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On a Degenerate System in Glaciology Giving Rise to a Free Boundary

1 Introduction

The mechanism whereby large ice sheets can surge periodically was recently studied by Fowler and Johnson ([4]) and Fowler ([3]). They proposed a two-dimensional ice sheet simplified model that includes basal ice sliding dependent on the basal water pressure and which consists in the following system:

$$
\begin{aligned}
h_t - \left[(\delta + Q)^S h^{R+1} |h_x|^{R-1} h_x\right]_x &= a(t,x) \\
Q_x - (\delta + Q)^S \left[h^{R+1}|h_x|^{R+1} - \mu h^R |h_x|^R \xi^{-1/2}\right] - \gamma + \lambda h^{-1} &\geq 0, \quad Q \geq 0 \\
\left(Q_x - (\delta + Q)^S \left[h^{R+1}|h_x|^{R+1} - \mu h^R |h_x|^R \xi^{-1/2}\right] - \gamma + \lambda h^{-1}\right) Q &= 0 \\
\xi_x &= (\delta + Q)^S h^R |h_x|^R
\end{aligned}
$$

over $\Omega \times (0,T)$ with $T > 0$, $\Omega = (0,1)$ (the scaled spatial domain), where R and S are some positive numbers satisfying $R > 1$ and $0 < S < 1$. The unknown variables are *the ice depth* $h(t,x)$, *the accumulated ice velocity* $\xi(t,x)$, and *the basal water flow* $Q(t,x)$. The *complementary formulation* for Q has been introduced in order to deal with the *cold* ($Q = 0$) and *temperate* ($Q > 0$) transition at the base. The points of $\Omega \times (0,T)$ separating those two zones are the free boundaries of the problem. Prescribing suitable boundary conditions for h, ξ, Q and an initial condition for h, Fowler and Schiavi ([5]) solved numerically the system by using a fully implicit backward finite difference scheme for h and an improved Euler method for Q and ξ. Numerical computations indicated a series of *surges* and showed that a front propagates backward during a surge. Moreover, their analysis suggested that the problem, as formulated, does not have smooth solutions. The main goal of this work is to present the mathematical analysis of the implicit backward scheme system showing the existence of a weak solution for the discretized system. Actually, this time discretized solution corresponds to the notion of *mild solution* of the evolution system as used in semigroup theory (see, e.g., Benilan ([1])).

2 The implicit discretized system

We define $p = R+1$, $m = (2R+1)/R$. We start by considering the following initial and boundary value problem: given h_0, h_D, ξ_D, Q_D and a strictly positive *accumulation rate* function $a(t,x)$, find three functions, h, Q and ξ satisfying

$$(\mathcal{S})\begin{cases} \partial_t h - \left[(\delta+Q)^S|(h^m)_x|^{p-2}(h^m)_x\right]_x = a & \text{in } \Omega\times(0,T),\\ \partial_x Q + \beta(Q) \ni (\delta+Q)^S h^p|h_x|^p - \mu\xi_x\xi^{-1/2} + \gamma - \lambda h^{-1} & \text{in } \Omega\times(0,T),\\ \partial_x \xi = (\delta+Q)^S h^{p-1}|h_x|^{p-1} & \text{in } \Omega\times(0,T),\\ h(t,1) = h_D(t,1), & t\in(0,T),\\ h_x(t,0)=0, Q(t,0)=Q_D(t), \xi(t,0)=\xi_D(t) & t\in(0,T),\\ h(0,x) = h_0(x) & \text{on } \Omega. \end{cases}$$

Here β denotes the maximal monotone graph defined by

$$\beta(r) = \emptyset \quad \text{if} \quad r<0, \quad \beta(0) = (-\infty,0], \quad \beta(r)=0 \quad \text{if} \quad r>0.$$

