ON A PROBLEM LAKING A CLASSICAL SOLUTION IN LUBRICATION THEORY.

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Modeling.

The main goal of this note is to present the mathematical treatment of a problem arising in hydrodynamic lubrication, relevant in the applications, which leads to a formulation laking a classical solution. So. the solvability must be necessarily boarded in terms of weak solutions. This type of arguments, justifying the needed of weak solutions. is typical of nonlinear hyperbolic equations. That we underline in this paper is that this situation also arise with some linear elliptic equations (which are relevant in the applications and not a merely mathematical exercise searched as a sophisticated counterexample).

Consider the problem of the lubricating the friction between a fixed rigid solid presenting some abrupt edges and a regular surface in movement by using an incompressible fluid in the separating region. This kind of problem frequently appears in different engineering applications, as in "feedbox" or "shaft-bearing" systems. We assume, for simplicity that the surface reduces to the one given by $z = 0$ and that it moves with a given velocity $(U_0, V_0, 0)$, (i.e. parallel to the own surface). Let $h(t, x, y)$ be the distance between the surface and the solid. That we want to describe is the fluid velocity $\mathbf{u} = (u, v, w)$ and pressure P. We suppose the fluid incompressible of density, ρ (a positive know constant) . Starting from the usual conservation principles

$$\begin{array}{ll} \text{mass conservation} & \rho_t + div\rho\mathbf{u} = 0, \\ \text{momentum conservation} & \rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \mu\Delta\mathbf{u}, \end{array}$$

using dimensional analysis and supposing h *small with respect the solid size*, we can simplify the momentum equation leading to the system

$$\begin{array}{ll} -P_x + \mu u_{zz} = 0 & \text{in the } x \text{ component,} \\ -P_y + \mu v_{zz} = 0 & \text{in the } y \text{ component,} \\ -P_z = 0 & \text{in the } z \text{ component.} \end{array}$$

The boundary conditions are

$$\begin{array}{lll} u = v = 0, \; w = h_t & \text{on} & z = h, \\ u - U_0 = v - V_0 = w = 0 & \text{on} & z = 0. \end{array}$$

Therefore, we have that

$$u = \frac{1}{2\mu}P_x z(z-h) + U_0(1-\frac{z}{h}), v = \frac{1}{2\mu}P_y z(z-h) + V_0(1-\frac{z}{h}).$$

The flow is given by

$$q_1 = \int_0^h u dz = \frac{U_0 h}{2} - \frac{h^3}{12\mu}P_x, q_2 = \int_0^h v dz = \frac{V_0 h}{2} - \frac{h^3}{12\mu}P_y.$$

Integrating in the mass equation, we get that

$$\begin{cases} \rho h_t + [\rho q_1]_x + [\rho q_2]_y = 0 & \text{in } \Omega, \\ P - P_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

If, for simplicity, we suppose that $h(t,\cdot) = h(\cdot)$ we arrive to the, so called *Reynolds equation*

$$(\mathcal{P}) \begin{cases} (\frac{U_0 h}{2} - \frac{h^3}{12\mu}P_x)_x + (\frac{V_0 h}{2} - \frac{h^3}{12\mu}P_y)_y = 0 & \text{in } \Omega, \\ P - P_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

In fact, in what follows, we shall always assume that

$$h \in L^\infty(\Omega), 0 < h_0 \le h(x,y) \le h_1, a.e. \text{ on } \Omega. \tag{1}$$

We point out that more general situations, in which the surface is more complicated, can be considered by expressing the pde in terms of a general coordinates system (α, β, z) associated to the surface, getting formulations of the type

$$\frac{\partial}{\partial\alpha}(g_\beta \frac{U_0 h}{2} - g_\beta \frac{h^3}{12\mu g_\alpha}\frac{\partial P}{\partial \alpha}) + \frac{\partial}{\partial\beta}(g_\alpha \frac{V_0 h}{2} - g_\alpha \frac{h^3}{12\mu g_\beta}\frac{\partial P}{\partial\beta}) = 0 \quad \text{in } \Omega,$$

An example of non classical solution.

