A quasilinear functional reaction-diffusion equation arising in climatology

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Dedicated to Jacques-Louis Lions
on occasion of his 70th birthday

0 Introduction

This work concerns the initial value problem

$$(0)\quad \begin{cases} c(x)\partial_t u + h(w)\partial_t w - \nabla \cdot [k(x)\,|\nabla u|^{p-2}\,\nabla u] + g(u) \\ \qquad\qquad = f(t,x,u,w),\ \ x \in M,\ \ t > 0, \\ w(t,x) = \int_{-T}^{0} \beta(s,x)u(t+s,x)\,\mathrm{d}s,\ \ t > 0,\ \ x \in M, \\ u(s,x) = u_0(s,x),\ \ -T \le s \le 0,\ \ x \in M, \end{cases}$$

on a two-dimensional compact oriented Riemannian manifold M which arises in the context of energy balance climate models with $M = \mathbf{S}^2$, the Euclidean unit sphere in $\mathbf{R}^3$. Here, the symbols ∇ and $\nabla\cdot$ denote the (weak) gradient and divergence on M where both concepts are understood with respect to the Riemannian metric of M (cf. [14], e.g.). Throughout $p \ge 2$ is assumed, and f is a bounded, possibly discontinuous function. Earlier work by the first author and L. Tello [14] has dealt with the corresponding reaction-diffusion problem without memory, i.e. $h \equiv 0$ and $f = f(x,t,u)$, whereas the effect of memory terms was investigated by the second author in [19] for linear diffusion, i.e. $p = 2$, and continuous f relying on a different framework ($h \equiv 0$, but c depending on w). The present contribution is committed to the program initiated by J.L. Lions in [23] and later continued (also in collaboration with other authors) in a series of works concerning the development of some mathematical methods needed in the study of climate and environment (see, e.g. [24], [25], [26], [27] and [12], [13]). Equation (0) rewritten in a more accessible form (2) can be treated in $\mathrm{L}^2(M)$ using the theory of evolution equations on Hilbert spaces approach as well as in a $C(M)$-setting (which requires to assume f and u_0 continuous functions. Roughly speaking, both cases lead to functional evolution equations that are governed by an m-accretive principal part A which generates a compact semigroup. Sobolev regularity in time is then a consequence of the strong solvability the Hilbert space approach yields, whereas the mild solvability in $C(M)$ paves the way for deriving global a priori bounds for the unique noncontinuable solution in the continuous case and for establishing the existence of a uniformly bounded global mild solution of (2), if f is discontinuous. The solution in the discontinuous case

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is obtained as the limit of a sequence of solutions for approximating continuous problems. An example, which goes back to the first author, shows that, even if $h \equiv 0$ and f does not depend on w, one cannot expect uniqueness without adding assumptions regarding the behavior of the solutions near their discontinuities (cf. [14] for more details and a uniqueness result).

There is a well-established existence theory for functional evolution equations of the form $\dot{u} + Au = F(t, u_t)$ in a Banach space X, where $A: X \supseteq \mathrm{dom}(A) \to X$ is m-accretive and generates a compact semigroup, b, $T > 0$ and $F: [0, b] \times C([-T, 0], X) \to X$ is continuous ($u_t: s \mapsto u(t+s)$ for $s \in [-T, 0]$). The reader is referred to chapters 4 and 5 in [32] for a brief account of the results we are going to utilize. Consequently, we focus here on establishing a priori bounds in the supremum norm for the mild solutions of (2), if f is continuous, bounds which only depend on the size of the supremum norm of f. This is achieved by employing a standard implicit difference scheme and cut off test functions which emulate in a sense the semidefiniteness properties at extrema in cases where regularity is missing.

As for the solution concept in the discontinuous case, there is not one distinguished "natural" definition. It is convenient to resort to a set-valued approach by filling in the gaps of f, if one is primarily interested in jump discontinuities. It turns out that (2) possesses a global solution in this set-valued sense which is the uniform limit on compacta of a sequence of solutions to equations with approximate locally Lipschitz selections of f replacing f. This approximation property is significant in the climatological context which will be presented in the following section. It suggests, roughly speaking, that Budyko-North type models are limiting cases of Sellers type models as long as one is interested in the evolution of the system over a finite period of time and can select solutions in the Budyko-North case appropriately.

The next section describes the model background, section 3 is devoted to the continuous case, and the general case is considered in the last section.

Throughout, we will employ standard notations as well as several fundamental results from the theory of evolution equations governed by m-accretive operators and refer to [32] for details. In particular, let $a \in \mathbf{R}$, $b \in (a, \infty]$, X be a Banach space and $\Gamma : I \times C([-T, 0], X) \to 2^X$ be a suitable "multi-valued" operator. A function $v : [a - T, b) \to X$ is called a mild (strong) solution of a functional evolution equation $\dot{u} + Au \in \Gamma(t, u_t)$, iff there exists a function $f : [a, b) \to X$ such that v is a mild (strong) solution of $\dot{u} + Au = f(t)$ on $[a, b)$ and $f(t) \in \Gamma(t, v_t)$ for $t \in [a, b)$ a.e..

As a final remark, let us point out that our focus is on the slow diffusion case in this paper. Clearly, sharper regularity results are valid for linear diffusion ($p = 2$) thanks to the theory of analytic semigroups, an aspect which is neglected here. We refer the interested reader to the recent monograph of [33] for a comprehensive overview on semilinear functional evolution equations of parabolic type.

1 Climatological background

Equation (0) arises from a one-layer energy balance climate model (EBM) which is to predict the evolution of a, say, ten-year mean of atmospheric temperature $u = u(t, x)$ at sea level. Here x denotes the position on the earth's surface M and t time in years. Starting point is the balance equation of energy which can be written in the form

$$\partial_t e(t, x) - \nabla \cdot \vec{\jmath}(t, x) = R_{\text{net}}(t, x) \tag{1}$$

where e, $\vec{\jmath}$ and R_{net} denote the internal energy flux, the flux due to horizontal heat transport and the net radiation flux, respectively. EBMs are designed by heuristically deriving functional expressions for e, $\vec{\jmath}$ and R_{net} in terms of u in such a way that the (in the model context) fundamental feedback mechanisms are accounted for. This approach goes back to Budyko [7] and Sellers [30] who proposed independently two such models in 1969. The interested reader is referred to [16] or [17] for an introduction and to the articles in [10] on EBMs for more recent developments.