The coefficients γ, μ, λ are $O(1)$ dimensionless parameters. The positive constant δ, $0<\delta\ll 1$ represents the ice shearing component in the flow when $Q=0$. Given a positive integer number N and letting $k=T/N$, the time step of the discretization, we denote by $I_n = I_{n,k} = (t_{n-1},t_n) = ((n-1)k, nk)$, $(n=1,..,N,\ t_n = nk)$ the associated sub-intervals of $(0,T)$. Let $V \doteq V_{h^m}\times V_\xi \times V_Q$ the Banach space defined by $V_{h^m} \doteq \{\phi\in W^{1,p}(\Omega) : \phi(1)=0\}$, $V_\xi \doteq \{\psi \in W^{1,p'}(\Omega) : \psi(0)=0\}$, $V_Q \doteq \{\eta\in W^{1,1}(\Omega) : \eta(0)=0\}$. We shall assume the following hypothesis on the data of the problem:

$$Q_D, \xi_D, h_D \in C[0,T], h_0\in C[0,1], h_D(0) = h_0(1) \tag{1}$$

$$\left.\begin{array}{l} M_D > h_D > m_D > 0, M_0 > h_0 > m_0 > 0, a>0, Q_D \geq 0 \text{ and } \xi_D > 0,\\ \text{for some constants } M_D > m_D > 0 \text{ and } M_0 > m_0 > 0. \end{array}\right\} \tag{2}$$

It is useful to introduce the following notation:

$$A = (\delta+Q)^S,\ B = h^p|h_x|^p,\ C = \mu\xi_x\xi^{-1/2},\ D = \lambda h^{-1},\ E = h^{p-1}|h_x|^{p-1}. \tag{3}$$

After defining the piecewise constant in time approximations of the data in the usual manner, we consider the elliptic discretized system

$$(\mathcal{S}_{k,n})\begin{cases} \partial_t^{-k} h_{k,n} - \left[A_{k,n}|(h_k^m)_x|^{p-2}(h_k^m)_x\right]_x = a_{k,n} & \text{in } \Omega,\\ \partial_x Q_{k,n} + \beta(Q_{k,n}) \ni (\delta+Q_{k,n})^S B_{k,n} - C_{k,n} + \gamma - D_{k,n} & \text{in } \Omega,\\ \partial_x \xi_{k,n} = A_{k,n}E_{k,n} & \text{in } \Omega,\\ h_{k,n}(t,1) = h_{D_{k,n}}(t,1) & t\in(0,T),\\ (h_{k,n})_x(t,0) = 0, Q_{k,n}(t,0) = Q_{D_{k,n}}(t), \xi_{k,n}(t,0) = \xi_{D_{k,n}}(t) & t\in(0,T),\\ h_{k,0}(x) = h_0(x) & \text{on } \Omega, \end{cases}$$

where

$$\partial_t^{-k} h_{k,n}(t,.) \doteq \frac{h_{k,n}(.) - h_{k,n-1}(.)}{k} \quad \forall n = 1,..N$$

and $A_{k,n}$, $B_{k,n}$, $C_{k,n}$, $D_{k,n}$ and $E_{k,n}$ are defined as in (3) replacing h, ξ and Q by $h_{k,n}$, $\xi_{k,n}$ and $Q_{k,n}$.

Definition 2.1 *Given a, h_D, Q_D, ξ_D, h_0 satisfying hypothesis (1), (2) and $a_{k,n}$, $h_{D_{k,n}}$, $Q_{D_{k,n}}$, $\xi_{D_{k,n}}$ the associated discretized functions, we say that $(h^m_{k,n}, \xi_{k,n}, Q_{k,n})$ is a weak solution of $(\mathcal{S}_{k,n})$ if*

$$(h^m_{k,n}(t,.), \xi_{k,n}(t,.), Q_{k,n}(t,.)) \in [h^m_D + V_{h^m}]\times[\xi_D + V_\xi]\times[Q_D+V_Q], \quad a.e.\ t\in(0,T),$$

there exists $b_{k,n}\in L^1(\Omega)$, with $b_{k,n}(x)\in\beta(Q_{k,n}(x))$ a.e. $x\in(0,1)$, and the following conditions hold:

$$\int_0^1 \partial_t^{-k} h^m_{k,n}(t)\phi + \int_0^1 (\delta+Q_{k,n})^S|(h^m_{k,n})_x|^{p-2}(h^m_{k,n})_x\phi_x = \int_0^1 a\,\phi$$

$$\int_0^1 \xi_{k,n}\psi_x + \int_0^1 (\delta+Q_{k,n})^S h^{p-1}_{k,n}|(h_{k,n})_x|^{p-1}\,\psi = \xi_{k,n}(1,t)\psi(1,t)$$

$$\int_0^1 Q_{k,n}\eta_x + \int_0^1 \left[(\delta+Q_{k,n})^S h^p_{k,n}|(h_{k,n})_x|^p + \gamma\right]\eta = Q_{k,n}(1,t)\eta(1,t) +$$

$$+\mu\int_0^1 (\xi_{k,n})_x(\xi_{k,n})^{-1/2}\eta + \lambda\int_0^1 h^{-1}_{k,n}\eta + \int_0^1 b_{k,n}\eta$$

for all test functions ϕ, ψ, $\eta\in V_h\times V_\xi\times V_Q$.

