0 \quad , \quad j = 0,1,\ldots,m-1 & \text{on } \Sigma,\\ (y^{n+1}-y^n)(0) = 0 & \text{in } \Omega \end{cases}$$

and therefore

$$y^{n+1} - y^n \in H^{1,2m}(Q) := H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^{2m}(\Omega))$$

and

$$\| y^{n+1} - y^n \|_{H^{1,2m}(Q)} \leq c_1 \| a\Delta^k(y^n - y^{n-1}) \|_{L^2(Q)}$$

(see, for instance, Theorem 4.6.1 of Lions-Magenes [16]). Then, since

$$H^{1,2m}(Q) \subset \mathcal{C}([0,T];H^m(\Omega))$$

with continuous embedding (see, for instance, Theorems 1.3.1 and 1.9.6 of Lions-Magenes [15]), there exists $c_2 = c_2(T)$ such that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,T];H_0^m(\Omega))} \leq c_2 \| a\Delta^k(y^n - y^{n-1}) \|_{L^2(Q)} .$$

Further, it is clear that we can choose $C_2 = C_2(T)$ such that for all $t \in [0,T]$

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,t];H_0^m(\Omega))} \leq C_2 \| a\Delta^k(y^n - y^{n-1}) \|_{L^2((0,t)\times\Omega)} .$$

Hence,

$$\| (y^{n+1} - y^n)(t) \|^2_{H_0^m(\Omega)} \leq (C_2 M)^2 \int_0^t \| \Delta^k(y^n - y^{n-1})(\tau) \|^2_{L^2(\Omega)}\, d\tau, \quad \text{for all } t \in [0,T]$$

and therefore, by using the Poincaré inequality, there exists a constant K, independent of M, such that

$$\| (y^{n+1} - y^n)(t) \|^2_{H_0^m(\Omega)} \leq (KM)^2 \int_0^t \| (y^n - y^{n-1})(\tau) \|^2_{H_0^m(\Omega)}\, d\tau, \quad \text{for all } t \in [0,T].$$

Then, for every $t \in [0,T]$ we deduce that

$$\begin{aligned} \| (y^{n+1} - y^n)(t) \|^2_{H_0^m(\Omega)} &\leq (K^2M^2)^{n-1} \int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}} \| (y^2 - y^1)(\tau_n) \|^2_{H_0^m(\Omega)}\, d\tau_n \ldots d\tau_1\\ &\leq (K^2M^2)^{n-1} \int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}} \| y^2 - y^1 \|^2_{\mathcal{C}([0,T];H_0^m(\Omega))}\, d\tau_n \ldots d\tau_1\\ &\leq (K^2M^2)^{n-1} \frac{t^{n-1}}{(n-1)!} \| y^2 - y^1 \|^2_{\mathcal{C}([0,T];H_0^m(\Omega))}\\ &\leq \frac{(K^2M^2T)^{n-1}}{(n-1)!} \| y^2 - y^1 \|^2_{\mathcal{C}([0,T];H_0^m(\Omega))}, \end{aligned}$$

which implies that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,T];H_0^m(\Omega))} \to 0 \quad \text{as } n \to \infty$$

and therefore, by (2.8), we deduce that

$$\| (y^{n+1} - y^n)_t \|_{L^2(0,T;H^{-m}(\Omega))} \to 0 \quad \text{as } n \to \infty.$$

Then, there exists $y \in W$ such that

$$y_n \to y \quad \text{in } W \quad \text{as } n \to \infty.$$

In order to prove that y satisfies (2.5) we point out that

$$\Delta^m y^n \to \Delta^m y \quad \text{in } L^2(0,T;H^{-m}(\Omega)) \quad \text{as } n \to \infty,$$

$$\Delta^k y^n \to \Delta^k y \quad \text{in } L^2(\Omega) \quad \text{as } n \to \infty,$$

and

$$y^n_t \to y_t \quad \text{in } L^2(0,T;H^{-m}(\Omega)) \quad \text{as } n \to \infty.$$

this implies (passing to the limit) that y is the solution of (2.5). In order to prove (2.6), we "multiply" in (2.5) by y. Then it is easy to see that

$$(2.9)\quad \begin{aligned} &\|y\|_{L^2(0,T;H_0^m(\Omega))} + \|y_t\|_{L^2(0,T;H^{-m}(\Omega))}\\ &\qquad \leq C\left(\|h\|_{L^2(0,T;H^{-m}(\Omega))} + \|y_0\|_{L^2(\Omega)} + \|y\|_{L^2(Q)}\right). \end{aligned}$$

Furthermore,

$$\| y(t) \|^2_{L^2(\Omega)} \leq \left(\| y(0) \|^2_{L^2(\Omega)} + c_2 \| h \|^2_{L^2(0,T;H^{-m}(\Omega))}\right) + c_3 \int_0^t \| y(s) \|^2_{L^2(\Omega)}\, ds.$$

Then, applying Gronwall's inequality (see, for instance, Lemma 4 of Haraux [11]), we deduce that

$$\| y(t) \|^2_{L^2(\Omega)} \leq \left(\| y(0) \|^2_{L^2(\Omega)} + c_2 \| h \|^2_{L^2(0,T;H^{-m}(\Omega))}\right) e^{c_3 t} \qquad \forall\, t \in [0,T].$$

From here, we obtain that

$$\| y \|_{L^2(Q)} \leq c_4 \left(\|h\|_{L^2(0,T;H^{-m}(\Omega))} + \|y_0\|_{L^2(\Omega)}\right)$$

which implies, together with (2.9), inequality (2.6). Now, thanks to (2.6) and the linearity of Problem (2.5), we deduce the uniqueness of solution.

Finally, if $h \in L^2(Q)$, since $y(\delta) \in H_0^m(\Omega)$ for all $\delta \in (0,T)$, taking $y(\delta)$ as initial datum and applying Theorem 4.6.1 of [16], we get (2.7). ■

As usual in Controllability Theory we shall use a *unique continuation* property for solutions of the *dual problem* (in our case Problem (2.4)).