Let us briefly discuss the significant qualitative features of the three flux terms beginning with the net radiation flux. R_{net} is equal to the difference between the absorbed and emitted radiation fluxes. The absorbed radiation flux is given by $Q(t, x)[1 - \alpha(x, u, w)]$ with Q the ten-year average of the incoming solar radiation flux and α the albedo, i.e. the relative portion of the incoming flux reflected to space. The function α depends on x due to land-water distribution, orography and vegetation zones, but more importantly on temperature which is utilized as an indicator for ice- and snow cover. Budyko and later North and collaborators (see, e.g. [29]) modeled the temperature dependence by means of a step function, which assigns a high value for the albedo to all temperatures below a certain threshold value, usually -10 C (ice- or snow cover occurs in that case), and a lower value, if u lies above that threshold. Sellers and Ghil used roughly speaking a continuous interpolation of such a step function. In linking the albedo to u alone, one neglects the very important long response times which the cryosphere exhibits. E.g. the expansion or the retreat of the huge continental ice sheets occurs with response times of thousands of years, a feature, which Bhattacharya, Ghil and Vulis [4] proposed to incorporate by substituting u by a long term average of u, here called w, e.g. $w(t, x) := \int_{-T}^{0} \beta(s, x) u(t + s, x)\, ds$ with $T \approx 10^4$ years. Of course, one can refine this procedure by having independent ice- and snow-lines. In that case, one understands ice-lines as the boundaries of regions that are covered by continental ice-sheets or huge glaciers (slow response times in comparison with the ten-year mean), whereas snow-lines refer to boundaries of regions where the variations in ice- or snow cover occur on the time-scale of u. This approach was chosen in [21], [18,19,20] and will be employed here, too. The emitted radiation flux is modeled either according to the Stefan-Boltzmann law with temperature in Kelvin (Sellers) or by a first order approximation of empirical

radiation data (Budyko). Our qualitative setting, a strictly increasing function g with $|g(y)| \to \infty$ as $|y| \to \infty$, comprises both cases.
The ten-year mean of the horizontal heat flux is described in EBMs by a diffusive approximation. Most papers use a linear diffusion operator, however following a suggestion of Stone in [31], the first author and collaborators have also considered "slow diffusion", i.e. $p > 2$ (cf. [9,10], [14], [11]) which leads to the generalized p-Laplacian appearing in (0).
Finally, as long as one disregards the latent energy stored in continental ice sheets and glaciers, the internal energy flux e is given by $c(x)u(t,x)$ with c the heat capacity which varies considerably with x due to the land-water distribution. However, a more accurate modeling suggests to set $e(t,x) = c(x)u(t,x) + \mathcal{H}(w(t,x))$ where c denotes the thermal inertia and $\mathcal{H}(w(t,x))$ stands for the latent energy density due to huge ice accumulations. This approach is closely related to the one for the Stefan problem (cf., e.g.,[22] and [28]) with the obvious change that $\mathcal{H}$ should depend on the long-term temperature mean w rather than on u in view of the time scales relevant for the latent fluxes. $\mathcal{H}$ is a nonnegative bounded decreasing function with derivative h having compact support. Using

$$\partial_t e(t,x) = \partial_t[c(x)u(t,x) + \mathcal{H}(w(t,x))] = c(x)\partial_t u(t,x) + h(w(t,x))\partial_t w(t,x)$$

and observing that $\partial_t w(t,x) = \int_{-T}^{0} \beta(s,x)\partial_1 u(t+s,x)\,ds$ in case that u is sufficiently smooth, one obtains

$$c(x)\partial_t u(t,x) + h(w(t,x))\beta(0,x)u(t,x) - h(w(t,x))\int_{-T}^{0}[\partial_s\beta(s,x)u(t+s,x)]\,ds$$

via integration by parts for $\partial_t e(t,x)$ supposing that $\beta(-T,\cdot) \equiv 0$, T the memory span of the system, e.g. $T \approx 10^4$ ys.
Collecting all terms apart from the absorbed radiation flux and a technical correction term in case of Budyko-North type models on the left hand side, one is led to

$$(2)\quad \begin{cases} c(x)\partial_t u(t,x) - \nabla\cdot[k(\cdot)\,|\nabla u(t,\cdot)|^{p-2}\,\nabla u(t,\cdot)](x) \\ \quad +h(w(t,x))\beta(0,x)u(t,x) \\ \quad -h(w(t,x))\int_{-T}^{0}[\partial_s\beta(s,x)u(t+s,x)]\,ds \\ \quad +g(u(t,x)) = f(t,x,u(t,x),w(t,x)),\ t>0,\ x\in M \text{ a.e.}, \\ w(t,x) = \int_{-T}^{0}\beta(s,x)u(t+s,x)\,ds,\ t>0,\ x\in M, \\ u(s,x) = u_0(s,x),\ s\in[-T,0],\ x\in M. \end{cases}$$

Throughout we will employ the following basic hypotheses which cover the various model settings.

(H0) M two-dimensional, compact, oriented Riemannian manifold without boundary; $T > 0$;

(H1) $c,\ k \in C^2(M)$ positive, $p \geq 2$, $\beta \in C^1([-T,0]\times M, \mathbb{R}_+)$, $\beta(-T,\cdot) \equiv 0$, $\beta(s,x) > 0$ for $s \in (-T,0]$ and $x \in M$, $\int_{-T}^{0}\beta(s,x)\,ds = 1$ for $x \in M$;

(H2) (a) $h : \mathbb{R} \to \mathbb{R}$ bounded Lipschitz function;
(b) $g \in C(\mathbb{R})$, $g(0) = 0$, g strictly increasing and odd, $\lim_{y\to\infty} g(y) = \infty$ and g satisfies an arbitrary polynomial growth condition if $p = 2$;
(c) $f\colon \mathbb{R}_+ \times M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ bounded.

Of course, we have to supply regularity hypotheses for f, e.g. one can consider f to be C^1 in case of a Sellers-type model, the situation we begin with.

2 Existence in the continuous case

We are going to establish global mild solvability, uniqueness and continuous dependence for (2) under hypotheses (H0)-(H2), if f belongs to $C(\mathbb{R}_+\times M\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and is a locally Lipschitz function. Moreover, requiring a minimum growth of g at ∞, a property of no climatological relevance, one can derive uniform boundedness of the solution.

2.1 A Hilbert Space Setting for Global Solvability.

Our goal is to rewrite (2) as an evolution problem

$$(3)\quad \begin{cases} \dot u(t) + Bu(t) = \hat F(t,u_t) & t > 0 \\ u(0) = u_0 \end{cases}$$

in $H := \mathrm{L}^2(M)$ equipped with the equivalent inner product

$$\langle\cdot,\cdot\rangle_H : (\phi,\psi) \mapsto \int_M \phi\psi c.$$

To this end let $G(s) = int_0^u g(s)ds$ and $J : H \to (-\infty,\infty]$ be defined by

$$J(\phi) = \begin{cases} \int_M [\frac{1}{p}k(\cdot)\,|\nabla\phi|^p + G\circ\phi] & \text{for } \phi \in W^{1,p}(M), \\ \infty & \text{otherwise.} \end{cases}$$

Notice that if $\phi \in W^{1,p}(M)$ then $G\circ\phi \in \mathrm{L}^1(M)$ since $p > 2$ (M being two-dimensional) implies $W^{1,p}(M) \hookrightarrow C(M)$ and, on the other hand, $W^{1,2}(M) \hookrightarrow \mathrm{L}^q(M)$ for all $q \in [1,\infty)$ leads to this property once that, in this case, g has a polynomial growth. It is known that J is a proper, lower semicontinuous, conv ex functional on H, hence its subdifferential $B := \partial J$ (w.r.t. $\langle\cdot,\cdot\rangle_H$) is m-accretive and dom$(B)$ is dense in H. Before defining $\hat{F}$ let us indicate the significance of utilizing the inner product $\langle\cdot,\cdot\rangle_H$. Denote by $\tilde{A}$ the subdifferential of

$$\Phi : \phi \mapsto \begin{cases} \int_M \frac{1}{p} k(\cdot)|\nabla\phi|^p & \text{for } \phi \in W^{1,p}(M), \\ \infty & \text{otherwise.} \end{cases}$$

with respect to the standard inner product on H —we use $\dot{\partial}$ in this section in order to distinguish the two subdifferentials. One has that dom$(\tilde{A}) \subset W^{1,p}(M)$ is dense in H and

$$\langle \tilde{A}\phi, \psi\rangle_{L^2} = \int_M k(\cdot)\,|\nabla\phi|^{p-2}\,\nabla\phi\cdot\nabla\psi \quad \text{for all } \phi \in \text{dom}(A) \text{ and } \psi \in W^{1,p}(M).$$