3 Existence of weak solutions via an iterative scheme

In order to prove the existence of a weak solution of $(S_{k,n})$, we shall use an iterative process which allows decoupling the system into three separate problems: $P(h^j_{k,n})$, $P(\xi^j_{k,n})$ and $P(Q^j_{k,n})$. Later we shall obtain *a priori* estimates which allow us to prove the convergence of such iterative schemes. The decoupled problem is the following: For each j we shall find three functions $(h^j_{k,n})^m$, $(\xi^j_{k,n})$ and $(Q^j_{k,n})$ satisfying

$$(S^j_{k,n})\begin{cases} \partial_t^{-k} h^j_{k,n} - \left[A^{j-1}_{k,n}|\partial_x[(h^j_{k,n})^m]|^{p-2}\partial_x[(h^j_{k,n})^m]\right]_x = a_{k,n} & \text{on } \Omega, \\ \partial_x \xi^j_{k,n} = A^{j-1}_{k,n}E^j_{k,n} & \text{in } \Omega, \\ \partial_x Q^j_{k,n} + \beta(Q^j_{k,n}) \ni (\delta + Q^j_{k,n})^S B^j_{k,n} - C^j_{k,n} + \gamma - D^j_{k,n} & \text{in } \Omega, \\ h^j_{k,n}(t,1) = h_{D_{k,n}}(t,1) & t\in(0,T), \\ (h^j_{k,n})_x(t,0) = 0, Q^j_{k,n}(t,0) = Q_{D_{k,n}}(t), \xi^j_{k,n}(t,0) = \xi_{D_{k,n}}(t) & t\in(0,T), \\ h_{k,0}(x) = h_0(x) & \text{in } \Omega. \end{cases}$$

In order to study this system we study separately three problems:
First step: Problem $P(h^j_{k,1})$. We introduce the change of unknown $w \doteq (h^j_{k,n})^m$. By defining $w_D \doteq (h_{D_{k,n}})^m$, $\hat{A} \doteq k A^{j-1}_{k,n}$ and $f \doteq k a_{k,n} + h_{k,n-1}$, for each j-step of the iterative process function, w must satisfy

$$P(h^j_{k,n})\begin{cases} -\partial_x\left(\hat{A}|w_x|^{p-2}w_x\right) + w^{1/m} = f & \text{in } \Omega, \\ w_x(0) = 0, w(1) = w_D. \end{cases}$$

Let $V_w := V_{h^m}$. As usual, given $\hat{A} \in L^\infty(\Omega)$, $\hat{A} > 0$ and $f \in L^\infty(\Omega)$, we say that w is a *bounded weak solution* of $P(h^j_{k,n})$ if $w \in w_D + V_w$ and it satisfies

$$-\int_\Omega \hat{A}|w_x|^{p-2}w_x\phi_x + \int_\Omega w^{1/m}\phi = \int_\Omega f\phi, \qquad \forall\phi\in V_w.$$

The existence of a unique approximate solution $w = (h^j_{k,n})^m \in w_D + V_w$ is a well-known result in the literature (see, e.g., the exposition made in Díaz ([2])). Notice that, in particular, $B^j_{k,n} \in L^1(\Omega)$, $E^j_{k,n} \in L^{p'}(\Omega)$. We can get some *a priori* estimates on w (i.e., on $(h^j_{k,n})^m$). First, by using the comparison principle and suitable super- and subsolutions built thanks to the assumptions on the data, we obtain

Lemma 3.1 *Let $h^j_{k,n}$ be a weak bounded solution of problem $P(h^j_{k,n})$. Then there exist two real positive numbers m^*, M^* (depending only on the data of the problem) such that*

$$0 < m^* \le h^j_{k,n}(t,x) \le M^* < +\infty \qquad \text{on } \Omega.$$

In particular, $D^j_{k,n} > 0$, a.e. $x\in\Omega$, $D^j_{k,n} \in L^\infty(\Omega)$ and $D^j_{k,n}$ is uniformly bounded in $L^\infty(\Omega)$ with respect to j.