Lemma 2.1. *Let ω be a nonempty open subset of Ω. Assume that*

$$\varphi \in L^2(0,T;H_0^m(\Omega)) \;\cap\; C([0,T];L^2(\Omega))$$

satisfies (2.4) and that $\varphi \equiv 0$ in $\mathcal{O} = \omega \times (0,T)$. Then $\varphi \equiv 0$ in Q.

Proof. From Proposition 2.1 (applied with backward time) we deduce that $\varphi \in L^2(0,T-\delta;H^{2m}(\Omega))$ for all $\delta \in (0,T)$. Then Lemma 2.1 follows from Theorem 3.2 of Saut-Scheurer [17]. ■

The following two results are easy adaptations (by using Lemma 2.1) of the similar ones given in [9], [10] for second order parabolic problems.

Proposition 2.2. *The functional $J(\cdot;a,y_d)$ is continuous, strictly convex on $L^2(\Omega)$ and verifies*

$$\liminf_{\|\varphi^0\|_{L^2(\Omega)}\to\infty} \frac{J(\varphi^0;a,y_d)}{\|\varphi^0\|_{L^2(\Omega)}} \geq \varepsilon. \tag{2.10}$$

Further $J(\cdot;a,y_d)$ attains its minimum at a unique point $\hat\varphi^0$ in $L^2(\Omega)$ and

$$\hat\varphi^0 = 0 \quad \Leftrightarrow \quad \| y_d \|_{L^2(\Omega)} \leq \varepsilon. \tag{2.11}$$

Proposition 2.3. *Let $\mathcal{M}$ be the mapping*

$$\begin{array}{rccc} \mathcal{M}: & L^\infty(Q)\times L^2(\Omega) & \to & L^2(\Omega) \\ & (a(t,x),y_d) & \longrightarrow & \hat\varphi^0. \end{array}$$

If B is a bounded subset of $L^\infty(Q)$ and K is a compact subset of $L^2(\Omega)$, then $\mathcal{M}(B\times K)$ is a bounded subset of $L^2(\Omega)$.

In order to characterize the duality of problem (2.4), we recall that given a convex and proper function $V: X\to\mathbb{R}\cup\{+\infty\}$ on the Banach space X, it is said that a element p_0 of V' belongs to the set $\partial V(x_0)$ (subdifferential of V at $x_0\in X$) if

$$V(x_0)-V(x)\leq (p_0, x_0-x) \quad \forall\, x\in X.$$

It is well known that that if V is Gateaux differentiable its differential coincides with its subdifferential and that x_0 minimizes V over X (or over a convex subset of X) if and only if $0\in\partial V(x_0)$. Finally, if V is a lower semicontinuous function, then $p_0\in\partial V(x_0)$ if and only if

$$(p_0,x)\leq \lim_{h\to 0^+}\frac{V(x_0+hx)-V(x_0)}{h}(<+\infty) \quad \forall\, x\in X.$$

(See, for instance, Aubin-Ekeland [3]). Coming back to the functional J we have:

Lemma 2.2. *For every $\varphi^0\in L^2(\Omega)$ ($\varphi^0\neq 0$), if φ is the solution of (2.4) satisfying $\varphi(T)=\varphi^0$, we have that*

$$\partial J(\varphi^0;a,y_d) = \{\xi\in L^2(\Omega),\ \exists\, v\in sgn(\varphi)\chi_{\mathcal{O}} \text{ satisfying}$$

$$\begin{aligned}\int_\Omega \xi(x)\theta^0(x)dx &= \left(\int_{\mathcal{O}}|\varphi(t,x)|d\Sigma\right)\left(\int_{\mathcal{O}} v(t,x)\theta(t,x)d\Sigma\right)\\ &+\varepsilon\int_\Omega \frac{\varphi^0(x)}{\|\varphi^0\|_{L^2(\Omega)}}\theta^0(x)dx - \int_\Omega y_d(x)\theta^0(x)dx \ \forall\theta^0\in L^2(\Omega)\},\end{aligned}$$

where θ is the solution of (2.4) satisfying $\theta(T)=\theta^0$.

Proof. It is an easy modification of Proposition 2.4 of [10].

Let us prove the approximate controllability property for an special version of the linear problem given in (2.2).

Theorem 2.1. *If $\| y_d \|_{L^2(\Omega)} > \varepsilon$ and $\hat\varphi$ is the solution of (2.4) corresponding to $\hat\varphi(T)=\hat\varphi^0$, with $\hat\varphi^0$ minimum of $J(\cdot;a,y_d)$. Then there exists $v\in sgn(\hat\varphi)\chi_{\mathcal{O}}$ such that the solution of*

$$\left\{\begin{array}{ll} y_t+(-\Delta)^m y+(-\Delta^k)(a(t,x)y) = \|\hat\varphi\|_{L^1(\mathcal{O})} v\chi_{\mathcal{O}} & \text{in } Q,\\ \dfrac{\partial^j y}{\partial\nu^j}=0 \quad (j=0\ldots(m-1)) & \text{on } \Sigma,\\ y(0)=0 & \text{in } \Omega,\end{array}\right. \tag{2.12}$$

satisfies

$$y(T) = y_d - \varepsilon\frac{\hat\varphi^0}{\|\hat\varphi^0\|_{L^2(\Omega)}},$$

and then $\| y(T)-y_d\|_{L^2(\Omega)}=\varepsilon$.

Remark 2.1. In the case $\|y_d\|_{L^2(\Omega)}\leq\varepsilon$, if we use the null control, we obtain $y=0$ and therefore $\|y(T)-y_d\|_{L^2(\Omega)}\leq\varepsilon$.

First of all we prove the existence and uniqueness to problem given by (2.2).