Set dom$(\hat{A})$:= dom$(\tilde{A})$ and define $\hat{A}$ by

$$\langle \hat{A}\phi, \psi\rangle_H = \langle \tilde{A}\phi, \psi\rangle_{L^2} \quad \text{for all } \phi \in \text{dom}(\hat{A}) \text{ and } \psi \in W^{1,p}(M).$$

The fact that $\tilde{A} = \dot{\partial}\Phi$ means that dom$(\tilde{A}) = \{\phi \in H : \dot{\partial}\Phi(\phi) \neq \emptyset\}$ and

$$\Phi(\phi) \leq \Phi(\psi) + \langle \tilde{A}\phi, \phi - \psi\rangle_{L^2(M)} \quad \text{for all } \phi \in \text{dom}(\tilde{A}) \text{ and } \psi \in H.$$

Now, dom$(\hat{A})$ = dom$(\tilde{A})$ and

$$\langle \hat{A}\phi, \phi - \psi\rangle_H = \langle \tilde{A}\phi, \phi - \psi\rangle_{L^2(M)},$$

hence $\partial\Phi = \hat{A}$. Likewise, one concludes that

$$\langle B\phi, \psi\rangle_H = \langle \tilde{A}\phi + g\circ\phi, \psi\rangle_{L^2(M)}.$$

A well-known result of Brezis ([5]) guarantees that if $v_0 \in H$, $b \in (0,\infty)$ and $z \in L^2([0,b],H)$, the (unique) mild solution of the initial value problem

$$\begin{cases} \dot{v} + Bv &= z \\ v(0) &= v_0 \end{cases}$$

(i.e., a function limit of step functions obtained by an approximation scheme as described in the proof of Theorem 1 below) is also a strong solution on $[0,b]$. Observing that therefore

$$0 = \langle \dot{v}(t) + Bv(t) - z(t), \psi\rangle_H = \langle c(\cdot)\dot{v}(t) + \tilde{A}v(t) + g\circ v(t) - c(\cdot)z(t), \psi\rangle_{L^2(M)}$$

holds for a.e. $t \in (0,b)$ and all $\psi \in H$ one concludes that v represents a solution of

$$c(x)\partial_t u(t,x) - \nabla\cdot[k(\cdot)|\nabla u(t,\cdot)|^{p-2}\nabla u(t,\cdot)](x) = c(x)z(t)(x) \qquad t \in [0,b], x \in M$$

With this in mind one sets

$$\hat{F}(t,\varphi)(x) := \frac{1}{c(x)}\left[h\left(\int_{-T}^0 \beta(s,x)\varphi(s,x)\,ds\right)\left(-\beta(0,x)\varphi(0,x) + \int_{-T}^0 \partial_s\beta(s,x)\varphi(s,x)ds\right) + f\left(t,x,\varphi(0,x),\int_{-T}^0 \beta(s,x)\varphi(s,x)ds\right)\right]$$
$$t>0,\ x \in M \text{ a.e.},\ \varphi \in C([-T,0],H)$$

and observes that

$$\|\hat{F}(t,\varphi)\|_{L^2(M)} \leq \frac{\|h\|_\infty}{\inf c}\left[\|\beta\|_\infty + T\|\partial_1\beta|_\infty\right]\|\varphi(s,\cdot)\|_{C([-T,0],L^2(M))} + \|f\|_\infty\left(\int_M 1_M\right)^{\frac{1}{2}}.$$

Standard arguments show that $\hat{F} : [0,\infty)\times C([-T,0],H) \to H$ is continuous if f is continuous. Thus, (3) falls into the scope of the existence theory for functional evolution equations with m-accretive principal part as described in [32; chap. 5], and with the preparatory remarks at hand, it is now easy to conclude.

Proposition 2.1 *Let (H0)-(H2) be fulfilled, $u_0 \in C([-T,0],H)$ and $f \in C(\mathbb{R}_+ \times M \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Then we have:*
(a) There exists a $t^* \in (0,\infty]$ and a $u \in C([-T,t^*),H)$ which is a local strong solution of (3),i.e. for any $0 < b < t^*$ $u|_{(0,b)} \in W^{1,1}_{loc}((0,b),H)$ and the following conditions hold:

$$\begin{cases} u(t) \in dom(B) \text{ for } t \in (0,b), \text{ a.e.} \\ \dot{u}(t) + Bu(t) = \hat{F}(t,u_t) \text{ holds for a.e. } t \in (0,b), \\ u(0) = u_0. \end{cases}$$

(b) Every local mild solution of (3) can be extended to a global mild solution on $[-T,\infty)$.

PROOF :
J is of compact type due to $W^{1,p}(M) \hookrightarrow H$ compactly, hence (a) follows from Corollary 5.3.1 of [32]. Moreover, the semigroup generated by B is compact thanks to a result of Brezis ([6], cf. [32; prop. 2.2.2]), thus Theorem 5.3.3 of [32] applies to (3). —— QED

Remark. A straightforward uniqueness proof is available only, if $h \equiv 0$ and f satisfies, say, a global Lipschitz conditions of the form

$$|f(t,x,y_1,z_1) - f(t,x,y_2,z_2)| \leq C_f\left[|y_1 - y_2| + |z_1 - z_2|\right]$$

for $(t,x,y_j,z_j) \in \mathbb{R}_+ \times M \times \mathbb{R} \times \mathbb{R}$ and $j = 1,2$ with C_f a positive constant. Again, there is no harm from a climatological point of view to require the global Lipschitz condition as soon as one opts for local versions.

2.2 A C(M) approach.

The disadvantage of the Hilbert space approach is that the notion of a solution u does neither imply the continuity of $(t,x) \mapsto u(t)(x)$ nor the uniform boundedness of this function on compact intervals. Also, one needs rather strong conditions to guarantee that an operator such as $\tilde{F}$ has good mapping properties. Therefore one wants to realize (2) in a continuous function space setting. The price one pays is that the theory does not provide for any time-differentiability of mild solutions in that context. Understanding $\tilde{A}$ as in the previous subsection one has thanks to $W^{1,p}(M) \hookrightarrow C(M)$ if $p > 2$ and from regularity theory in case $p = 2$ that $\mathrm{dom}(\tilde{A}) \subset C(M)$, hence one can introduce an operator A by setting

$$\mathrm{dom}(A) := \left\{\phi \in \mathrm{dom}(\tilde{A})\colon \tilde{A}\phi \in C(M)\right\} \text{ and } A\phi = \frac{1}{c(\cdot)}\tilde{A}\phi \quad \forall \phi \in \mathrm{dom}(A).$$