A second *a priori* estimate can be obtained by using an energy method.

Lemma 3.2 *Function $(h^1_{k,n})^m - (h_{D_{k,n}})^m$ is uniformly bounded in the energy space V_w. In particular, $B^j_{k,n}$ and $E^j_{k,n}$ are uniformly bounded in their respective spaces.*

Second step: Problem $P(\xi^j_{k,n})$. Let $h^j_{k,n}$ be the weak solution of problem $P(h^j_{k,n})$. Then $A^{j-1}_{k,n} \in L^\infty(\Omega)$, $E^j_{k,n} \in L^{p'}(\Omega)$ and $A^{j-1}_{k,n}E^j_{k,n} \doteq (\delta + Q^{j-1}_{k,r})^S (h^j_{k,r})^{p-1}\,|(h^j_{k,r})_x|^{p-1} \in L^{p'}(\Omega)$. We consider the problem

$$P(\xi^j_{k,n})\begin{cases} (\xi^j_{k,n})_x = A^{j-1}_{k,n}E^j_{k,n} & \text{in } \Omega, \\ \xi^j_{k,n}(0) = \xi_{D_{k,n}}. \end{cases}$$

Definition 3.1 *We shall say that $\xi^j_{k,n} \in W^{1,p'}(\Omega)$ is a weak solution of problem $P(\xi^j_{k,n})$ if $\xi^j_{k,n} - \xi_{D_{k,n}} \in V_\xi$ and*

$$\int_0^1 \xi^j_{k,n}\psi_x + \int_0^1 A^{j-1}_{k,n}E^j_{k,n}\psi = \xi_{D_{k,n}}(1)\psi(1), \qquad \forall\psi\in V_\xi.$$

It is straightforward to show the existence of a unique function $\xi^j_{k,n}$ weak solution of problem $P(\xi^j_{k,n})$. Since $(\xi^j_{k,n})_x(x) \ge 0$, a.e. $x\in\Omega$, a direct integration leads to the following result:

Lemma 3.3 $\xi^j_{k,n}(x) > 0$, $\forall x\in\Omega$. *In particular, $C^j_{k,n} \ge 0$, a.e. $x\in\Omega$.*

Third step: Problem $P(Q^j_{k,n})$. Let $h^j_{k,n}$ and $\xi^j_{k,n}$ be the weak solutions of $P(h^j_{k,n})$ and $P(\xi^j_{k,n})$, respectively. We consider $B^j_{k,n}$, $C^j_{k,n}$ y $D^j_{k,n}$ defined by (3). Notice that $D^j_{k,n} \in C([0,1])$, $C^j_{k,n} \in L^{p'}(\Omega)$, but $B^j_{k,n}$ is, in general, merely in $L^1(\Omega)$. We introduce the problem

$$P(Q^j_{k,n})\begin{cases} \partial_x Q^j_{k,n} + \beta(Q^j_{k,n}) \ni (\delta + Q^j_{k,n})^S B^j_{k,n} + \gamma - C^j_{k,n} - D^j_{k,n} & \text{in } \Omega, \\ Q^j_{k,n}(0) = Q_{D_{k,n}}. \end{cases}$$

Definition 3.2 *We shall say that* $Q^j_{k,n} \in Q_{D_{k,n}} + V_Q \subset W^{1,1}(\Omega)$ *is a weak solution if there exist a function* $z \in L^1(\Omega)$ *such that* $z(x) \in \beta(Q^j_{k,n}(x))$, *a.e.* $x \in \Omega$ *and*

$$\int_0^1 Q^j_{k,n}\eta_x + \int_0^1 (\delta + Q^j_{k,n})^S B^j_{k,n}\eta + \gamma \int_0^1 \eta = \\ = \int_0^1 C^j_{k,n}\eta + \int_0^1 D^j_{k,n}\eta + \int_0^1 z\eta + Q^j_{k,n}(1)\eta \tag{4}$$

for each $\eta \in V_Q$.

We have

Theorem 3.1 *There exists a unique weak solution of* $P(Q^j_{k,n})$.