Proposition 2.4. *Assumed $y_0\in L^2(\Omega)$, $h\in L^2(0,T;H^{-m}(\Omega))$ and $a(t,x)\in L^\infty(Q)$, there exists a unique function $y\in W$ satisfying*

$$\left\{\begin{array}{lll} y_t+(-\Delta)^m y+\Delta^k(a(t,x)y)=h & & \text{in } Q,\\ \dfrac{\partial^j y}{\partial\nu^j}=0 \quad , & j=0,1,\ldots,m-1 & \text{on } \Sigma,\\ y(0)=y_0 & & \text{in } \Omega.\end{array}\right. \tag{2.13}$$

Moreover, we have the estimate

$$\|y\|_{L^2(0,T;H_0^m(\Omega))}+\|y_t\|_{L^2(0,T;H^{-m}(\Omega))}\leq C\left(\|h\|_{L^2(0,T;H^{-m}(\Omega))}+\|y_0\|_{L^2(\Omega)}\right), \tag{2.14}$$

where the constant C depends only on M (provided that Ω, T and m are kept fixed).

Proof. For all $n\in\mathbb{N}$ we define again y^{n+1} as the solution of the iterative problem

$$\left\{\begin{array}{lll} y_t^{n+1}+(-\Delta)^m y^{n+1}=h-\Delta^k(a(t,x)y^n) & & \text{in } Q,\\ \dfrac{\partial^j y^{n+1}}{\partial\nu^j}=0 \quad , & j=0,1,\ldots,m-1 & \text{on } \Sigma,\\ y^{n+1}(0)=y_0 & & \text{in } \Omega,\end{array}\right.$$

where $y^0(t):=0$ for all $t\in[0,T]$. The existence of a solution $y^n\in W$ can be found, for instance, in Theorem 3.4.1 of Lions-Magenes [15]. Thus, for all $n\in\mathbb{N}\backslash\{0,1\}$, $y^{n+1}-y^n$ is solution of

$$\left\{\begin{array}{lll} (y^{n+1}-y^n)_t+(-\Delta)^m(y^{n+1}-y^n)=-\Delta^k[a(t,x)(y^n-y^{n-1})] & & \text{in } Q,\\ \dfrac{\partial^j(y^{n+1}-y^n)}{\partial\nu^j}=0 \quad , & j=0,1,\ldots,m-1 & \text{on } \Sigma,\\ (y^{n+1}-y^n)(0)=0 & & \text{in } \Omega\end{array}\right. \tag{2.15}$$

and therefore (see again Theorem 3.4.1 of Lions-Magenes [15]) $y^{n+1}-y^n\in W$ and

$$\| y^{n+1}-y^n\|_W\leq c_1\| a(y^n-y^{n-1})\|_{L^2(Q)}\,. \tag{2.16}$$

Then, since $W \subset \mathcal{C}([0,T];L^2(\Omega))$ with continuous embedding (see, for instance, [12] or [15]), we have that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,T];L^2(\Omega))} \leq c_2 \| a(y^n - y^{n-1}) \|_{L^2(Q)} .$$

Further, as in the proof of Proposition 2.1, we can choose $C_2 = C_2(T)$ such that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,t];L^2(\Omega))} \leq C_2 \| a(y^n - y^{n-1}) \|_{L^2((0,t)\times\Omega)}, \quad \text{for all } t \in [0,T].$$

Hence,

$$\| (y^{n+1} - y^n)(t) \|^2_{L^2(\Omega)} \leq (C_2 M)^2 \int_0^t \| (y^n - y^{n-1})(\tau) \|^2_{L^2(\Omega)} d\tau, \quad \text{for all } t \in [0,T]$$

Then, for every $t \in [0,T]$ we deduce that

$$\begin{aligned} \| (y^{n+1} - y^n)(t) \|^2_{L^2(\Omega)} &\leq (C_2^2 M^2)^{n-1} \int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{n-1}} \| (y^2 - y^1)(\tau_n) \|^2_{L^2(\Omega)} d\tau_n \dots d\tau_1 \\ &\leq (C_2^2 M^2)^{n-1} \int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{n-1}} \| y^2 - y^1 \|^2_{\mathcal{C}([0,T];L^2(\Omega))} d\tau_n \dots d\tau_1 \\ &\leq (C_2^2 M^2)^{n-1} \frac{t^{n-1}}{(n-1)!} \| y^2 - y^1 \|^2_{\mathcal{C}([0,T];L^2(\Omega))} \\ &\leq \frac{(C_2^2 M^2 T)^{n-1}}{(n-1)!} \| y^2 - y^1 \|^2_{\mathcal{C}([0,T];L^2(\Omega))}, \end{aligned}$$

which implies that

$$\| y^{n+1} - y^n \|_{\mathcal{C}([0,T];L^2(\Omega))} \to 0 \quad \text{as} \quad n \to \infty$$

and therefore, by (2.16), we deduce that

$$\| y^{n+1} - y^n \|_W \to 0 \quad \text{as} \quad n \to \infty.$$

Then, there exists $y \in W$ such that

$$y_n \to y \quad \text{in } W \quad \text{as } n \to \infty.$$

The end of the proof is similar to the end of the proof of Proposition 2.1. ∎

Proof of Theorem 2.1. Using the subdifferentiability of $J(.;a,y_d)$ at $\widehat{\varphi}^0$ ($\neq 0$ by (2.11)), we know that

$$0 \in \partial J(\widehat{\varphi}^0),$$

which is equivalent, from Lemma 2.2, to the existence of $v \in sgn(\widehat{\varphi})\chi_{\mathcal{O}}$, such that

$$(2.17) \quad -\| \widehat{\varphi} \|_{L^1(\mathcal{O})} \left(\int_{\mathcal{O}} v(x,t)\theta(x,t)dxdt \right) = \frac{\varepsilon}{\| \widehat{\varphi}^0 \|_{L^2(\Omega)}} \int_\Omega \widehat{\varphi}^0(x)\theta^0(x)dx - \int_\Omega y_d(x)\theta^0(x)dx.$$