When $h(x,y)$ is discontinuous (which corresponds to the case of solids with abrupt edges) is possible to show that no classical solution of $(\mathcal{P})$ may exits. This is specially easy to present in the onedimensional case (i.e. an uniform solid which is understood as unbounded).
We start by recalling the notion of weak solution:

Definition 1 *We say that P is a weak solution of $(\mathcal{P})$ if $P = u + P_0$, with $u \in H_0^1(\Omega)$ satisfying that*

$$\int_\Omega \frac{h^3}{12\mu}\nabla u \cdot \nabla\psi \, d\sigma = \int_\Omega \frac{h}{2}(U_0, V_0)\cdot\nabla\psi d\sigma, \ \forall\psi \in H_0^1(\Omega). \tag{2}$$

A standard application of the Lax-Milgram theorem allows to prove the existence and uniqueness of a weak solution P of (1). In the special discontinuous onedimensional case we have

Proposition 1 *Let $\Omega = (0, L)$ and*

$$h(x) = \begin{cases} h_0 & \text{if } x \in (0, \frac{L}{2}) \\ h_1 & \text{if } x \in (\frac{L}{2}, L). \end{cases} \tag{3}$$

Then the weak solution P is non of class C^2 and so is not a classical solution.

Proof: The one dimensional Reynolds equation becomes

$$\begin{cases} [\frac{U_0 h}{2} - \frac{h^3}{12\mu}P(x)_x]_x = 0 & \text{in } \Omega, \\ P - P_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

and the (unique) weak solution is explicitly given by:

$$P(x) = \begin{cases} -6\mu\frac{2k + U_0 h_0}{h_0^3}x + P_0 & \text{if } 0 < x < \frac{L}{2} \\ 6\mu\frac{-2k + U_0 h_1}{h_1^3}(L - x) + P_0 & \text{if } \frac{L}{2} < x < L \end{cases} \tag{5}$$

where

$$K = \frac{LU_0}{4}[\frac{1}{h_1^2} - \frac{1}{h_0^2}][\frac{1}{h_0^3} - \frac{1}{h_1^3}]^{-1}.$$

Then, obviously, $P \not\in C^1(\Omega)$ (nevertheless, it is easy to see that function given by (5) satisfies that $P \in W^{1,\infty}(0, L)$). ∎

$W^{1,\infty}$ regularity of weak solution for discontinuous separation functions.

The main contribution of this paper concerns the study of the regularity of the weak solution of $(\mathcal{P})$ associated to, eventually discontinuous, separation functions $h(x,y)$ satisfying assumption (1). For the sake of the exposition, we shall restrict ourselves to the special case of $\Omega = (0, L) \times (0, B)$ and

$$h(x,y) = \begin{cases} h_0 & \text{if } x \in (0, \frac{L}{2}) \\ h_1 & \text{if } x \in (\frac{L}{2}, L) \end{cases} \tag{6}$$

where $0 < h_0 < h_1$, nevertheless our results remain valid under a greater generality.
The regularity $C^{0,\alpha}(\Omega)$, $\forall\alpha \in [0, 1)$, of the weak solution of $(\mathcal{P})$ is a direct consequence of the regularity theory (see, e.g. Kinderlehrer-Stampacchia [5; Th.9.2]. The $W^{1,p}(\Omega)$ regularity is a more delicate question due to the lake of continuity of h. As far as we know, there is not any general result in the literature that could be applied directly to this case.
The main result of this paper shows that, in fact, $P \in W^{1,\infty}(\Omega)$.

Theorem 1 *Let P be the weak solution of $(\mathcal{P})$. Then function $u := P - P_0$ is such that $u \in W_0^{1,\infty}(\Omega)$.*

The proof will use some previous lemmata:

Lemma 1 *Consider problem*

$$(\mathcal{P}_\varepsilon) \begin{cases} -\, div\, (\frac{h_\epsilon^3}{12\mu}\nabla u_\epsilon) = -\frac{d}{dx}(\frac{h_\epsilon U_0}{2}) & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$h_\epsilon(x,y) = \begin{cases} h_0 & \text{if } 0 \le x \le \frac{L}{2}, \\ \frac{1}{\epsilon}(x - \frac{L}{2} + \epsilon h_0) & \text{if } \frac{L}{2} \le x \le \frac{L}{2} + \epsilon(h_1 - h_0), \\ h_1 & \text{if } \frac{L}{2} + \epsilon(h_1 - h_0) \le x \le L. \end{cases} \tag{7}$$

Then $u_\epsilon \in W_0^{-1,q}(\Omega)$ $\forall\ 1 \le q < \infty$.