It can be shown that A is accretive, hence m-accretive in view of its definition. Moreover, $\mathrm{dom}(A)$ is dense in $C(M)$ thanks to the fact that M does not have a boundary. Assuming that f is continuous one can define a mapping

$$F \in C([0,\infty) \times C([-T,0], C(M)) \to C(M)$$

by

$$\begin{aligned}F(t,\varphi)(x) := \frac{1}{c(\cdot)}\Big[&-h\left(\int_{-T}^{0}\beta(s,x)\varphi(s,x)\,ds\right)\beta(0,x)\varphi(0,x)\\ &+h\left(\int_{-T}^{0}\beta(s,x)\varphi(s,x)\,ds\right)\int_{-T}^{0}\partial_s\beta(s,x)\varphi(s,x)ds\\ &+f(t,x,\varphi(0,x)),\int_{-T}^{0}\beta(s,x)\varphi(s,x)ds) - g(\varphi(0,x))\Big]\end{aligned}$$

for $t > 0$, $x \in M$ and $\varphi \in C([-T,0], C(M))$. Then (2) corresponds to the initial value problem

$$(4) \qquad \begin{cases}\dot{u}(t) + Au(t) = F(t,u_t) & t > 0\\ u(0) = u_0\end{cases}$$

and one obtains:

Proposition 2.2 *Let (H0)-(H2) be fulfilled, $u_0 \in C([-T,0], C(M))$ and $f \in C([0,\infty) \times M \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$. Then we have:*
(a) There exists a local mild solution of (4), and every noncontinuable mild solution of (4) is defined on $[-T,\infty)$.
(b) Suppose that f satisfies a local Lipschitz condition of the form: For each $r > 0$ there exists a $C_f(r) \in (0,\infty)$ with

$$|f(t,x,y_1,z_1) - f(t,x,y_2,z_2)| \le C_f(r)\,[|y_1 - y_2| + |z_1 - z_2|]$$

whenever $(t,x,y_j,z_j) \in [0,\infty) \times M \times \mathbf{R} \times \mathbf{R}$, $j = 1,\ 2$ and

$$\max\{t, |y_1|, |y_2|, |z_1|, 1z_2|\} \le r.$$

Then the mild solution is uniquely determined and depends continuously on u_0

PROOF :

Part (a) follows by quite the same reasoning as for Proposition 1. As for (b), it is a matter of routine to derive that (H2) and the Lipschitz condition required for f imply:

For each b, $r \in \mathbf{R}_+$ there exists a $C(b,r) \in \mathbf{R}_+$ such that

$$\|F(t,\varphi_1) - F(t,\varphi_2)\|_\infty \le C(b,r)\|\varphi_1 - \varphi_2\|_\infty$$

for all $t \in [0,b]$ and φ_1, $\varphi_2 \in C([-T,0], C(M))$ with $\|\varphi_j\|_\infty \le r$ for $j = 1,\ 2$.

Now, let u and v be two mild solutions of $\dot{u} + Au = F(\cdot, u_\cdot)$ on $[-T,\infty)$. A theorem of Benilan ([3], cf. [32; Theorem 1.7.5]) implies that

$$\begin{aligned}\|u(t) - v(t)\|_\infty &\le \|u(0) - v(0)\|_\infty + \textstyle\int_0^t [u(\tau) - v(\tau), F(\tau,u_\tau) - F(\tau,v_\tau)]_+ d\tau\\ &\le \|u(0) - v(0)\|_\infty + \textstyle\int_0^t \|F(\tau,u_\tau) - F(\tau,v_\tau)\|_\infty d\tau\end{aligned}$$

for each $t \in \mathbf{R}_+$. Here, $[\cdot,\cdot]_+$ denotes the normalized upper semi-inner product on $C(M)$, i.e.

$$[\phi,\psi]_+ = \begin{cases}\max\{\psi(x)\mathrm{sgn}(\phi(x)) : x \in M, |\phi(x)| = \|\phi\|_\infty\} & \text{if } \phi \not\equiv 0\\ \|\psi\|_\infty & \text{if } \phi \equiv 0.\end{cases}$$

Choose $b > 0$ and $r \in \mathbf{R}_+$ such that $\max\{\|u|_{[-T,b]}\|_\infty, \|v|_{[-T,b]}\|_\infty\} \le r$. Then

$$\begin{aligned}\|u(t) - v(t)\|_\infty &\le \|u(0) - v(0)\|_\infty + C(b,r)\textstyle\int_0^t \|u_\tau - v_\tau\|_\infty d\tau\\ &\le \|u_0 - v_0\|_\infty + C(b,r)\textstyle\int_0^t \|u_\tau - v_\tau\|_\infty d\tau\end{aligned}$$

for all $t \in [0,b]$. Thus, Gronwall's inequality yields

$$\|u_t - v_t\|_\infty \le \|u_0 - v_0\|_\infty \exp(C(b,r)t)$$

for $t \in [0,b]$, which implies uniqueness and continuous dependence by standard arguments.
—— QED

The same reasoning can be employed to ensure uniqueness in the Hilbert space setting for mild solutions with continuous initial conditions.

Corollary 2.3 *Let (H0)-(H2) be fulfilled, and let $u_0 \in C([-T,0], C(M))$ and $f \in C([0,\infty) \times M \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$. Assume that there exists a $C_f \in \mathbf{R}_+$ with*

$$|f(t,x,y_1,z_1) - f(t,x,y_2,z_2)| \le C_f\,[|y_1 - y_2| + |z_1 - z_2|]$$

whenever $(t,x,y_j,z_j) \in [0,\infty) \times M \times \mathbf{R} \times \mathbf{R}$, $j = 1,\ 2$. Then the unique global mild solution of (4) guaranteed by Proposition 2. is the unique mild solution of (3), hence the unique strong solution of (3).

PROOF :

Let u be the mild solution guaranteed by part (b) of the previous proposition. Since $C(M) \hookrightarrow L^2(M)$ and $\mathrm{Graph}(A) \subset \mathrm{Graph}(\frac{1}{c(\cdot)}\tilde{A})$ (notations from 3.1.), u is a mild solution of $\dot{v} + \frac{1}{c(\cdot)}\tilde{A}v = F(t, u_t)$, hence of (3). Now, let v be a mild solution of (3), then the inner product version of Benilan's result used before yields

$$(5) \qquad \|u(t) - v(t)\|^2 \leq \int_0^t < u(\tau) - v(\tau), \hat{F}(\tau, u_\tau) - \hat{F}(\tau, v_\tau) > d\tau$$

for all $t \in \mathbf{R}_+$, and it suffices to establish for $b \in \mathbf{R}_+$ that there exists of a $\hat{C}_u(b) \in \mathbf{R}_+$ with

$$\langle u(\tau) - v(\tau), \hat{F}(\tau, u_\tau) - \hat{F}(\tau, v_\tau)\rangle \leq C_u(b)\|u_\tau - v_\tau\|^2_{C([-T,0],H)}.$$