Proof. We approximate the maximal monotone graph β and function $b(Q) \doteq (\delta + Q)^S$ by some sequences of Lipschitz functions generating some approximating regularized problems of solutions $(Q^j_{k,n})_\epsilon \in W^{1,1}(\Omega)$ (the existence of solutions of such problems is consequence of a Banach fixed point argument). Moreover, we get that $||(Q^j_{k,n})_\epsilon||_{L^\infty(\Omega)} \leq C$ and $||(\partial_x Q^j_{k,n})_\epsilon||_{L^1(\Omega)} \leq C$. Passing to the limit in the weak formulation of the regularizing problems it is possible to show that $(Q^j_{k,n})_\epsilon \to Q^j_{k,n}$ as $\epsilon \to \infty$ *strongly* in $L^2(\Omega)$ with $Q^j_{k,n}$ solution of $P(Q^j_{k,n})$.

4 Convergence

We already obtained the *a priori* estimates

$$||(h^j_{k,n})_x||_{L^p(\Omega)} \leq C, \quad ||(\xi^j_{k,n})_x||_{L^{p'}(\Omega)} \leq C, \quad ||Q^j_{k,n}||_{L^\infty(\Omega)} \leq C, \quad ||(Q^j_{k,n})_x||_{L^1(\Omega)} \leq C;$$

uniformly in j, $\forall\, k, n$ fixed and $\forall t \in I_{k,n}$. So

$$||A^{j-1}_{k,n}||_{L^\infty(\Omega)} \leq C, \qquad ||B^j_{k,n}||_{L^1(\Omega)} \leq C, \qquad ||C^j_{k,n}||_{L^{p'}(\Omega)} \leq C,$$

$$||D^j_{k,n}||_{L^\infty(\Omega)} \leq C, \qquad ||E^j_{k,n}||_{L^{p'}(\Omega)} \leq C.$$

By applying Poincaré inequality, and Sobolev and Lebesgue theorems we get the following result:

Lemma 4.1 *Let* $\{(h^j_{k,n})^m\}$, $\{\xi^j_{k,n}\}$ *and* $\{Q^j_{k,n}\}$ *be the sequences of solutions of problems* $P(h^j_{k,n})$, $P(\xi^j_{k,n})$ *and* $P(Q^j_{k,n})$, *respectively. Then,* $\forall\, k, n$ fixed *and* $\forall t \in I_{k,n}$ *we have*

$$h^j_{k,n} \to h_{k,n}, \quad \xi^j_{k,n} \to \xi_{k,n}, \quad Q^j_{k,n} \to Q_{k,n} \text{ strongly in } L^q(\Omega),\ \forall\, q \geq 1, \text{ when } j \to \infty.$$

Moreover, $h^j_{k,n} \to h_{k,n}$, $\xi^j_{k,n} \to \xi_{k,n}$ *strongly in* $C^0([0,1])$.

An analysis of system $(S^j_{k,n})$ reveals that the difficult term, in order to pass to the limit in the weak formulation, is the product $(\delta + Q^j)^S |h^j_x|^p$ representing the frictional heating due to viscous dissipation. Nevertheless, we have

Lemma 4.2 $(\delta + Q^{j-1})^S |h^j_x|^p \to (\delta + Q)^S |h_x|^p$, strongly *in* $L^1(\Omega)$, *when* $j \to \infty$.

Proof. We consider $w^j = (h^j)^m$ with h^j solution of problem $P(h^j_{k,n})$. Without loss of generality we can suppose that $w_D \equiv 0$. Multiplying by $(h^j)^m$ and integrating by parts, we have

$$\int_\Omega (\delta + Q^{j-1})^S |w^j_x|^p dx = -\frac{1}{k}\int_\Omega (w^j)^{\frac{1}{m}+1} + \int_\Omega f^j w^j\, dx$$

with $f^j = f^j_{k,n} = a_{k,n} + \frac{1}{k}(w^j_{k,n-1})^{1/m} \in L^\infty(\Omega)$, $||f^j||_{L^\infty(\Omega)} \leq C$, uniformly in j. Using Lebesgue theorem

$$\int_\Omega (w^j)^{\frac{1}{m}+1}\, dx \quad \to \quad \int_\Omega w^{\frac{1}{m}+1}\, dx, \text{ when } j \to \infty,$$

and we deduce that

$$\int_\Omega f^j w^j\, dx \quad \to \quad \int_\Omega f w\, dx, \text{ when } j \to \infty \tag{5}$$

(remember that $f^j \to f$, $w^j \to w$ *strongly* in $L^2(\Omega)$). Then we deduce that

$$\int_\Omega (\delta + Q^{j-1})^S |w^j_x|^p dx \quad \to \quad -\frac{1}{k}\int_\Omega w^{\frac{1}{m}+1}\, dx + \int_\Omega f w\, dx, \text{ when } j \to \infty.$$