According Troianiello [8: Theorem 3.7 and Theorem 3.14] the proof of the above lemma reduces to prove that the normal derivative $\frac{\partial u_\epsilon}{\partial \vec{n}}$ is a bounded function.. This is proved in the following result:

Lemma 2 $\frac{\partial u_\epsilon}{\partial \vec{n}} \in L^\infty(\partial\Omega)$.

Proof: Define the pd operator $L_\epsilon(\cdot) = -\operatorname{div}\left(\frac{h_\epsilon^3}{12\mu}\nabla(\cdot)\right)$. Then

$$L_\epsilon(u_\epsilon) = -\frac{\partial}{\partial x}\left(\frac{h_\epsilon U_0}{2}\right) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{L}{2}, \\ \frac{U_0}{2\epsilon} & \text{if } \frac{L}{2} \le x \le \frac{L}{2} + \epsilon(h_1 - h_0), \\ 0 & \text{if } \frac{L}{2} + \epsilon(h_1 - h_0) \le x \le L. \end{cases}$$

Let $\bar{u}_\epsilon(x,y) = cx(x-L)y(y-B)$ with $c = \frac{12\mu U_0}{h_0^3 L^2\epsilon}$. A routine computation shows that

$$L_\epsilon(\bar{u}_\epsilon) = \begin{cases} -2c\frac{h_0}{12\mu}y(y-B) - 2c\frac{h_0}{12\mu}x(x-L) \\ \quad \text{on } 0 \le x \le \frac{L}{2}, \\ -cy(y-B)[2h_\epsilon^3 + \frac{3}{\epsilon}(2x-L)h_\epsilon^2] - c\frac{h_\epsilon^3}{12\mu}x(x-L) \\ \quad \text{on } \frac{L}{2} \le x \le \frac{L}{2} + \epsilon(h_1 - h_0), \\ -2c\frac{h_1}{12\mu}y(y-B) - 2c\frac{h_1}{12\mu}x(x-L) \\ \quad \text{on } \frac{L}{2} + \epsilon(h_1 - h_0) \le x \le L. \end{cases}$$

Since $L_\epsilon(\bar{u}_\epsilon) > L_\epsilon(u_\epsilon) \ge L_\epsilon(0) = 0$ in Ω, and $\bar{u}_\epsilon = u_\epsilon = 0$ on $\partial\Omega$, we get that $\bar{u}_\epsilon$ is a supersolution and 0 is a subsolution of problem $(\mathcal{P}_\epsilon)$. Thus, by the comparison principle $\bar{u}_\epsilon \ge u_\epsilon \ge 0$ in Ω. In consequence

$$\frac{\partial \bar{u}_\epsilon}{\partial \vec{n}} \le \frac{\partial u_\epsilon}{\partial \vec{n}} \le 0,$$

which implies that

$$-c(B^2 + L^2) \le \frac{\partial u_\epsilon}{\partial \vec{n}} \le 0 \quad \text{and so} \quad \frac{\partial u_\epsilon}{\partial \vec{n}} \in L^\infty(\partial\Omega). \qquad \blacksquare$$

In a second step, for any $p > 1$ we introduce the auxiliary problem

$$(\mathcal{P}_{\epsilon,p}) \begin{cases} -\operatorname{div}\left(\frac{h_\epsilon^3}{12\mu}\nabla w_\epsilon\right) = -\operatorname{div}\left(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\right) & \text{in } \Omega, \\ \frac{h_\epsilon^3}{12\mu}\frac{\partial w_\epsilon}{\partial \vec{n}} + a_\epsilon(x,y)w_\epsilon = |\nabla u_\epsilon|^{p-2}\frac{\partial u_\epsilon}{\partial \vec{n}} & \text{on } \partial\Omega, \end{cases}$$

where

$$G_\epsilon(x) := \int_0^x \frac{6\mu U_0}{h_\epsilon^2(s)}ds - k_0, \tag{8}$$

$$k_0 := -\frac{\left(\|\frac{\partial u_\epsilon}{\partial \vec{n}}\|^p_{L^\infty(\partial\Omega)} + |U_0|h_1\right)C^2(\Omega)h_0^3}{6\mu}, \tag{9}$$

$$a_\epsilon(x,y) := G_\epsilon^{-1}\left(\frac{h_\epsilon^3}{12\mu}\frac{\partial u_\epsilon}{\partial \vec{n}} - \frac{h_\epsilon^3}{12\mu}\frac{\partial G_\epsilon}{\partial \vec{n}}\right). \tag{10}$$