One knows that $u|_{[-T,b]}$ is bounded and obtains therefore for example, for the usual norm in $L^2(M)$:

$$\left\| h\left(\int_{-T}^0 \beta(s,\cdot)v(\tau+s,\cdot)ds\right)\beta(0,\cdot)v(\tau,\cdot) - h\left(\int_{-T}^0 \beta(s,\cdot)u(\tau+s,\cdot)ds\right)\beta(0,\cdot)u(\tau,\cdot)\right\|$$

$$\leq \|h(\int_{-T}^0 \beta(s,\cdot)v(\tau+s,\cdot)ds)\beta(0,\cdot)v(\tau,\cdot) - h(\int_{-T}^0 \beta(s,\cdot)v(\tau+s,\cdot)ds)\beta(0,\cdot)u(\tau,\cdot)\|$$

$$+\|h(\int_{-T}^0 \beta(s,\cdot)v(\tau+s,\cdot)ds)\beta(0,\cdot)u(\tau,\cdot) - h(\int_{-T}^0 \beta(s,\cdot)u(\tau+s,\cdot)ds)\beta(0,\cdot)u(\tau,\cdot)\|$$

$$\leq \|h\|_\infty\|\beta\|_\infty\|v(\tau) - u(\tau)\|$$

$$+[h]_{\mathrm{Lip}}\|\beta\|_\infty\|u|_{[-T,b]}\|_\infty\| \int_{-T}^0 \beta(s,\cdot)v(\tau+s,\cdot)ds - \int_{-T}^0 \beta(s,\cdot)u(\tau+s,\cdot)ds\|$$

$$\leq \left[\|h\|_\infty\|\beta\|_\infty + T[h]_{\mathrm{Lip}}\|\beta\|^2_\infty\|u|_{[-T,b]}\|_\infty\right]\|u_\tau - v_\tau\|_{C([-T,0],L^2(M))}.$$

Here, $[h]_{\mathrm{Lip}}$ denotes the smallest Lipschitz factor of h. Likewise, the term

$$h\left(\int_{-T}^0 \beta(s,\cdot)v(s,\cdot)ds\right) \int_{-T}^0 \partial_s\beta(s,\cdot)v(s,\cdot)ds - h\left(\int_{-T}^0 \beta(s,\cdot)u(s,\cdot)ds\right) \int_{-T}^0 \partial_s\beta(s,\cdot)u(s,\cdot)ds$$

can be treated, whereas it is routine to establish a global Lipschitz condition for the term involving f thanks to the required global Lipschitz condition. Consequently, we get from (5) and the estimate for the right hand side that

$$\|u_t - v_t\|^2_{C([-T,0],H)} \leq C_u(b)\int_0^t \|u_\tau - v_\tau\|^2_{C([-T,0],H)} \qquad \forall t \in [0,b],$$

which implies $u_t = v_t$ for $t \in [0,b]$ thanks to $u_0 = v_0$. —— QED

Remark. (a) The corollary tells us that the global mild solution of (4) is also a strong solution in the L^2-sense. Consequently,

$$u \in C([-T,\infty), C(M)) \cap W^{1,1}_{\mathrm{loc}}([0,\infty), L^2(M))$$

and $u(t) \in \mathrm{dom}(A)$ for a.e. $t \in (0,\infty)$, hence $(t,x) \mapsto u(t,x)$ can be considered a weak solution of (2), and it is therefore possible to apply regularity results for degenerate quasilinear parabolic equations (cf.[15]).
(b) It is not too difficult to see that a rather similar approach should work in an L^∞-setting (cf. [14] for a related problem without delays).

2.3 Global boundedness.

In order to establish the uniform boundedness of the solutions, we require at least linear growth of g at infinity. Again this hypothesis can always be realized for the climatological applications by modifying the expression for g outside the climatologically relevant range, if necessary. We have:

Theorem 2.4 *Let (H0)-(H2) be fulfilled, $u_0 \in C([-T,0], C(M))$ and $f \in C([0,\infty) \times M \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$. Suppose that*

$$\liminf_{y\to\infty} \frac{g(y)}{y} > \|h\|_\infty \left[\|\beta(0,\cdot)\|_\infty + \sup_{x\in M}\int_{-T}^0 |\partial_1\beta(s,x)|ds\right].$$

Then every (global) mild solution u of (4) is bounded.

PROOF :

The notations of the previous two subsections are used. Select $r \in (\|u_0\|_\infty, \infty)$ with

$$g(y) - \|h\|_\infty\left[\|\beta(0,\cdot)\|_\infty + \sup_{x\in M}\int_{-T}^0 |\beta(s,x)|\,ds\right](y+1) - \|f\|_\infty \geq \|c\|_\infty$$

for all $y \geq r$. Assume that $\|u\|_\infty > r$. Then there exists a $\rho \in (r, r+1)$ and a $\bar{t} \in (0,\infty)$ with $\|u(t)\|_\infty < \rho$ for $t \in [0,\bar{t})$ and $\|u(\bar{t})\|_\infty = \rho$. Set $z(t) = F(t,u_t)$ and note $z \in C([0,\infty), C(M))$. Therefore $u|[0,\bar{t}]$ is a mild solution of

$$(6) \qquad \begin{cases} \dot{v} + Av = z & \text{on } [0,\bar{t}], \\ u(0) = u_0(0). \end{cases}$$

Choose $\varepsilon \in \left(0, \min\{\frac{\rho-r}{4}, \frac{r-\|u_0\|_\infty}{4}, \frac{1}{2}\}\right)$ with $\|u(t,\cdot)\|_\infty \geq \frac{3\rho+r}{4}$ for $t \in [\bar{t}-\varepsilon, \bar{t}]$ and $\|u(t,\cdot)\|_\infty \leq \frac{3\|u_0\|_\infty + r}{4}$ for $t \in [0,\varepsilon]$. Since $u|[0,\bar{t}]$ is a mild solution, one finds an ε-discretization $D_A(\varepsilon, t_0, t_1, \ldots, t_n, z_0, z_1, \ldots, z_n)$ for (6) on $[0,\bar{t}]$ and a solution v of that discretization such that $\|u(t) - v(t)\|_\infty \leq \varepsilon$ for $t \in [t_0, t_n]$.

Recall that $D_A(\varepsilon, t_0, t_1, \dots, t_n, z_0, z_1, \dots, z_n)$ is called an ε-discretization of (6) on $[0, \bar{t}]$ iff

- $0 \le t_0 \le t_1, \dots, t_n \le \bar{t}$, $t_0 \le \varepsilon$, $t_j - t_{j-1} \le \varepsilon$ for $1 \le j \le n$, $\bar{t} - t_n \le \varepsilon$;
- $z_0, z_1, \dots, z_n \in C(M)$;
- $\sum_{1 \le j \le n} \int_{t_{j-1}}^{t_j} \|z(s) - z_j\|_\infty \, ds \le \varepsilon$.