But multiplying in $P(h^j_{k,n})$ by w and integrating in Ω, we get

$$\int_\Omega (\delta + Q)^S |w_x|^p\, dx = -\frac{1}{k}\int_\Omega w^{\frac{1}{m}+1} + \int_\Omega f w\, dx.$$

Hence

$$\int_\Omega (\delta + Q^{j-1})^S |w^j_x|^p dx \to \int_\Omega (\delta + Q)^S |w_x|^p\, dx.$$

As a consequence, we get the strong convergence of $(h^m_j)_x$ to $(h^m)_x$ in $L^p(\Omega)$.

Lemma 4.3 *We have that* $w^j_x \to w_x$ strongly *in* $L^p(\Omega)$ *and*

$$(\delta + Q^j)^S |h^j_x|^p \to (\delta + Q)^S |h_x|^p$$

strongly *in* $L^1(\Omega)$.

Proof. Subtracting the equations verified by w_j and w, and multiplying by $w_j - w$ we get

$$I_j \doteq \int_\Omega (\delta + Q^{j-1})^S \left[|w_x^j|^{p-2} w_x^j - |w_x|^{p-2} w_x\right] (w_x^j - w_x)\, dx$$

$$= \int_\Omega \left[(\delta + Q)^S - (\delta + Q^{j-1})^S\right] \left(|w_x|^{p-2} w_x\right) (w_x^j - w_x)\, dx -$$

$$-\frac{1}{k} \int_\Omega \left((w^j)^{\frac{1}{m}} - w^{\frac{1}{m}}\right) (w^j - w)\, dx + \int_0^1 (f^j - f)(w^j - w).$$

By the previous lemma and (5) we deduce that $I_j \to 0$ if $j \to \infty$. Finally, as $p > 2$ (remember that $p = R + 1$ and $R > 1$), Q^j are uniformly bounded in $L^\infty(\Omega)$. Moreover, it is well-known (see, e.g., Díaz ([2], Lemma 4.10)) that there exists $C > 0$ (independently of j) such that

$$C \int_\Omega |w_x^j - w_x|^p \leq \int_\Omega (\delta + Q^{j-1})^S \left[|w_x^j|^{p-2} w_x^j - |w_x|^{p-2} w_x\right] (w^j - w)\, dx.$$

So, $w_x^j \to w_x$, *strongly* in $L^p(\Omega)$. Moreover, since $\{(\delta + Q^j)^S\}$ is uniformly bounded in $L^\infty(\Omega)$ and $|h_x^j|^p \to |h_x|^p$ *strongly* in $L^1(\Omega)$, we obtain the second conclusion. Thus, we have proved

Theorem 4.1 *The sequence* $(h_{k,n}^j, \xi_{k,n}^j, Q_{k,n}^j)$ *converges, when* $j \to \infty$, *to* $(h_{k,n}, \xi_{k,n}, Q_{k,n})$ *solution of* $(S_{k,n})$.

References

[1] BENILAN, PH., *Equations d'evolution dans un espace de Banach quelconque et applications*, Thesis, Univ. Paris XI, Orsay, 1972.

[2] DÍAZ, J.I., *Nonlinear Partial Differential Equations and Free Boundaries.* Ed. Pitman, London, 1985.

[3] FOWLER, A.C., Glaciers and ice sheets, in *The Mathematics of Models for Climatology and Environment,* J.I. Díaz ed., NATO ASI Series I 48, Springer, Berlin, 1997, pp.301-336.

[4] FOWLER, A. AND JOHNSON, C., Ice-sheet surging and ice-stream formation, *Ann. Glaciol.*, **23**, 68-73.

[5] FOWLER, A.C. AND SCHIAVI, E., A theory of ice sheet surges. To appear in *Journal of Glaciology.*

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Singular Finite Element Solutions of the Axisymmetric Extrudate-Swell Problem

We solve the axisymmetric, creeping Newtonian extrudate-swell problem for the case of zero surface tension. Both the standard and the singular finite element methods are used and the convergence of the numerical solutions with mesh refinement is studied. The numerical results show that the singular finite elements accelerate the convergence of the free surface considerably; they perform well when coarse or moderately refined meshes are used.