On the other hand, as $y \in W$, if we "multiply" by θ in (2.12) we obtain, by (2.4), that

$$(2.18) \qquad \int_\Omega y(T,x)\theta^0(x)dxdt = \| \widehat{\varphi} \|_{L^1(\mathcal{O})} \left(\int_{\mathcal{O}} v(x,t)\theta(x,t)dxdt \right)$$

Then, from (2.17) and (2.18), we obtain

$$\int_\Omega y(T,x)\theta^0(x)dxdt = \int_\Omega (y_d(x) - \varepsilon \frac{\widehat{\varphi}^0(x)}{\| \widehat{\varphi}^0 \|_{L^2(\Omega)}})\theta^0(x)dxdt \quad \forall\, \theta^0 \in L^2(\Omega)$$

and we conclude that $y(T) = y_d - \varepsilon \dfrac{\widehat{\varphi}^0}{\| \widehat{\varphi}^0 \|_{L^2(\Omega)}}$. ∎

Now we are ready to prove a linear version of Theorem 1.1 for problem (2.2)

Corollary 2.1. *Let* $\| y_d \|_{L^2(\Omega)} > \varepsilon$ *and* $\widehat{\varphi}$ *the solution of (2.4) corresponding to* $\widehat{\varphi}(T) = \widehat{\varphi}^0$, *with* $\widehat{\varphi}^0$ *minimum of* $J(\cdot;a,y_d - y(T;a,0))$, *where in general* $y(t;a,u)$ *denotes the solution of (2.2) corresponding to the control* u. *Then there exists* $v \in sgn(\widehat{\varphi})\chi_{\mathcal{O}}$ *such that the solution of*

$$\begin{cases} y_t + (-\Delta)^m y + (-\Delta^k)(a(t,x)y) = h + h(a) + \| \widehat{\varphi} \|_{L^1(\mathcal{O})} v\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad (j = 0 \dots (m-1)) & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

satisfies

$$\| y(T) - y_d \|_{L^2(\Omega)} \leq \varepsilon.$$

Proof. We put $y = L + Y$, where $L = L(a)$ satisfies

$$(2.19) \qquad \begin{cases} L_t + (-\Delta)^m L + (-\Delta^k)(a(t,x)L) = h + h(a) & \text{in } Q, \\ \dfrac{\partial^j L}{\partial \nu^j} = 0 \quad (j = 0 \dots (m-1)) & \text{on } \Sigma, \\ L(0) = y_0 & \text{in } \Omega \end{cases}$$

and $Y = Y(a)$ is taken associated to the approximate controllability problem

$$\begin{cases} Y_t + (-\Delta)^m Y + (-\Delta^k)(a(t,x)Y) = u(a)\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j Y}{\partial \nu^j} = 0 \quad (j = 0 \dots (m-1)) & \text{on } \Sigma, \\ Y(0) = 0 & \text{in } \Omega, \end{cases}$$

with desired state $y_d - L(T)$, i.e. such that $\| Y(T) - (y_d - L(T)) \| \leq \varepsilon$. Notice that the existence of such a control $u(a)$ is consequence of Theorem 2.1. In particular, if $\| y_d - L(T) \| \leq \varepsilon$, we can take $u(a) \equiv 0$ and if $\| y_d - L(T) \| > \varepsilon$, then we take $u(a) = \| \widehat{\varphi}(a) \|_{L^1(\Omega)} v(a)$, where $v(a) \in sgn(\widehat{\varphi}(a))\chi_{\mathcal{O}}$ and $\widehat{\varphi}(a)$ is the solution of (2.4) with initial value $\mathcal{M}(\ (a(x,t), y_d - L(T))\ )$ defined in Proposition 2.3. It is obvious that such function y and such control $u(a)$ lead to the conclusion. ∎

3. Controllability for the nonlinear problem

As mentioned before, we shall use a fixed point argument to prove Theorem 1.1. In fact we shall deal with multivalued operators. Let us recall a well-known result: the Kakutani's fixed point Theorem. The usual continuity assumption in other fixed pont theorems is replaced here by the following notion:

Definition 3.1. *Let X, Y two Banach spaces and, $\Lambda : X \to \mathcal{P}(Y)$ a multivalued function. We say that Λ is upper hemicontinuous at $x_0 \in X$, if for every $p \in Y'$, the function*

$$x \to \sigma(\Lambda(x), p) = \sup_{y \in \Lambda(x)} < p, y >_{Y' \times Y}$$

is upper semicontinuous at x_0. We say that the multivalued function is upper hemicontinuous on a subset K of X, if it satisfies this properties for every point of K.

Theorem 3.1. ***(Kakutani's fixed point Theorem).*** *Let $K \subset X$ be a convex and compact subset and $\Lambda : K \to K$ an upper hemicontinuous application with convex, closed and nonempty values. Then, there exists a fixed point x_0, of Λ.*

For a proof see, for instance, Aubin [2].

Proof of Theorem 1.1. We fix $y_d \in L^2(\Omega)$ and $\varepsilon > 0$. By using Corollary 2.1, for each $z \in L^2(Q)$ and $\varepsilon > 0$ it is possible to find two functions $\varphi(z) \in L^1(Q)$ and $v(z) \in sgn(\varphi(z))\chi_{\mathcal{O}}$ such that the solution $y = y^z$ of

$$(3.1) \qquad \begin{cases} y_t + (-\Delta)^m y + (-\Delta)^k (g(z)y) = h + h(g(z)) + u\chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \ , \ j = 0, 1, \dots m-1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

(where $u = u(z) = |\varphi(z)|_{L^1(\mathcal{O})} v(z))$ satisfies

$$(3.2) \qquad |y(T) - y_d|_{L^2(\Omega)} \leq \varepsilon.$$

Here $\varphi(z)$ is the solution of (2.4) with initial value $M(\ (g(z), y_d - L(z;T))\ )$ (see Proposition 2.3) and $a(t,x) = g(z)$, where is $L(z;T)$ the solution of (2.19), with $a = g(z)$, at time T .