Moreover, a step function $v : [t_0, t_n] \to C(M)$ satisfying $v|(t_{j-1}, t_j]$ constant for $1 \le j \le n$, is called a solution of this ε-discretization of (6) on $[0, \bar{t}]$, iff $\mathrm{ran}(v) \subset \mathrm{dom}(A)$ and

$$\frac{v(t_j) - v(t_{j-1})}{t_j - t_{j-1}} + Av_j = z_j. \qquad 1 \le j \le n.$$

Since z is continuous, one can assume

$$\sup_{0 \le t \le b} \|h(t) - \sum_{1 \le j \le n} z_j \mathbf{1}_{(t_{j-1}, t_j]}\|_\infty \le \frac{\varepsilon}{1 + \bar{t}}$$

without loss of generality. One obtains thanks to these choices that $\|u(t_0)\|_\infty \le r - \varepsilon$ and $\|u(t_n)\|_\infty > \frac{r+3\rho}{4}$, hence there is an $i \in \{1, \dots, n\}$ with $\|v(t_j)\|_\infty < \frac{r+\rho}{2}$ for $0 \le j \le i-1$ and $\|v(t_i)\|_\infty \ge \frac{r+\rho}{2}$. Suppose that there is a $\bar{x} \in M$ with $v(t_i, \bar{x}) = \|v(t_i)\|_\infty \ge \frac{r+\rho}{2}$. Since $v(t_i) \in \mathrm{dom}(A) \subset W^{1,p}(M)$, it follows that $\zeta := (v(t_i) - \frac{r+\rho}{2})^+ \in W^{1,p}(M)$ and

$$\int_M Av(t_i)\zeta = \int_{\{\zeta > 0\}} k(\cdot)|\nabla v(t_i)|^p,$$

thus

$$0 \le \int_M \frac{v(t_i) - v(t_{j-i})}{t_i - t_{i-1}} \zeta \le \int_M h_{i\zeta = \int_{\{\zeta > 0\}} h_i \zeta}.$$

Now, $v(t_i)(x) > \frac{r+\rho}{2}$ implies $u(t_i)(x) > \frac{3r+\rho}{4} > r$, hence

$$F(t_i, u_{t_i})(x) \le -g(r) + \|f\|_\infty$$

$$+\|h\|_\infty \left(\|\beta(0, \cdot)\|_\infty |u(t_i, x)| + \|\partial_1 \beta(\cdot, x)\|_{L^1([-T,0])} \sup_{t_i - T \le s \le t_i} |u(s, x)| \, ds \right)$$

$$\le -g(r) + \|f\|_\infty + \|h\|_\infty \left[\|\beta(0, \cdot)\|_\infty + \sup_{x \in M} \|\partial_1 \beta(\cdot, x)\|_{L^1([-T,0])} \right] \rho \le -\|c\|_\infty;$$

thus $z_i(x) < 0$ for these x. Consequently, $\int_{\{\zeta > 0\}} z_i \zeta < 0$, which is a contradiction. Therefore, there is a $\underline{x} \in M$ with $v(t_i, \underline{x}) = -\|v(t_i)\|_\infty \le -\frac{r+\rho}{2}$. Set $\eta := (-v(t_i) - \frac{r+\rho}{2})^+$ and observe that

$$\int_M Av(t_i)\eta = -\int_{\{\eta > 0\}} k(\cdot)|\nabla v(t_i)|^p \le 0.$$

Arguing as in the previous case one now obtains

$$0 \ge \int_M \frac{v(t_i) - v(t_{j-i})}{t_i - t_{i-1}} \eta \ge \int_M z_i \eta = \int_{\{\eta > 0\}} z_i \eta$$

and

$$c(x) F(t_i, u_{t_i})(x) \ge g(r) - \|f\|_\infty - \|h\|_\infty \left[\|\beta(0, \cdot)\|_\infty + \sup_{x \in M} \|\partial_1 \beta(\cdot, x)\|_{L^1([-T,0])} \right] \rho$$

$$\ge \|c\|_\infty$$

in view of $u(t_i)(x) < -\frac{3r+\rho}{4} < -r$ whenever $\eta(x) > 0$, which shows $\int_{\{\eta > 0\}} z_i \eta > 0$, again a contradiction. Thus, $\|u\|_\infty \le r$. —— QED

Remarks.

(a) We record from the proof for later purposes. If r_0 is chosen such that

$$g(y) - \|h\|_\infty \left[\|\beta(0, \cdot)\|_\infty + \sup_{x \in M} \int_{-T}^0 |\beta(s, x)| ds \right] (y + 1) - \|f\|_\infty \ge \|c\|_\infty$$

for all $y \ge r_0$ and $r > r_0$, then $\|u\|_\infty \le r$ for all mild solutions u of (4) with $\|u_0\| \le r$. Observe that the proof shows $\|u\|_\infty \le r + \delta$ for every $\delta > 0$.

(b) Actually, a careful analysis of the arguments of the proof shows that $\|u(t, \cdot)\|_\infty < r$ for all $t \in (0, \infty)$. Assuming that f satisfies the local Lipschitz condition of part (b) of Proposition 2. and is independent of t (autonomous case) one can conclude that the closed ball $\mathcal{B}_{r_0}$ in $C([-T, 0], C(M))$ with center $\varphi \equiv 0$ and radius r_0 absorbs all solutions of (9) uniformly on bounded subsets of initial conditions, i.e. for each $r \in (0, \infty)$ there exists a $t_0 \in (0, \infty)$ such that for each $u_0 \in \mathcal{B}_r$ one has $u_t \in \mathcal{B}_{r_0}$ provided that $t \ge t_0$ and u is the soluion of (6) for that u_0. This yields that the solution semiflow possesses a compact attractor in that case.

(c) *First Open Problem.* Assume that one deals with temperature in Kelvin, $p = 2$ and $h \equiv 0$. It is well-known [21] that solutions u of (6) are nonnegative in that case whenever $u_0 \ge 0$. Quite similar arguments as those employed in the last part of the previous proof should yield the same statement for $p > 2$. In the general case one has $\mathrm{supp}(h) \subset (0, \infty)$ for models with temperature in Kelvin, but the invariance of the nonnegative cone is not at all obvious from the above proof. Since this invariance is a natural climatological postulate, can it be proven for the situation under consideration?

3 Existence if f has discontinuities

One obtains as already mentioned in Section 2. albedo functions with jump discontinuities at the snow-line from Budyko-North type albedo parameterization.

This leads in the framework under consideration to a function f with a jump discontinuity along a hyperplane. More precisely, one may think of f to be given by

$$f(t,x,u,w) = \begin{cases} f^\flat(t,x,u,w) & \text{if } t \geq 0,\ x \in M,\ u \in (-\infty, \check{u}],\ w \in \mathbf{R}, \\ f^\sharp(t,x,u,w) & \text{if } t \geq 0,\ x \in M,\ [\check{u}, \infty),\ w \in \mathbf{R}, \end{cases}$$

where $\check{u} \in \mathbf{R}$, $f^\sharp \in C^2(\mathbf{R}_+ \times M \times (-\infty, \check{u}] \times \mathbf{R}, \mathbf{R})$ and $f^\flat \in C^2(\mathbf{R}_+ \times M$ $[\check{u}, \infty) \times \mathbf{R}, \mathbf{R})$ with $0 < \inf f^\flat$, $\sup f^\flat < \inf f^\sharp$ and $\sup f^\sharp < \infty$.
There are several solution concepts for differential equations with (jump) discontinuous "right hand sides", e.g. one fills in the gaps, a process which leads to multivalued equations. This approach has recently been employed by the first author and collaborators in dealing with models without memory, and will be utilized here, too. It turns out that one then can treat more general right hand sides than the one indicated by the example previously outlined. We assume (H0), (H1), (H2) (a) and (b) and