In order to solve $(\mathcal{P}_{\epsilon,p})$ we introduce the Hilbert space $V(\Omega) := \{\phi \in H^1(\Omega)$ such that $\int_{\partial\Omega} a_\epsilon \phi d\sigma_2 = 0\}$ associated to the scalar product $<\phi,\psi>_V := \int_\Omega \nabla\phi\cdot\nabla\psi d\sigma$. We have

Lemma 3 *Problem $(\mathcal{P}_{\epsilon,p})$ has a unique weak solution $w_\epsilon \in V(\Omega)$.*

Proof: First of all, it is easy to see that the norm $\|\cdot\|_V$ is equivalent to the usual norm $\|\cdot\|_{H^1}$ over V.
Define, now, the bilinear form $A_\epsilon : V(\Omega) \times V(\Omega) \to I\!R$ by

$$A_\epsilon(\varphi,\psi) := \int_\Omega \frac{h_\epsilon^3}{12\mu}\nabla\psi\cdot\nabla\phi d\sigma + \int_{\partial\Omega} a_\epsilon \psi\phi d\sigma_2.$$

Then we have that A_ϵ is a continuous and coercive bilinear form V, and so, by Lax-Milgram Theorem, there exists a unique $w_\epsilon \in V(\Omega)$ weak solution of $(\mathcal{P}_{\epsilon,p})$. $\blacksquare$

The third step consists in substituting w_ϵ, weak solution of $(\mathcal{P}_{\epsilon,p})$, as test functions in (2) (we point out that any function in $H^1(\Omega)$ can be expressed as the sum of a function in $V(\Omega)$ plus a constant). We have

Lemma 4 *There exists a positive constant $K(\Omega, h_0, U_0, \mu)$ such that*

$$\| u_\epsilon \|_{W_c^{1,p}(\Omega)} \le K(\Omega, h_0, U_0, \mu) \quad \forall 1 \le p < \infty. \tag{11}$$

Proof: We star by taking w_ϵ as test function in the weak formulation of $(\mathcal{P}_{\epsilon,p})$. We get

$$\int_\Omega \frac{h_\epsilon^3}{12\mu}\nabla u_\epsilon\cdot\nabla w_\epsilon d\sigma = \int_\Omega \nabla w_\epsilon\cdot\left(\frac{h_\epsilon U_0}{2},0\right)d\sigma + \int_{\partial\Omega}\frac{h_\epsilon^3}{12\mu}\left(\frac{\partial u_\epsilon}{\partial \vec{n}} - <(h_\epsilon U_0, 0),\vec{n}>\right)w_\epsilon d\sigma_2. \tag{12}$$

Taking, also, u_ϵ as test function in the weak formulation of $(\mathcal{P}_{\epsilon,p})$ we obtain:

$$\int_\Omega \frac{h_\epsilon^3}{12\mu}\nabla u_\epsilon\cdot\nabla w_\epsilon d\sigma = \int_\Omega |\nabla u_\epsilon|^p d\sigma. \tag{13}$$

Therefore,

$$\int_\Omega |\nabla u_\epsilon|^p d\sigma = \int_\Omega \nabla w_\epsilon\cdot\left(\frac{h_\epsilon U_0}{2},0\right)d\sigma + \int_{\partial\Omega}\left(\frac{h_\epsilon^3}{12\mu}\frac{\partial u_\epsilon}{\partial \vec{n}} - <\left(\frac{h_\epsilon U_0}{2},0\right),\vec{n}>\right)w_\epsilon d\sigma_2.$$

But, by construction, $\left(\frac{h_\epsilon U_0}{2},0\right) = \frac{h_\epsilon^3}{12\mu}\nabla G_\epsilon$. Thus

$$\int_\Omega \nabla w_\epsilon\cdot(h_\epsilon U_0, 0)d\sigma = \int_\Omega \frac{h_\epsilon^3}{12\mu}\nabla G_\epsilon\cdot\nabla w_\epsilon d\sigma.$$