Lemma 3.1. *The set*

$$\{y_d - L(z;T), \ z \in L^2(Q)\},$$

is relatively compact in $L^2(\Omega)$.

Proof of Lemma 3.1. Applying Proposition 2.4 it is easy to see that the set of solutions $L(z)$ of

$$(3.3) \qquad \begin{cases} L_t + (-\Delta)^m L + (-\Delta)^k (g(z)y) = h + h(g(z)) & \text{in } Q, \\ \dfrac{\partial^j L}{\partial \nu^j} = 0 \ , \ j = 0, 1, \dots m-1 & \text{on } \Sigma, \\ L(0) = y_0 & \text{in } \Omega, \end{cases}$$

satisfy

$$(3.4) \qquad \| L(z) \|_W \leq K(1 + \| y_0 \|_{L^2(\Omega)} + \| h \|_{L^2(0,T;H^{-m}(\Omega))}) \ \ \forall \ z \in L^2(Q)$$

with $K > 0$ independent of z. Recall that $\| g(z) \|_{L^\infty(Q)} \leq M$ with M independent of z. Now, let $L(z_n)$ be a sequence of solutions (3.3) with $z_n \in L^2(Q)$. We must prove that there exists a subsequence (that we rewrite as $L(z_n)$), such that

$$\| L(z_n;T) - L(z_{n+1};T) \|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

By a compactness result due to Aubin [1], we know that

$$W \subset L^2(0,T; H^{m-1}(\Omega)) \ \ \text{with compact embedding.}$$

Therefore, by (3.4), we can suppose that

$$\| L(z_n) - L(z_{n+1}) \|_{L^2(0,T;H^{m-1}(\Omega))} \to 0 \quad \text{as } n \to \infty.$$

Further, it is easy to prove that $L(z_n) - L(z_{n+1})$ satisfies

$$\| L(z_n;T) - L(z_{n+1};T) \|^2_{L^2(\Omega)}$$

$$\leq - \int_0^T \langle D^k\left(g(z_n)L(z_n) - g(z_{n+1})L(z_{n+1})\right), D^k\left(L(z_n) - L(z_{n+1})\right)\rangle_{H^{-k}(\Omega)\times H_0^k(\Omega)} dt$$

$$+ \int_0^T \langle D^k\left(g(z_n)s_0 - g(z_{n+1})s_0\right), D^k\left(L(z_n) - L(z_{n+1})\right)\rangle_{H^{-k}(\Omega)\times H_0^k(\Omega)} dt.$$

Then, by (3.4), since $k \leq m-1$ (notice that $k = 0$ if $m = 1$),

$$\| L(z_n;T) - L(z_{n+1};T) \|^2_{L^2(\Omega)} \leq \widetilde{K} \| L(z_n) - L(z_{n+1}) \|^2_{L^2(0,T;H^{m-1}(\Omega))} \to 0 \quad \text{as } n \to \infty$$

and the proof ends. ∎

Completion of Proof of Theorem 1.1. From Lemma 3.1, we obtain that $y_d - L(z;T)$ belongs to a compact set for all $z \in L^2(Q)$ and so, by using Propositions 2.3 and 2.1, we obtain that

$$(3.5) \qquad \{\| \varphi(z) \|_{L^1(\mathcal{O})} v(z), \ z \in L^2(Q)\} \ \ \text{is bounded in } L^\infty(Q)$$

Thus

$$(3.6) \qquad K_1 = \sup_{z \in L^2(Q)} \| \varphi(z) \|_{L^1(\mathcal{O})} < \infty.$$

Obviously, $u = u(z)$ satisfies

$$(3.7) \qquad \| u \|_{L^2(Q)} \leq K_2.$$

Therefore, if we define the operator

$$\Lambda : L^2(Q) \to \mathcal{P}(L^2(Q))$$

by

$$\Lambda(z) = \{y \text{ satisfies (3.1), (3.2) for some } u \text{ satisfying (3.7) }\},$$

we have seen that for each $z \in L^2(Q)$, $\Lambda(z) \neq \emptyset$. In order to apply Kakutani's fixed point theorem, we have to check that the next properties hold:

(i) There exists a compact subset U of $L^2(Q)$, such that for every $z \in L^2(Q)$, $\Lambda(z) \subset U$.

(ii) For every $z \in L^2(Q)$, $\Lambda(z)$ is a convex, compact and nonempty subset of $L^2(Q)$.

(iii) Λ is upper hemicontinuous.

The proof of these properties is as follows:

(i) From Proposition 2.4 we know that, there exists a bounded subset U of W such that for every $z \in L^2(Q)$, $\Lambda(z) \subset U$. Now, to see that we can choose U compact we shall prove that the set

$$\mathcal{Y} = \{y \text{ satisfying (3.1) for some } z \in L^2(Q) \text{ and } u \text{ satisfying (3.7)}\}$$

is a relatively compact subset of $L^2(Q)$. But this is easy to prove by using that

$$W \subset L^2(Q) \text{ with compact embedding} \tag{3.8}$$

(see Lions [12] or Simon [18]).