1 Introduction

In this work, we revisit the singular finite element method (SFEM) developed by Georgiou *et al.* for solving Newtonian flow problems with boundary stress singularities [1, 2]. In the SFEM, special elements incorporating the radial form of the local solution by means of singular basis functions are employed in a small region around the singularity, while standard elements are used in the rest of the domain. The idea of incorporating the form of the local singularity solution into the numerical scheme was borrowed from analogous methods used in fracture mechanics (see, e.g., [1] and references therein). The basic motive behind using singular methods is to improve the accuracy and the rate of convergence of the solution with mesh refinement, which are rather unsatisfactory with standard numerical methods, especially in the neighborhood of the singularity. The poor performance of the standard FEM is attributed to the fact that the calculated pressure and stresses cannot be infinite at the singular point, as required by the local asymptotic solution, and are thus tainted by spurious oscillations. This difficulty is overcome with the SFEM.

Georgiou *et al.* applied the SFEM to the planar Newtonian extrudate-swell problem which describes the extrusion of a viscous fluid through a die into an inviscid medium [2]. This is a well-known free surface problem; at low Reynolds numbers, the fluid swells as it comes out of the die. Another important characteristic of this flow is the presence of a stress singularity at the exit of the die, resulting from the sudden change in the boundary condition from the wall of the die to the free surface of the extrudate. The extrudate-swell problem is extremely important in polymer processing and has thus been the focus of a plethora of experimental and numerical studies in the last twenty-five years [3].

The singular finite element calculations for the planar Newtonian extrudate-swell problem have revealed that the spurious stress oscillations that characterize the stresses in the standard finite element solution are eliminated [2]. Similar observations have been made when solving the planar Newtonian stick-slip and 2:1 expansion problems [1, 2]. The former problem is the special case of the extrudate-swell problem in the

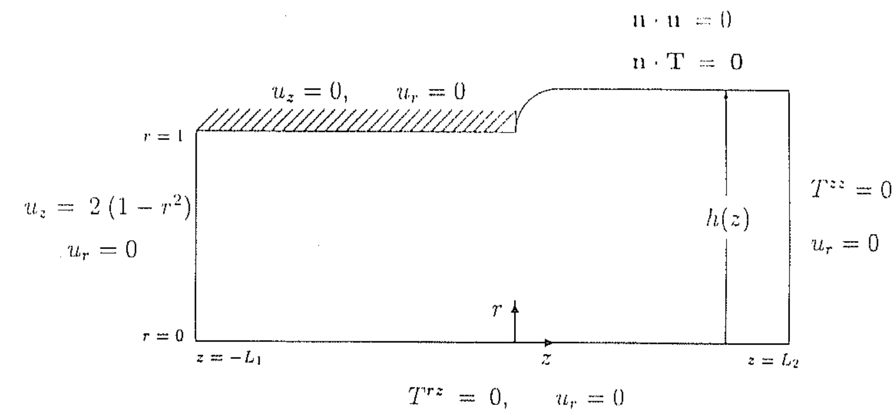

Figure 1: Geometry and boundary conditions for the extrudate-swell problem.

limit of infinite surface tension, in which the free surface becomes completely flat. In the case of the planar extrudate-swell problem, the convergence of the free surface profile with mesh refinement is considerably accelerated by using the singular finite elements [2].

The two main drawbacks of the SFEM have also been addressed in Ref. [1]. First, extensive mesh refinement is not possible with the SFEM. As the mesh is refined, the singular elements become smaller in size, and, consequently, the size of the region over which the singularity is given special attention is reduced. Second, the method can be implemented only if the radial form of the local solution is known, at least approximately. This implies that the method is not applicable to many important problems such as most viscoelastic flow problems in which the inaccuracies, stemming from the failure to approximate satisfactorily the stress behavior near the singularity, are, in general, more severe.

In this paper, we solve the round Newtonian extrudate-swell problem at zero Reynolds number (creeping flow) and zero surface tension, using both the standard and the singular finite element methods. We systematically study the convergence of the numerical solutions with mesh refinement. Our objective is to compare the performance of the two methods and to obtain accurate estimates of the position of the free surface and the extrudate-swell ratio. These results can be quite useful in testing other numerical methods proposed in the literature.

2 Governing Equations

The flow geometry and the dimensionless governing equations and boundary conditions for the steady-state axisymmetric extrudate-swell problem are depicted in Figure 1. The scaling parameter for lengths is the radius R, the velocity vector u is

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