(H3) $\Gamma: \mathbf{R}_+ \times M \times \mathbf{R} \times \mathbf{R} \to 2^{\mathbf{R}}$ upper semicontinuous and bounded; $\Gamma(t,x,y,z)$ a nonempty compact interval for (t,x,y,z)

in dealing with the "set-valued" version

$$(7)\quad \begin{cases} c(x)\partial_t u(t,x) - \nabla \cdot [k(\cdot)\,|\nabla u(t,\cdot)|^{p-2}\,\nabla u(t,\cdot)](x) \\ \qquad +h(w(t,x))\beta(0,x)u(t,x) \\ \qquad -h(w(t,x)) \int_{-T}^{0} [\partial_s \beta(s,x) u(t+s,x)]\,ds \\ \qquad +g(u(t,x)) \in \Gamma(t,x,u(t,x),w(t,x)) \qquad t>0,\ x \in M \text{ a.e.}, \\ w(t,x) = \int_{-T}^{0} \beta(s,x)u(t+s,x)\,ds,\ t>0,\ x \in M, \\ u(s,x) = u_0(s,x),\ s \in [-T,0],\ x \in M. \end{cases}$$

of (2). In the sequel we are going to employ several concepts and results from set-valued analysis. We refer to [2] and [8] for details.
As pointed out in [8; Ex. 1.3.] one finds an upper semicontinuous (single-valued) function $\bar{\gamma}: \mathbf{R}_+ \times M \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and a lower semicontinuous function $\underline{\gamma}: \mathbf{R}_+ \times M \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ with

$$\Gamma(t,x,y,z) = [\underline{\gamma}(t,x,y,z), \bar{\gamma}(t,x,y,z)] \text{ for } (t,x,y,z) \in \mathbf{R}_+ \times M \times \mathbf{R} \times \mathbf{R}.$$

Moreover, the graph of Γ, i.e.

$$\{((t,x,y,z),\gamma): \gamma \in \Gamma(t,x,y,z),\ (t,x,y,z) \in \mathbf{R}_+ \times M \times \mathbf{R} \times \mathbf{R}\},$$

is closed [8; prop. 1.2.(b)].

Theorem 3.1 *Let (H0), (H1), (H2) (a) and (b) and (H3) be fulfilled. Suppose that g satisfies*

$$\liminf_{y\to\infty} \frac{g(y)}{y} > \|h\|_\infty \left[\|\beta(0,\cdot)\|_\infty + \sup_{x\in M} \int_{-T}^{0} |\partial_1 \beta(s,x)|ds \right].$$

Then, for every $u_0 \in C([-T,\infty), C(M))$, there exists a bounded (L^2)-mild solution $u \in C([-T,\infty), C(M))$.

PROOF :

The Approximate Selection Theorem for upper semicontinuous mappings (cf.[2; Theorem 9.2.1.] and [1; Theorem 1, sect. 1.12.]) guarantees the existence of a sequence $(\gamma_j)_{j\in\mathbf{N}}$ of bounded locally Lipschitz functions on $\mathbf{R}_+ \times M \times \mathbf{R} \times \mathbf{R}$ such that $\operatorname{dist}(((t,x,y,z),\gamma_j(t,x,y,z)), \operatorname{Graph}(\Gamma)) \to 0$ as $j \to \infty$ for every $(t,x,y,z) \in \mathbf{R}_+ \times M \times \mathbf{R} \times \mathbf{R}$. Part (b) of Prop. 2. guarantees for each $j \in \mathbf{N}$ the existence of a unique global mild solution u_j of (4) with $f = \gamma_j$ in the definition of F. The sequence $(u_j)_{j\in\mathbf{N}}$ is uniformly bounded as part (a) of the Remarks to Theorem 2. reveals. In fact, denote by C a bound for the absolute values of numbers in the convex hull of

$$\operatorname{Im}(\Gamma) := \bigcup_{(t,x,y,z)\in\mathbf{R}_+\times M\times\mathbf{R}\times\mathbf{R}} \Gamma(t,x,y,z)$$

and choose $r \in (\|u_0\|_\infty, \infty)$ with

$$g(y) - \|h_\infty \left[\|\beta(0,\cdot)\|_\infty + \sup_{x\in M} \int_{-T}^{0} |\beta(s,x)|ds \right] (y+1) - C_\Gamma \geq \|c\|_\infty$$

for all $y \geq r$, then $\sup_{j\in\mathbf{N}} \|u_j\|_\infty \leq r$. Let us write w_j for

$$(t,x) \mapsto \int_{-T}^{0} \beta(s,x) u_j(t+s,x)\,ds$$

and set

$$k_j(t,x) := \frac{1}{c(x)} \Big(\gamma_j(t,x,u_j(t,x),w_j(t,x)) - h(w_j(t,x))\beta(0,x)u_j(t,x) + h(w_j(t,x)) \int_{-T}^{0} [\partial_s \beta(s,x) u_j(t+s,x)]\,ds - g(u_j(t,x)) \Big)$$

for all $(t,x,j) \in \mathbf{R}_+ \times M \times \mathbf{N}$. Noting that u_j solves mildly the initial value problem

$$(8)\quad \begin{cases} \dot{v} + Av = k_j \\ v(0) = u_0(0,\cdot) \end{cases}.$$

in C(M) for $j \in \mathbf{N}$, where A has the same meaning as in Section 3.2., that $(k_j)_{j\in\mathbf{N}}$ is uniformly bounded and that A generates a compact semigroup on $C(M)$, one can employ a theorem of Baras (cf. [32;Theorem 2.3.2.]), which implies for $n \in \mathbf{N}$ that $\{u_j|_{[0,n]\times M} : j \in \mathbf{N}\}$ is relatively compact in $C([0,n], C(M))$ for $n \in \mathbf{N}$. Therefore Cantor's diagonal argument yields a subsequence —also called (u_j)— which converges uniformly on compact intervals to a function $u_\infty \in$

$C([-T,\infty), C(M))$. Note that $u_\infty|_{[-T,0]} = u_0$ and $\|u_\infty\|_\infty \le r$. Fixing $n \in \mathbf{N}$ one observes that $(k_j|_{[0,n]\times M})_{j\in\mathbf{N}}$ is a bounded sequence in $L^2([0,n]\times M)$, hence possesses a weakly convergent subsequence. Now, employing Cantor's diagonal argument once more, one can suppose by passing to a suitable subsequence, if necessary, that there exists an $k_\infty\colon \mathbf{R}_+\times M \to \mathbf{R}$ with $k_\infty|_{[0,n]\times M} \in L^2([0,n]\times M)$ for $n\in\mathbf{N}$ and $k_j|_{[0,n]\times M} \rightharpoonup k_\infty|_{[0,n]\times M}$ (weakly in $L^2([0,n]\times M)$). Moreover, one observes that u_j solves mildly the initial value problem

$$(9)\qquad \begin{cases} \dot v + \hat A v = k_j \\ v(0) = u_0(0,\cdot) \end{cases}$$

in H for $j\in\mathbf{N}$, where H and $\hat A$ have the same meaning as in Section 3.1.. Since $\hat A$ also generates a compact semigroup one can appeal to [32; Corollary 2.3.1] and obtains that u_∞ is a mild solution of (9) for $j=\infty$. Therefore it remains to show that

$$\begin{aligned}\hat\gamma_\infty(t,x) := c(x)k_\infty(t,x) - \Big[h(w_\infty(t,x))\beta(0,x)u_\infty(t,x) \\ - h(w_\infty(t,x))\int_{-T}^0 [\partial_s\beta(s,x)u_\infty(t+s,x)]\,\mathrm{d}s + g(u_\infty(t,x))\Big] \\ \in \Gamma(t,x,u_\infty(t,x),w_\infty(t,x))\end{aligned}$$

holds for a.e. $(t.x)\in[0,n]\times M$ and all $n\in\mathbf{N}$.
Assume this is not true. Then there exists an $n\in\mathbf{N}$ and a (without loss of generality) compact set $E\subseteq[0,n]\times M$ of positive measure with

$$\hat\gamma_\infty(t,x)\notin\Gamma(t,x,u_\infty(t,x),w_\infty(t,x)) \text{ for } (t,x)\in E.$$