(ii) We have already seen that for every $z \in L^2(Q)$, $\Lambda(z)$ is a nonempty subset of $L^2(Q)$. Further $\Lambda(z)$ is obviously convex, because $B(y_d, \varepsilon)$ and $\{u \in L^2(Q) : \text{satisfying (3.7)}\}$ are convex sets. Then, we have to see that $\Lambda(z)$ is a compact subset of $L^2(Q)$. In (i) we have proved that $\Lambda(z) \subset U$ with U compact. Let $(y^n)_n$ be a sequence of elements of $\Lambda(z)$ which converges in $L^2(Q)$ to $y \in U$. We have to prove that $y \in \Lambda(z)$. We know that there exist $u^n \in L^2(Q)$ satisfying (3.7) such that

$$\begin{cases} y^n_t + (-\Delta)^m y^n + (-\Delta)^k (g(z) y^n) = h + h(g(z)) + u^n \chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y^n}{\partial \nu^j} = 0 \ , \ j = 0, 1, \ldots, m-1 & \text{on } \Sigma, \\ y^n(0) = y_0 & \text{in } \Omega, \\ |y^n(T) - y_d|_2 \le \varepsilon. \end{cases} \tag{3.9}$$

Now, by using that the controls u^n are uniformly bounded, we deduce that $u^n \to u$ in the weak topology of $L^2(Q)$ and u satisfies (3.7) (see Proposition III.5 of Brezis [5]). Then, using (3.9) and Proposition 2.4 we can see that $(y^n)_n$ converges to y in the weak topology of W (and so, by (3.8), strongly in $L^2(Q)$). Therefore, passing to the limit in (3.9) we obtain

$$\begin{cases} y_t + (-\Delta)^m y + (-\Delta)^k (g(z) y) = h + h(g(z)) + u \chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \ , \ j = 0, 1, \ldots, m-1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

Further, $v^n = y - y^n$ is solution of

$$\begin{cases} v^n_t + (-\Delta)^m v^n + (-\Delta)^k (g(z) v^n) = (u - u^n) \chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j v^n}{\partial \nu^j} = 0 \ , \ j = 0, 1, \ldots, m-1 & \text{on } \Sigma, \\ v^n(0) = 0 & \text{in } \Omega \end{cases} \tag{3.10}$$

and satisfies $v^n \in W$ (see Proposition 2.4). Further, if we "multiply" in (3.10) by v^n and integrate, we obtain that

$$\| v^n(T) \|^2_{L^2(\Omega)} \le k \int_Q (u - u^n) \chi_{\mathcal{O}} v^n dx dt \to 0 \quad \text{as } n \to \infty.$$

Thus $y^n(T)$ converges to $y(T)$ in the strong topology of $L^2(\Omega)$ and $\| y(T) - y_d \|_2 \le \varepsilon$. This prove that $y \in \Lambda(z)$ and concludes the proof of (ii).

(iii) We must prove that for every $z_0 \in L^2(Q)$

$$\limsup_{z_n \xrightarrow{L^2(Q)} z_0} \sigma(\Lambda(z_n), k) \le \sigma(\Lambda(z_0), k), \ \forall\, k \in L^2(Q).$$

We have seen in (ii) that $\Lambda(z)$ is a compact set, which implies that for every $n \in \mathbb{N}$ there exists $y^n \in \Lambda(z_n)$ such that

$$\sigma(\Lambda(z_n), k) = \int_Q k(x,t) y^n(x,t) dx dt.$$

Now, by (i), $(y^n)_n \subset U$ (compact set of $L^2(Q)$). Then, there exists $y \in L^2(Q)$ such that (after extracting a subsequence) $y^n \to y$ in $L^2(Q)$. We shall prove that $y \in \Lambda(z_0)$. We know that there exist $u^n \in L^2(Q)$ satisfying (3.7) such that

$$\begin{cases} y^n_t + (-\Delta)^m y^n + (-\Delta)^k (g(z_n) y^n) = h + h(z_n) + u^n \chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y^n}{\partial \nu^j} = 0 \ , \ j = 0, 1, \ldots, m-1 & \text{on } \Sigma, \\ y^n(0) = y_0 & \text{in } \Omega, \\ |y^n(T) - y_d|_2 \le \varepsilon. \end{cases} \tag{3.11}$$

Then there exists $u \in L^2(Q)$ satisfying (3.7) such that $u^n \to u$ in the weak topology of $L^2(\mathcal{O})$. On the other hand, by using the smoothing effect of the parabolic linear equation (in a similar way to the proof of (ii)) and that $g \in L^\infty(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$, we deduce that y satisfies (3.1) and (3.2) with $z = z_0$ for some $u \in L^2(Q)$ satisfying (3.7), which implies that $y \in \Lambda(z_0)$. Then, for every $k \in L^2(Q)$,

$$\sigma(\Lambda(z_n), k) = \int_Q k(x,t) y^n(x,t) dx dt \to \int_Q k(x,t) y(x,t) dx dt$$

$$\le \sup_{\overline{y} \in \Lambda(z_0)} \int_Q k(x,t) \overline{y}(x,t) dx dt = \sigma(\Lambda(z_0), k),$$

which proves that Λ is upper hemicontinuous and conclude the proof of (iii).

Finally, if we restrict Λ to $K = conv(U)$ (the convex envelope of U), which is a compact set of $L^2(Q)$, it satisfies the assumptions of Kakutani's fixed point theorem. Then, Λ has a fixed point $y \in K$. Further, by construction, there exists a control $u \in L^2(Q)$ satisfying (3.7) such that

$$\begin{cases} y_t + (-\Delta)^m y + (-\Delta)^k (f(y)) = h + u \chi_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \ , \ j = 0, 1, \ldots m-1 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \\ |y(T) - y_d|_2 \le \varepsilon. \end{cases} \tag{3.12}$$

Therefore, y is the solution that we were looking for. ∎

Remark 3.1. Several generalizations seem possible. For instance, the equation of (1.1) could be replaced by other ones with a more general nonlinearity

$$y_t + (-\Delta)^m y + \sum_{i=0}^{k} (-\Delta)^i f_i(y) = h + v \chi_\omega$$

or a more general lower order differential operator

$$y_t + (-\Delta)^m y + L(f(y)) = h + v\chi_\omega,$$

with L suitable linear partial differential operator of degree lower than $2m$. The key point in those generalizations is that the unique continuation result of Lemma 2.1, for the associated dual problem, remains true thanks to Theorem 3.2 of Saut-Scheurer [17] and the rest of arguments of the proof of Theorem 1.1 apply.

4. Non-controllability for superlinear problems

In this section we assume $k = 0$. We shall prove a result of non-controllability for a superlinear nonlinear term with $\overline{\omega} \subset \Omega$.