Taking G_ϵ (which belongs to V) as test function in $(\mathcal{P}_{\epsilon,p})$ we obtain:

$$\int_\Omega \frac{h_\epsilon^3}{12\mu}\nabla G_\epsilon\cdot\nabla w_\epsilon d\sigma = \int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\cdot\nabla G_\epsilon d\sigma -$$

$$-\int_{\partial\Omega} G_\epsilon|\nabla u_\epsilon|^{p-2}\frac{\partial u_\epsilon}{\partial \vec{n}}d\sigma_2 + <G_\epsilon, \frac{h_\epsilon^3}{12\mu}\frac{\partial w_\epsilon}{\partial \vec{n}}>_{H^{1/2},H^{-1/2}}$$

$$\int_\Omega |\nabla u_\epsilon|^p d\sigma = \int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\cdot\nabla G_\epsilon d\sigma - \int_{\partial\Omega} G_\epsilon|\nabla u_\epsilon|^{p-2}\frac{\partial u_\epsilon}{\partial \vec{n}}d\sigma_2 +$$

$$+ < G_\epsilon, \frac{h_\epsilon^3}{12\mu}\frac{\partial w_\epsilon}{\partial \vec{n}} >_{H^{1/2},H^{-1/2}} + \int_{\partial\Omega}(\frac{h_\epsilon^3}{12\mu}\frac{\partial u_\epsilon}{\partial \vec{n}} - < (h_\epsilon U_0, 0), \vec{n} >) w_\epsilon d\sigma_2.$$

Using the definition of w_ϵ we arrive to

$$\int_\Omega |\nabla u_\epsilon|^p d\sigma = \int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \cdot \nabla G_\epsilon d\sigma,$$

$$\int_\Omega |\nabla u_\epsilon|^p d\sigma \leq \|\nabla G_\epsilon\|_{(L^\infty)^2}\int_\Omega |\nabla u_\epsilon|^{p-1} d\sigma.$$

In consequence

$$\|\nabla u_\epsilon\|^p_{(L^p)^2} \leq \frac{U_0 12\mu}{h_0^2}\|\nabla u_\epsilon\|^{p-1}_{(L^{p-1})^2} \leq C(\Omega)\frac{U_0 12\mu}{h_0^2}\|\nabla u_\epsilon\|^{p-1}_{(L^p)^2},$$

i.e. $\|\nabla u_\epsilon\|_{(L^p)^2} \leq C(\Omega)\frac{U_0 12\mu}{h_0^2}$ and so we conclude that $\|u_\epsilon\|_{W_0^{1,p}} \leq C(\Omega)\frac{U_0 12\mu}{h_0^2}$,$\forall p \in [1,\infty)$.

■

End of proof of Theorem 1.

Since $\| u_\epsilon \|_{W_0^{1,p}(\Omega)} \leq C$ then $u_{\epsilon_{i(p)}} \rightharpoonup v \in W_0^{1,p}(\Omega)$. Passing to the limit in the weak formulation

$$\int_\Omega \frac{h_\epsilon^3}{12\mu}\nabla u_\epsilon \cdot \nabla\psi d\sigma = \int_\Omega \nabla\psi \cdot (h_\epsilon U_0, 0) d\sigma \quad \forall\, \psi \in H_0^1(\Omega).$$

we get

$$\int_\Omega \frac{h^3}{12\mu}\nabla v \cdot \nabla\psi d\sigma = \int_\Omega \nabla\psi \cdot (hU_0, 0) d\sigma \quad \forall\, \psi \in H_0^1(\Omega)\ .$$

So, by uniqueness of the solution of $(\mathcal{P})$, we deduce that $u = v$. Then, by estimate 11, $u \in W_0^{1,p}(\Omega)\ \ \forall 1 \leq p < \infty$. Moreover, $\| u \|_{W_0^{1,p}(\Omega)} \leq C(\Omega, h_0)$. Since $\| u \|_{W_0^{1,\infty}} = lim_{p\to\infty} \| u \|_{W_0^{1,p}}$ we conclude that $u \in W_0^{1,\infty}(\Omega)$.

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