Set

$$\hat\gamma_j(t,x) := \gamma_j(t,x,u_j(t,x),w_j(t,x)) \text{ for } (t,x)\in[0,n]\times M.$$

Since

$$\begin{aligned}(t,x)\mapsto\ & h(w_j(t,x))\beta(0,x)u_j(t,x) \\ & -h(w_j(t,x))\textstyle\int_{-T}^0[\partial_s\beta(s,x)u_j(t+s,x)]\,\mathrm{d}s + g(u_j(t,x))\end{aligned}$$

converges to

$$\begin{aligned}(t,x)\mapsto\ & h(w_\infty(t,x))\beta(0,x)u_\infty(t,x) \\ & -h(w_\infty(t,x))\textstyle\int_{-T}^0[\partial_s\beta(s,x)u_\infty(t+s,x)]\,\mathrm{d}s + g(u_\infty(t,x))\end{aligned}$$

uniformly on $[0,n]\times M$ and $k_j|_{[0,n]\times M}\rightharpoonup k_\infty|_{[0,n]\times M}$, it follows that $\hat\gamma_j\rightharpoonup\hat\gamma_\infty$ in $L^2([0,n\times M)$. Using the representation

$$\Gamma(t,x,y,z) = [\underline{\gamma}(t,x,y,z),\bar\gamma(t,x,y,z)] \text{ for } (t,x,y,z)\in\mathbf{R}_+\times M\times\mathbf{R}\times\mathbf{R}$$

with $\underline{\gamma}$ lower semicontinuous and $\bar\gamma$ upper semicontinuous one obtains that at least one of the sets $\{(t,x)\in E : \hat\gamma_\infty(t,x) > \bar\gamma(t,x,u_\infty(t,x),w_\infty(t,x))\}$ and $\{(t,x)\in E : \hat\gamma_\infty(t,x) < \underline{\gamma}(t,x,u_\infty(t,x),w_\infty(t,x))\}$ has positive measure. Let us consider the first case. Setting $\bar\gamma_\infty(t,x) := \bar\gamma(t,x,u_\infty(t,x),w_\infty(t,x))$ for all $(t,x)\in E$ one has that $\int_E\hat\gamma_\infty > \int_E\bar\gamma_\infty$. On the other hand, one knows from the choice of γ_j that

$$\operatorname{dist}((t,x,u_j(t,x),w_j(t,x)),\hat\gamma_j(t,x)),\operatorname{Graph}(\Gamma))\to 0 \text{as } j\to\infty \text{ for } (t,x)\in E.$$

This implies that for each $(t,x)\in E$, there are sequence $(s_j)\in[0,\infty)^{\mathbf{N}}$, $(m_j)\in M^{\mathbf{N}}$, $(y_j)\in\mathbf{R}^{\mathbf{N}}$, $(z_j)\in\mathbf{R}^{\mathbf{N}}$ and (ς_j) with

$$\begin{cases}\varsigma_j\in\Gamma(s_j,m_j,y_j,z_j), \\ \operatorname{dist}((s_j,m_j,y_j,z_j),(t,x,u_j(t,x),w_j(t,x))\to 0\end{cases}$$

and

$$|\hat\gamma_j(t,x)-\varsigma_j|\to 0.$$

Thus, $s_j\to t$, $m_j\to x$, $y_j\to u_\infty(t,x)$ and $z_j\to w_\infty(t,x)$. Now,

$$\varsigma_j\le\bar\gamma(s_j,m_j,y_j,z_j) \text{ for } j\in\mathbf{N}$$

and $\bar\gamma$ upper semicontinuous imply that

$$\limsup_{j\to\infty}\varsigma_j\le\limsup_{j\to\infty}\bar\gamma(s_j,m_j,y_j,z_j)\le\bar\gamma_\infty(t,x),$$

consequently $\limsup_{j\to\infty}\hat\gamma_j(t,x)\le\bar\gamma_\infty(t,x)$ for $(t,x)\in E$. Since $(\hat\gamma_j(t,x))$ is uniformly bounded and the measure of E is finite, one obtains by means of Lebesgue's dominated convergence theorem that

$$\limsup_{j\to\infty}\int_E\hat\gamma_j\le\int_E\limsup_{j\to\inf ty}\hat\gamma_j\le\int_E\bar\gamma_\infty$$

and from $\hat\gamma_j\rightharpoonup\hat\gamma_\infty$ in $L^2([0,n\times M)$ that $\lim_{j\to\infty}\int_E\hat\gamma_j=\int_E\hat\gamma_\infty$. Thus,

$$\int_E(\hat\gamma_\infty-\bar\gamma_\infty)\le 0,$$

which is a contradiction. The second case can be treated likewise. —— QED

Remarks. 1. The proof of Theorem 2. reveals that for each $u_0\in C([-T,0]\times M,\mathbf{R})$, one can find at least one global L^2-mild solution $u\in C([-T,\infty]\times M,\mathbf{R})$ with the following approximation property. There exists a sequence of locally Lipschitz approximate selections of Γ such that the unique mild solutions in $C(M)$ to these approximate selections converge uniformly on compacta to u. This is significant from the underlying climatological point of view. It means, roughly speaking, that at least one solution to the initial value problem from a Budyko-North type model setting is the limit of a sequence of solutions to approximating Sellers type initial value problems with steeper and steeper slopes of the albedo at the snow-line.

2. We cannot expect uniqueness as the first author already showed for equations without memory and $h \equiv 0$. An open question is whether all solutions of the initial value problem, if there are more than one, can be approximated in that way by Sellers type "solutions". One should be aware that there is no straightforward solution concept for the discontinuous case, cf. the discussion in [8] for ordinary differential equations. Therefore it is very much the meaning for the specific application which should determine the choice of the respective concept.
3. A main reason for utilizing a $C(M)$ setting is that this approach allows to establish global boundedness very naturally. One could also think of using a fixed point theorem for set-valued mappings, but again one would need an extra argument for deriving the global boundedness of the solution. Actually, it is not obvious that (7) has only bounded global mild solutions under the assumptions of Theorem 2.
4. The fact that u is a mild solution in $L^2(M)$ and $B.$ (as in Section 3.1.) is the subdifferential of a proper lower semicontinuous convex functional, allows to employ a Theorem of Brezis ([5], cf. [32; Theorem 1.9.3]) which shows that u is actually a strong solution of (9) in case $j = \infty$ and gives various growth estimates. In particular, one obtains $u(t, \cdot) \in \mathrm{dom}(B)$ for a.e. $t \in (0, \infty)$ (spatial regularity), $u \in W^{1,1}_{\mathrm{loc}}((0,\infty), L^2(M))$ and $t \mapsto \sqrt{t}\dot{u}(t) \in L^2_{\mathrm{loc}}([0,\infty), L^2(M))$.

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