Theorem 4.1. *Let $p > 1$ and let $y(t;u) = y \in L^2(0,T;H^m(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega))$ a function satisfying*

$$\begin{cases} y_t + (-\Delta)^m y + |y|^{p-1}y = u\chi_\omega & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

associated to any "natural" boundary condition and with control $u \in L^2(Q)$. Then we can choose $y_d \in L^2(\Omega)$ and $\varepsilon > 0$ such that

$$\| y(T;u) - y_d \|_{L^2(\Omega)} > \varepsilon \quad \text{for any} \quad u \in L^2(Q). \tag{4.1}$$

In order to prove Theorem 4.1 we introduce, previously, some auxiliar functions. Given $R > 0$ we define, on $\mathbb{R}^N$, the functions

$$\xi_R(x) = (R^2 - |x|^2)/R \ \text{ if } |x| < R, \quad \xi_R(x) = 0 \ \text{ if } |x| \geq R$$

and

$$d_R(x) = R - |x| \ \text{ if } |x| < R, \quad d_R(x) = 0 \ \text{ if } |x| \geq R. \tag{4.2}$$

It is clear that

$$d_R(x) \leq \xi_R(x) \leq 2d_R(x) \tag{4.3}$$

for all $x \in \mathbb{R}^N$.

The following result was proved in Bernis [4].

Proposition 4.1. *Let $s \geq 2m$ and $R > 0$. Then, for each $\varepsilon > 0$ there exist a constant C depending only on N, m, s and ε (thus independent of R) such that the following inequality holds for all $y \in H^m_{loc}(\mathbb{R}^N)$:*

$$((-\Delta)^m y, \xi_R^s y)_{H^{-m}_{loc}(\mathbb{R}^N) \times H^m_c(\mathbb{R}^N)} \geq (1-\varepsilon) \int_{\mathbb{R}^N} \xi_R^s |D^m y|^2 dx - C \int_{\mathbb{R}^N} \xi_R^{s-2m} y^2 dx.$$

Remark 4.1. *Since $s \geq 2m$, $\xi_R^s \in W_c^{2m,\infty}(\mathbb{R}^N)$. Hence $\xi_R^s \in C_c^m(\mathbb{R}^N)$ (see e.g. Corollary IX.13 of [5]) and $\xi_R^s u \in H_c^m(\mathbb{R}^N)$ (see e.g. Note IX.4 of [5]).*

Corollary 4.1. *Let $s \geq 2m$ and $R > 0$ such that $\overline{B_R} \subset \Omega$. Then, for each $\varepsilon > 0$ there exist a constant C depending only on N, m, s and ε (thus independent of R) such that the following inequality holds for all $y \in H^m(\Omega)$:*

$$((-\Delta)^m y, \xi_R^s y)_{H^{-m}(\Omega) \times H_0^m(\Omega)} \geq (1-\varepsilon) \int_\Omega \xi_R^s |D^m y|^2 dx - C \int_\Omega \xi_R^{s-2m} y^2 dx.$$

Proof. Let $\overline{y} \in H^m(\Omega)$ such that $\overline{y} = y$ in Ω (such $\overline{y}$ exists by standar results: see, e.g., Chapter IX of Brezis [5]). Then, by Proposition 4.1, the inequality holds for $\overline{y}$, but as $\overline{B_R} \subset \Omega$ we obtain the result. ■

Theorem 4.2. *Let $p > 1$, $r = p+1$, $y_0 \in L^2(\Omega)$ and $u \in L^{r'}(Q)$. Then any solution $y \in L^r(Q) \cap L^2(0,T;H^m(\Omega))$ of*

$$\begin{cases} y_t + (-\Delta)^m y + |y|^{p-1}y = u & \text{in } \mathcal{D}'(Q), \\ y(0) = y_0 & \text{on } \Omega, \end{cases} \tag{4.4}$$

with any "natural" boundary condition, satisfies the local estimate

$$\sup_{0<t<T} \int_{B_R} y(x,t)^2 dx + \int_{B_R \times (0,T)} (|D^m y|^2 + |y|^r) dx dt$$

$$\leq K \left(1 + \int_{B_{R_1} \times (0,T)} |u|^{r'} dx dt + \int_{B_{R_1}} y_0^2 dx \right)$$

if $\overline{B_{R_1}} \subset \Omega$ and $0 < R \leq R_1$. Moreover, the constant K depends only on N, m, p, R, R_1 and T.

Proof of Theorem 4.2. We take $X_r = L^r(Q) \cap L^2(0,T;H_0^m(\Omega))$. Then the equation of (4.4) is satisfied in $X_r' = L^{r'}(Q) + L^2(0,T;H^{-m}(\Omega))$. Then, if $s \geq 2m$, we can multiply (4.4) by $\xi_R^s y$ with the duality product $(\cdot,\cdot)_{X_r' \times X_r}$ and we obtain

$$\frac{1}{2} \int_{B_R} \xi_R^s y(x,T)^2 dx + ((-\Delta)^m y, \xi_R^s y)_{L^2(0,T;H^{-m}(\Omega)) \times L^2(0,T;H_0^m(\Omega))}$$

$$+ (|y|^{p-1}y, \xi_R^s y)_{L^{r'}(Q) \times L^r(Q)}$$

$$= \frac{1}{2} \int_{B_R} \xi_R^s y_0(x)^2 dx + (u, \xi_R^s y)_{L^{r'}(Q) \times L^r(Q)}.$$

Now, from Corollary 4.1 it follows that

$$\begin{aligned} &\frac{1}{2} \int_{B_R} \xi_R^s y(x,T)^2 dx + \int_{B_R \times (0,T)} \xi_R^s (|D^m y|^2 + |y|^r) dx dt \\ &\leq C \int_{B_R} \xi_R^s y_0(x)^2 dx + C \int_{B_R \times (0,T)} \xi_R^{s-2m} y^2 dx dt + C \int_{B_R \times (0,T)} \xi_R^s u y dx dt. \end{aligned} \tag{4.5}$$

By (4.2) and (4.3) we can replace in (4.5) $\xi_R(x)$ by $R - |x|$ (modifying the constants). Further, writing $s - 2m = 2s/r + (s(r-2)/r) - 2m$, we can apply Hölder's or Young's inequality with exponents $q = r/2$ and $q' = r/r-2$ and we obtain

$$\int_{B_R \times (0,T)} (R - |x|)^{s-2m} y^2 dx dt$$

$$\leq \varepsilon \int_{B_R \times (0,T)} (R - |x|)^s |y|^r dx dt + K(\varepsilon, q) \int_{B_R \times (0,T)} (R - |x|)^{s-\gamma} dx dt$$

with

$$K(\varepsilon,q)=\frac{1}{q'(q\varepsilon)^{q'/q}}\quad\text{and}\quad \gamma=\frac{2mr}{r-2}\quad.$$

Hence, if we choose $s>\gamma-1$, the last integral is finite and equal to $\widetilde{C}R^{s+N-\gamma}$. On the other hand, we can apply again Young's inequality and we have

$$\int_{B_R\times(0,T)}(R-|x|)^s uy\,dxdt$$
$$\leq \varepsilon\int_{B_R\times(0,T)}(R-|x|)^s|y|^r dxdt+k(\varepsilon,r)\int_{B_R\times(0,T)}(R-|x|)^s|u|^{r'}dxdt.$$

Thus, by changing the constants, we deduce that

$$\frac{1}{2}\int_{B_R}(R-|x|)^s y(x,T)^2dx+\int_{B_R\times(0,T)}(R-|x|)^s(|D^m y|^2+|y|^r)dxdt$$
$$\leq C\left(\int_{B_R}(R-|x|)^s y_0(x)^2dx+R^{s+N-\gamma}+\int_{B_R\times(0,T)}(R-|x|)^s|u|^{r'}dxdt\right).$$

Finally, by replacing R by R_1 and by taking into account that $R_1-|x|\geq R_1-R$ and $R_1-|x|\leq R_1$ if $|x|\leq R$ we deduce the result with

$$K=\max\left\{C\left(\frac{R_1}{R_1-R}\right)^s,\ \frac{CR_1^{s+N-\gamma}}{(R_1-R)^s}\right\}.$$ ■

Proof of Theorem 4.1. It is a trivial consequence of Theorem 4.2 since, if R_1 satisfies $\overline{B_{R_1}}\subset\Omega\backslash\omega$, then

$$\|\ y(u;T)\ \|^2_{L^2(\Omega)}\leq K(1+\|\ y_0\ \|^2_{L^2(\Omega)})\quad \forall u\in L^{r'}(Q).$$

Therefore, taking y_d with $\|\ y_d\ \|_{L^2(\Omega)}$ large enough, we obtain (4.1) for $\varepsilon>0$ small enough. ■

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References

1. **Aubin, J.P.**: Un théorème de compacité. *C. R. Acad. Sci.*, Paris, Serie I, T. 256, pp. 5042–5044, (1963).
2. **Aubin, J.P.**: *L'analyse non linéaire et ses motivations économiques.* Masson. (1984).
3. **Aubin, J.P. and Ekeland, I.**: *Applied nonlinear Analysis.* Wiley-Interscience Publication, New York, (1984).
4. **Bernis, F.**: Elliptic and Parabolic Semilinear Problems without Conditions at infinity. *Arch. Rat. Mech. Anal.*, Vol. 106, N. 3, pp. 217–241, (1989).
5. **Brézis, H.**:*Analyse Fonctionnelle: Théorie et applications.* Masson, Paris, (1987).
6. **Cahn, J.W. and Hilliard, J.E.**: Free energy of a nonuniform system. I. Interfacial free energy. *J. Chem. Phys.* N. 28, pp. 258–267, (1958).
7. **Díaz, J.I. and Ramos, A.M.**: Positive and negative approximate controllability results for semilinear parabolic equations. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales*, Madrid, LXXXIX, 1º–2º, pp. 11–30 (1995).
8. **Elliot, C.M. and Songmu, Z.**: On the Cahn-Hilliard Equation. *Arch. Rat. Mech. Anal.* N. 96, pp. 339–357, (1986).
9. **Fabre, C., Puel, J.P. and Zuazua, E.**: Contrôlabilité approchée de l'équation de la chaleur semi-linéaire. *C. R. Acad. Sci. Paris*, t. 315, Série I, pp. 807–812, (1992).

10. **Fabre, C., Puel, J.P. and Zuazua, E.**: Approximate controllability of the semilinear heat equation, *Proceedings of the Royal Society of Edinburgh*, 125A, pp. 31–61, (1995).
11. **Haraux, A.**: *Nonlinear Evolution Equations.* Lecture Notes in Mathematics. Springer-Verlag, Heidelberg, (1981).
12. **Lions, J.L.**: *Contrôle optimal de systemes gouvernés par des equations aux derivées partielles.* Dunod, Paris, (1968).
13. **Lions, J.L.**: *Quelques méthodes de résolution des problèms aux limites non linéares.* Dunod, Paris, (1969).
14. **Lions, J.L.**: Remarques sur la contrôlabilité approchée. In Proceedings of *Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos*, Universidad de Malaga, pp. 77–88, (1990).
15. **Lions, J.L. and Magenes, E.**: *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod, Paris, (1968).
16. **Lions, J.L. and Magenes, E.**: *Problèmes aux limites non homogènes et applications*, Vol. 2, Dunod, Paris, (1968).
17. **Saut, J.C. and Scheurer, B.**: Unique continuation for some evolution equations. *J. Differential Equations*, Vol. 66, N. 1, pp. 118–139, (1987).
18. **Simon, J.**: Compact Sets in the Space $L^p(0,T;B)$. *Annali di Matematica Pura ed Applicata.* Serie 4, N. 146, pp.65–96, (1987).

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